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Irreducible Lie Algebra Extensions of the Poincaré Algebra

I. Extensions with Abelian Kernels

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Abstract. We use cohomology of Lie algebras to analyse the abelian extensions of the Poincaré algebra $\mathcal{P}$. We study particularly the irreducible and truly irreducible extensions: some irreducibility criteria are proved and applied to obtain a classification of types of irreducible abelian extensions of $\mathcal{P}$. We give a characterization of the minimal essential extensions in terms of truly irreducible extensions.

Introduction

The investigation of Lie algebra extensions of the Poincaré algebra has a short history. The only contribution to this analysis is essentially a paper by GALINDO [1]. The more difficult problem of group extensions of the Poincaré group had been discussed formerly by MICHEL [2], in connection with the mixing of internal and space-time symmetry groups. The group extension problem is very hard, especially from the topological point of view, even in the case where only Lie group extensions are considered. This immediately brings about the consideration of Lie algebra extensions. In general, one cannot state that the extensions so obtained have corresponding Lie group extensions. With some connectedness requirements this correspondence can be established [3]. The study of Lie algebra extensions shows up the intrinsic, rather than topological difficulties of the problem. Some manifestations of Lie algebras as fundamental structures in physics suggest also the idea of such an analysis, independently of the corresponding group problem.

We recall in Section I how the cohomology theory of CHEVALLEY-EILENBERG [4] provides for the determination of Lie algebra extensions with abelian kernels [5].

In Section II an important theorem of HOCHSCHILD-SERRE [6] is applied to the study of the abelian extensions of the Poincaré algebra $\mathcal{P}$.

The structure of the Lie algebra obtained by extending $\mathcal{P}$ is analysed in Section III. The irreducibility and true irreducibility of the abelian extensions of $\mathcal{P}$ are examined in Section IV. A classification of types of irreducible abelian extensions and a characterization of the minimal essential extensions are given.

Some Conventions

We denote Lie algebras by capital script letters: $\mathscr{A}, \mathscr{B}, \ldots$ and the underlying vector spaces by the corresponding capital print-like characters: $A, B, \ldots$

If $\mathscr{G}$ is a Lie algebra, A_Φ denotes the $\mathscr{G}$-module structure induced by the representation $\Phi: \mathscr{G} \to \mathrm{End}_F(A)$ on the vector space A. We symbolise the usual exceptional cases of morphisms $\varphi: \mathfrak{A} \to \mathfrak{B}$ of a given algebraic structure as follows:

epimorphism: $\varphi: \mathfrak{A} \twoheadrightarrow \mathfrak{B}$,

monomorphism: $\varphi: \mathfrak{A} \rightarrowtail \mathfrak{B}$,

isomorphism: $\varphi: \mathfrak{A} \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{B}$ or $\mathfrak{A} \approx \mathfrak{B}$.

The direct sum of $\mathfrak{A}$ and $\mathfrak{B}$ is denoted by $\mathfrak{A} \oplus \mathfrak{B}$ and for a semidirect sum of two Lie algebras $\mathscr{A}$ and $\mathscr{B}$ we use the symbol $\mathscr{A} \ni\!\!\!\!+ \mathscr{B}$, if $\mathscr{B}$ is the ideal. $\mathfrak{A} \times \mathfrak{B}$ will stand for the (direct) product of $\mathfrak{A}$ and $\mathfrak{B}$, and $A_\Phi \otimes B_\Phi$, for the tensor product (relative to the field $\boldsymbol{F}$ considered) of the $\mathscr{G}$-modules A_Φ and B_Φ.

Let $\boldsymbol{L}_n(G, K)$ be the vector space of the n-linear maps $G^n \to K$ $\forall\, n > 0$. We define $\boldsymbol{L}_0(G, K) = K$ and write $\boldsymbol{L}_1(G, K) = \boldsymbol{L}(G, K)$. If $\boldsymbol{A}_n(G, K) \subset \boldsymbol{L}_n(G, K)$ is the subspace of the n-linear alternating maps, then $\boldsymbol{A}_1(G, K) = \boldsymbol{L}(G, K)$. We define $\boldsymbol{A}_0(G, K) = K$. $\boldsymbol{N}^+$ will stand for the set of positive integers and $\boldsymbol{N} = \boldsymbol{N}^+ \cup \{0\}$. The symbol $\supset$ ($\supseteqq$) denotes proper (improper) set or extension inclusion throughout the paper.

Only Lie algebras, modules and vector spaces of finite dimension over a field $\boldsymbol{F}$ of characteristic 0 are considered. These restrictions are tacitly understood throughout the paper. Whenever we view $\boldsymbol{F}$ as a $\mathscr{G}$-module we understand it with trivial action: $g \cdot f = 0\ \forall (g \in \mathscr{G}; f \in \boldsymbol{F})$, i.e. $\boldsymbol{F}$ is seen as a trivial $\mathscr{G}$-module.

We define in a $\mathscr{G}$-module A_Φ the invariant vector x by $g \cdot x = 0\ \forall\, g \in \mathscr{G}$. The invariant vectors of A_Φ make up a trivial submodule $A_\Phi^{\mathscr{G}}$.

We call Poincaré algebra $\mathscr{P}$ the real Lie algebra of the Poincaré group. Therefore, whenever we mention the Poincaré algebra, the field $\boldsymbol{F}$ in consideration is $\boldsymbol{R}$.

Let us use the standard symbols $\mathfrak{D}^{(j_1, j_2)}$ for the irreducible finite complex representations of the Lorentz algebra $\mathscr{L}$ (i.e. the real Lie algebra of the Lorentz group). The irreducible finite real representations of $\mathscr{L}$ are then written:

$$\mathfrak{D}^{\{j_1, j_2\}} = \mathfrak{D}^{(j_1, j_2)} \oplus \mathfrak{D}^{(j_2, j_1)} \text{ if } j_1 > j_2 ; \mathfrak{D}^{\{j, j\}} = \mathfrak{D}^{(j, j)} .$$

I. Cohomology of Lie Algebras and Extensions with Abelian Kernels

I. 1. The Chevalley – Eilenberg Cohomology [4]

Let $\mathscr{G}$ be a Lie algebra and V_Φ the $\mathscr{G}$-module associated with the representation $\Phi:\mathscr{G} \to \mathrm{End}_F(V)$. We define the vector spaces:

$$C^n(\mathscr{G}, V_\Phi) = \{f_n \mid f_n \in \boldsymbol{A}_n(G, V)\} \;\forall\; n \in \boldsymbol{N}.$$

The alternating maps f_n are said V_Φ-cochains of degree n or simply $(n; \mathscr{G}, V_\Phi)$-cochains.

On each $C^n(\mathscr{G}, V_\Phi)$ we define the structure of a $\mathscr{G}$-module [4], [6]: $n = 0$: $C^0(\mathscr{G}, V_\Phi) = V_\Phi$ is already a $\mathscr{G}$-module,

$$\begin{aligned}\forall\; n \in \boldsymbol{N}^+: (g \cdot f_n)(g_1, \ldots, g_n) &= \Phi(g)\, f_n(g_1, \ldots, g_n)\\ &\quad - \sum_{i=1}^{n} f_n(g_1, \ldots, g_{i-1}, [g, g_i], g_{i+1}, \ldots, g_n)\\ &\quad \forall (f_n \in C^n(\mathscr{G}, V_\Phi); g, g_1, \ldots, g_n \in \mathscr{G}).\end{aligned} \tag{I.1}$$

We consider the linear maps:

$$i_n(g): C^n(\mathscr{G}, V_\Phi) \to C^{n-1}(\mathscr{G}, V_\Phi) \;\forall\; (g \in \mathscr{G}; n \in \boldsymbol{N}^+)$$

such that

$$\begin{aligned}(i_n(g)\, f_n)(g_1, \ldots, g_{n-1}) = (f_n)_g(g_1, \ldots, g_{n-1}) &= f_n(g, g_1, \ldots, g_{n-1})\\ \forall (f_n \in C^n(\mathscr{G}, V_\Phi); g_1, \ldots, g_{n-1} \in \mathscr{G}).&\end{aligned} \tag{I.2}$$

If $n = 0$: $i_0(g)\, f_0 = 0 \;\forall (g \in \mathscr{G}; f_0 \in C^0(\mathscr{G}, V_\Phi))$. $\quad$ (I.3)

Then there exists one and only one linear map $\delta_n: C^n(\mathscr{G}, V_\Phi) \to C^{n+1}(\mathscr{G}, V_\Phi)$ $\forall\; n \in \boldsymbol{N}$ such that

$$(\delta_n f_n)_g = g \cdot f_n - \delta_{n-1}(f_n)_g \qquad \forall (g \in \mathscr{G}; n \in \boldsymbol{N}^+) \tag{I.4}$$

and

$$(\delta_0 f_0)(g) = \Phi(g)\, f_0 \qquad \forall\; g \in \mathscr{G}. \tag{I.5}$$

δ_n is referrred to as the coboundary operator and reads explicitly:

$$\begin{aligned}(\delta_n f_n)(g_1, \ldots, g_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \Phi(g_i)\, f_n(g_1, \ldots, \hat{g}_i, \ldots, g_{n+1})\\ &\quad + \sum_{j<k} (-1)^{j+k} f_n([g_j, g_k], g_1, \ldots, \hat{g}_j, \ldots, \hat{g}_k, \ldots, g_{n+1}),\end{aligned} \tag{I.6}$$

where the ^ sign indicates the omission of the argument below it. The coboundary operator satisfies the identities

$$\delta_n(g \cdot f_n) = g \cdot (\delta_n f_n) \qquad \forall (n \in \boldsymbol{N}; g \in \mathscr{G}; f_n \in C^n(\mathscr{G}, V_\Phi)) \tag{I.7}$$

and

$$\delta_{n+1}\delta_n = 0 \qquad \forall\; n \in \boldsymbol{N}. \tag{I.8}$$

Once (I.8) is verified we can define the vector spaces:

$$
\begin{aligned}
Z^n(\mathscr{G}, V_\Phi) &= \operatorname{Ker}\delta_n && \forall\, n \in \boldsymbol{N}\,,\\
B^n(\mathscr{G}, V_\Phi) &= \operatorname{Im}\delta_{n-1} && \forall\, n \in \boldsymbol{N}^+,\\
B^0(\mathscr{G}, V_\Phi) &= \{0\}\,,
\end{aligned}
$$

and the quotients $H^n(\mathscr{G}, V_\Phi) = Z^n(\mathscr{G}, V_\Phi)/B^n(\mathscr{G}, V_\Phi)\ \forall\, n \in \boldsymbol{N}$.

$Z^n(\mathscr{G}, V_\Phi)$ is the vector space of the V_Φ-cocycles of degree n or $(n;\ \mathscr{G}, V_\Phi)$-cocycles; $B^n(\mathscr{G}, V_\Phi)$ the vector space of the V_Φ-coboundaries of degree n or $(n;\ \mathscr{G}, V_\Phi)$-coboundaries. The quotient $H^n(\mathscr{G}, V_\Phi)$ is the cohomology space of degree n of $\mathscr{G}$ over V_Φ.

The $\mathscr{G}$-module structure of $C^n(\mathscr{G}, V_\Phi)$ induces a $\mathscr{G}$-module structure on $Z^n(\mathscr{G}, V_\Phi)$ and $B^n(\mathscr{G}, V_\Phi)\ \forall\, n \in \boldsymbol{N}$ by virtue of identity (I.7). By passing to the quotient we obtain a $\mathscr{G}$-module structure on $H^n(\mathscr{G}, V_\Phi)$ $\forall\, n \in \boldsymbol{N}$.

If $f_i \in C^i(\mathscr{G}, V_\Phi)$; $g_1, g_2, g_3 \in \mathscr{G}$, then:

$$
\begin{aligned}
(\delta_0 f_0)\,(g_1) &= \Phi(g_1)\, f_0\\
(\delta_1 f_1)\,(g_1, g_2) &= \Phi(g_1)\, f_1(g_2) - f_1([g_1, g_2]) - \Phi(g_2)\, f_1(g_1)\\
(\delta_2 f_2)\,(g_1, g_2, g_3) &= \Phi(g_1)\, f_2(g_2, g_3) + \Phi(g_2)\, f_2(g_3, g_1) + \Phi(g_3)\, f_2(g_1, g_2)\\
&\quad - f_2([g_1, g_2], g_3) - f_2([g_2, g_3], g_1) - f_2([g_3, g_1]), g_2)\,.
\end{aligned}
$$

The linear maps $f_1 \in \boldsymbol{L}(G, V)$ such that $\delta_1 f_1 = 0$ are referred to as *crossed homomorphisms of* $\mathscr{G}$ *into* V_Φ and the linear maps $\delta_0 f_0 \in \boldsymbol{L}(G, V)$ as *principal crossed homomorphisms*.

I.2. Extensions of Lie Algebras with Abelian Kernels

Let $\mathscr{A}$ and $\mathscr{B}$ be Lie algebras. We call a short exact sequence

$$
\mathscr{A} \overset{\alpha}{\rightarrowtail} \mathscr{E}(\mathscr{B}, \mathscr{A}) \overset{\beta}{\twoheadrightarrow} \mathscr{B}\,,
$$

where $\mathscr{E}$ is a Lie algebra, an *extension of* $\mathscr{B}$ *by* $\mathscr{A}$ [7].

The kernel $\mathscr{K}$ of β is called the kernel of the extension.

Two extensions

$$
\mathscr{A} \overset{\alpha}{\rightarrowtail} \mathscr{E}(\mathscr{B}, \mathscr{A}) \overset{\beta}{\twoheadrightarrow} \mathscr{B} \quad \text{and} \quad \mathscr{A} \overset{\alpha'}{\rightarrowtail} \mathscr{E}'(\mathscr{B}, \mathscr{A}) \overset{\beta'}{\twoheadrightarrow} \mathscr{B}
$$

are equivalent if there is an homomorphism $\gamma : \mathscr{E} \to \mathscr{E}'$ such that the following diagram is commutative:

$$
\begin{tikzcd}
 & \mathscr{E}(\mathscr{B}, \mathscr{A}) \arrow[dd, "\gamma"] \arrow[dr, two heads, "\beta"] & \\
\mathscr{A} \arrow[ur, tail, "\alpha"] \arrow[dr, tail, "\alpha'"'] & & \mathscr{B}\,. \\
 & \mathscr{E}'(\mathscr{B}, \mathscr{A}) \arrow[ur, two heads, "\beta'"'] &
\end{tikzcd}
$$

Necessarily $\mathscr{E} \approx \mathscr{E}'$ and the relation between the two extensions of $\mathscr{B}$ by $\mathscr{A}$ is an equivalence relation.

An extension of $\mathscr{B}$ by $\mathscr{A}$ can be described also as a pair $(\mathscr{E}, \varrho)$ where ϱ is an epimorphism $\mathscr{E} \twoheadrightarrow \mathscr{B}$ and $\mathscr{A} = \operatorname{Ker} \varrho$. Each such pair determines a short exact sequence

$$\mathscr{A} \rightarrowtail \mathscr{E} \xrightarrow{\varrho} \mathrel{\mkern-14mu}\twoheadrightarrow \mathscr{B} \approx \mathscr{E}/\mathscr{A} ,$$

and every extension of $\mathscr{B}$ by $\mathscr{A}$ is equivalent to one so obtained. In this paper we shall always use this definition.

The following exceptional cases of extensions are of particular importance:

1. $(\mathscr{E}, \varrho)$ is *inessential* if there exists a supplementary Lie algebra of $\operatorname{Ker} \varrho$ in $\mathscr{E}$.

2. $(\mathscr{E}, \varrho)$ is *trivial* if it is inessential and the supplementary Lie algebra of $\operatorname{Ker} \varrho$ is an ideal.

3. $(\mathscr{E}, \varrho)$ is *central* if $\operatorname{Ker} \varrho$ is contained in the center of $\mathscr{E}$.

In the following we shall consider only *extensions with abelian kernels*, abreviated as *abelian extensions.*

A section of $(\mathscr{E}, \varrho)$ over $\mathscr{B}$ is a linear map $\sigma: \mathscr{B} \dashrightarrow \mathscr{E}$ such that $(\varrho \circ \sigma)(b) = b \ \forall\, b \in \mathscr{B}$.

We can associate uniquely to any abelian extension $(\mathscr{E}, \varrho)$ of $\mathscr{B}$ by $\mathscr{A}$ a representation $\Phi: \mathscr{B} \to \operatorname{End}_F(A)$. We choose any section σ of $(\mathscr{E}, \varrho)$ over $\mathscr{B}$ and define $\Phi(b) \in \operatorname{End}_F(A) \ \forall\, b \in \mathscr{B}$ as follows: $\Phi(b)\, a = [\sigma(b), a]$ $\forall\, a \in A$ ($[\quad,\quad]$ is the Lie product of $\mathscr{E}$). It is easy to verify that Φ is independent of the choice of σ and that the relation $[\Phi(b), \Phi(b')]\, a$ $= \Phi([b, b'])\, a \ \forall\ (b, b' \in \mathscr{B}; a \in A)$ holds, using the Jacobi identity. We call Φ the representation associated with the extension $(\mathscr{E}, \varrho)$ of $\mathscr{B}$ by $\mathscr{A}$. Φ induces canonically on A the $\mathscr{B}$-module structure A_Φ.

Given the $\mathscr{B}$-module A_Φ and the structure of abelian Lie algebra $\mathscr{A}$ on A, there generally exist several extensions $(\mathscr{E}, \varrho)$ of $\mathscr{B}$ by $\mathscr{A}$ such that the associated representation is Φ. These extensions are called the *extensions of* $\mathscr{B}$ *by* Φ or *extensions of* $\mathscr{B}$ *by* A_Φ.

We can define a vector space structure $\operatorname{Ext}(\mathscr{B}, A_\Phi)$ on the set of equivalence classes of extensions of $\mathscr{B}$ by A_Φ. The zero element of this space is the class of inessential extensions.

Let $(\mathscr{E}, \varrho)$ be a representative element of the class $\{(\mathscr{E}, \varrho)\}$ of equivalent extensions of $\mathscr{B}$ by A_Φ, and σ a section of $(\mathscr{E}, \varrho)$ over $\mathscr{B}$.

Then $\varrho([\sigma(b), \sigma(b')]) = [b, b'] = \varrho(\sigma([b, b'])) \ \ \forall\, b, b' \in \mathscr{B}$ and an element $f_2(b, b') \in A_\Phi$ exists such that

$$[\sigma(b), \sigma(b')] = \sigma([b, b']) + f_2(b, b') \ \forall\, b, b' \in \mathscr{B} . \tag{I.9}$$

$f_2 \in \boldsymbol{A}_2(B, A)$ is called the *factor set* corresponding to the section σ. The Jacobi identity requires $f_2 \in Z^2(\mathscr{B}, A_\Phi)$.

If we choose another section σ' of $(\mathscr{E}, \varrho)$ over $\mathscr{B}$ we have

$$\sigma'(b) - \sigma(b) = f_1(b) \in A_\Phi \ \forall\, b \in \mathscr{B} \tag{I.10}$$

and $f_1 \in C^1(\mathscr{B}, A_\Phi)$.

It follows that

$$f_2'(b, b') = f_2(b, b') + (\delta_1 f_1)(b, b') \ \forall\, b, b' \in \mathscr{B}, \tag{I.11}$$

if f_2' is the factor set corresponding to the section σ'. Then f_2' and f_2 belong to the same class of $Z^2(\mathscr{B}, A_\Phi)$ and the choice of different sections of $(\mathscr{E}, \varrho)$ over $\mathscr{B}$ leaves the cohomology class of the factor set unchanged.

Conversely: given a factor set $f_2 \in Z^2(\mathscr{B}, A_\Phi)$ we can determine a corresponding extension $(\mathscr{E}, \varrho)$ of $\mathscr{B}$ by A_Φ. Obviously $E \approx B \oplus A$ and we can then identify the elements of E with the couples (b, a), where $b \in B$ and $a \in A$, on account of the canonical isomorphism $B \times A \approx B \oplus A$.

We define: $\varrho(b, a) = b$ and $\sigma(b) = (b, 0) \ \forall\,(a \in A; b \in B)$ (this corresponds to the choice of a normalised section). The Lie algebra product is then defined by the bilinear alternating map $\omega: E \times E \to E$ such that

$$\begin{aligned} \omega((b_1, a_1), (b_2, a_2)) &= [(b_1, a_1), (b_2, a_2)] \\ &= ([b_1, b_2], \Phi(b_1)\, a_2 - \Phi(b_2)\, a_1 + f_2(b_1, b_2)) \\ &\quad \forall\,(b_1, b_2 \in B; a_1, a_2 \in A) \end{aligned} \tag{I.12}$$

as is easily verified.

Then

$$\operatorname{Ext}(\mathscr{B}, A_\Phi) \approx H^2(\mathscr{B}, A_\Phi) \tag{I.13}$$

and we can infer immediately that an extension of $\mathscr{B}$ by A_Φ is inessential if and only if $H^2(\mathscr{B}, A_\Phi) = \{0\}$.

We have the following interpretation of the cohomology spaces $H^i(\mathscr{B}, A_\Phi)$ $i = 0, 1, 2$:

$H^0(\mathscr{B}, A_\Phi) = A_\Phi^{\mathscr{B}}$ is the vector space of the invariant vectors of A_Φ;

$H^1(\mathscr{B}, A_\Phi)$ is the vector space of the crossed homomorphisms $f_1 \in \boldsymbol{L}(B, A)$ modulo the principal crossed homomorphisms;

$H^2(\mathscr{B}, A_\Phi)$ is isomorphic to the vector space of equivalence classes of extensions of $\mathscr{B}$ by A_Φ.

II. The Hochschild-Serre Theorem and the Abelian Extensions of the Poincaré Algebra

II.1. Relative Cohomology and the Hochschild-Serre Theorem

Let $\mathscr{G}$ again be a Lie algebra and V_Φ the $\mathscr{G}$-module associated with the representation $\Phi: \mathscr{G} \to \operatorname{End}_F(V)$.

If $\mathscr{H}$ is a subalgebra of $\mathscr{G}$, $f_n \in C^n(\mathscr{G}, V_\Phi)$ is called *orthogonal* to $\mathscr{H}$ provided that:

$$\begin{aligned} h \cdot f_n &= 0 \quad \forall\, h \in \mathscr{H}, \\ (f_n)_h &= 0 \quad \forall\, h \in \mathscr{H}. \end{aligned} \tag{II.1}$$

The orthogonality of f_n to $\mathscr{H}$ implies that of $\delta_n f_n$ and as a consequence we can define the vector spaces:

$$C^n(\mathscr{G}, \mathscr{H}, V_\Phi) = \{f_n \mid f_n \in C^n(\mathscr{G}, V_\Phi);\, h \cdot f_n = 0,\, (f_n)_h = 0 \;\forall h \in \mathscr{H}\}\,,$$

$$Z^n(\mathscr{G}, \mathscr{H}, V_\Phi) = Z^n(\mathscr{G}, V_\Phi) \cap C^n(\mathscr{G}, \mathscr{H}, V_\Phi)\,,$$

$$B^n(\mathscr{G}, \mathscr{H}, V_\Phi) = \delta_{n-1} C^{n-1}(\mathscr{G}, \mathscr{H}, V_\Phi) \qquad \forall\, n \in \boldsymbol{N}^+,$$

$$B^0(\mathscr{G}, \mathscr{H}, V_\Phi) = \{0\}\,.$$

The relative cohomology space of degree n of $\mathscr{G}$ mod $\mathscr{H}$ is given by:

$$H^n(\mathscr{G}, \mathscr{H}, V_\Phi) = Z^n(\mathscr{G}, \mathscr{H}, V_\Phi)/B^n(\mathscr{G}, \mathscr{H}, V_\Phi)\,.$$

The relative cohomology spaces are very important in the case where $\mathscr{H}$ is a Levi subalgebra of $\mathscr{G}$ since the following factorization theorem can be applied.

Theorem 1 *(Hochschild-Serre). Let $\mathscr{G}$ be a Lie algebra and V_Φ a $\mathscr{G}$-module. Suppose that $\mathscr{F}$ is an ideal of $\mathscr{G}$ such that $\mathscr{G}/\mathscr{F}$ is semisimple. Then:*

$$H^n(\mathscr{G}, V_\Phi) \approx \sum_{i+j=n} H^i(\mathscr{G}/\mathscr{F}, \boldsymbol{F}) \otimes H^j(\mathscr{F}, V_\Phi)^{\mathscr{G}} \qquad \forall\, n \in \boldsymbol{N} \quad \text{(II.2)}$$

($\mathscr{G}$-module isomorphism).

Proof. [6], pp. 602–603.

If $\mathscr{G}/\mathscr{F}$ is semisimple, there is a subalgebra $\mathscr{S}$ of $\mathscr{G}$ such that $\mathscr{G}/\mathscr{F} \approx \mathscr{S}$ by virtue of the canonical epimorphism $\mathscr{G} \twoheadrightarrow \mathscr{G}/\mathscr{F}$. $H^i(\mathscr{S}, \boldsymbol{F})$ and $H^i(\mathscr{G}/\mathscr{F}, \boldsymbol{F})$ can then obviously be identified and it is easy to verify that $H^j(\mathscr{F}, V_\Phi)^{\mathscr{G}} \approx H^j(\mathscr{G}, \mathscr{S}, V_\Phi)$. Therefore we write:

$$H^n(\mathscr{G}, V_\Phi) \approx \sum_{i+j=n} H^i(\mathscr{S}, \boldsymbol{F}) \otimes H^j(\mathscr{G}, \mathscr{S}, V_\Phi) \qquad \forall\, n \in \boldsymbol{N}\,. \quad \text{(II.3)}$$

We have that $H^0(\mathscr{S}, \boldsymbol{F}) \approx \boldsymbol{F}$ and $H^1(\mathscr{S}, \boldsymbol{F}) = \{0\}$, since a Lie algebra $\mathscr{S}$ is semisimple if and only if the condition $H^1(\mathscr{S}, V_\Phi) = \{0\}$ is verified for every $\mathscr{S}$-module V_Φ. Also $H^2(\mathscr{S}, \boldsymbol{F}) = \{0\}$, because of the fact that all extensions by a trivial $\mathscr{S}$-module of a semisimple Lie algebra $\mathscr{S}$ are trivial.

Remarks. 1. More generally: $H^2(\mathscr{S}, V_\Phi) = H^1(\mathscr{S}, V_\Phi) = \{0\}$ for every semisimple Lie algebra $\mathscr{S}$ and every $\mathscr{S}$-module V_Φ. This is the cohomological translation of the two Lemmas of WHITEHEAD [8].

2. Let $\mathscr{S}$ be a semisimple Lie algebra, then $H^3(\mathscr{S}, \boldsymbol{F}) = \{0\}$ is not necessarily true.

We consider the case of the real Lie algebra $\boldsymbol{so}_3$ of the group $\boldsymbol{SO}_3$. Let $\{x_l\}_{l=1,2,3}$ be the standard basis of $\boldsymbol{so}_3$ with $[x_i, x_j] = \varepsilon_{ijk} x_k$. Then:

a) $H^0(\boldsymbol{so}_3, \boldsymbol{R}) \approx Z^0(\boldsymbol{so}_3, \boldsymbol{R}) = C^0(\boldsymbol{so}_3, \boldsymbol{R}) = \boldsymbol{R}$.

b) $Z^1(\boldsymbol{so}_3, \boldsymbol{R}) = \{0\}$, since $(\delta_1 f_1)\,(x_i, x_j) = f_1([x_j, x_i]) = 0\;\forall$ $(f_1 \in Z^1(\boldsymbol{so}_3, \boldsymbol{R});\, x_i, x_j \in \{x_l\})$ implies $f_1 = 0 \;\forall\, f_1 \in Z^1(\boldsymbol{so}_3, \boldsymbol{R})$. Therefore $B^1(\boldsymbol{so}_3, \boldsymbol{R}) = \{0\}$ and $H^1(\boldsymbol{so}_3, \boldsymbol{R}) = \{0\}$.

c) $Z^2(\boldsymbol{so}_3, \boldsymbol{R}) = C^2(\boldsymbol{so}_3, \boldsymbol{R})$ since $(\delta_2 f_2)(x_i, x_j, x_k) = \sum_{\text{cycl.}} - f_2([x_i, x_j], x_k) = 0 \;\forall\; (f_2 \in C^2(\boldsymbol{so}_3, \boldsymbol{R});\, x_i, x_j, x_k \in \{x_l\})$. It is easy to verify the relation: $B^2(\boldsymbol{so}_3, \boldsymbol{R}) = C^2(\boldsymbol{so}_3, \boldsymbol{R})$. Hence $H^2(\boldsymbol{so}_3, \boldsymbol{R}) = \{0\}$.

d) $Z^3(\boldsymbol{so}_3, \boldsymbol{R}) = C^3(\boldsymbol{so}_3, \boldsymbol{R})$, as $\delta_3 f_3 = 0 \;\forall\; f_3 \in C^3(\boldsymbol{so}_3, \boldsymbol{R})$. $B^3(\boldsymbol{so}_3, \boldsymbol{R}) = \delta_2 C^2(\boldsymbol{so}_3, \boldsymbol{R}) = \{0\}$ and then $H^3(\boldsymbol{so}_3, \boldsymbol{R}) \approx C^3(\boldsymbol{so}_3, \boldsymbol{R}) \neq \{0\}$ (the mixed product $(x|y|z)$ is a trilinear alternating form over $\boldsymbol{so}_3$).

e) $C^n(\boldsymbol{so}_3, \boldsymbol{R}) = \{0\} \;\forall\; n > 3$, hence $H^n(\boldsymbol{so}_3, \boldsymbol{R}) = \{0\}$ too.

By the Hochschild-Serre theorem we have in particular for the $\mathscr{G}$-modules $H^i(\mathscr{G}, V_\Phi)$ $(i = 0, 1, 2)$:

$$\begin{aligned} H^0(\mathscr{G}, V_\Phi) &\approx \boldsymbol{F} \otimes H^0(\mathscr{G}, \mathscr{S}, V_\Phi) \approx V_\Phi^{\mathscr{G}}, \\ H^1(\mathscr{G}, V_\Phi) &\approx \boldsymbol{F} \otimes H^1(\mathscr{G}, \mathscr{S}, V_\Phi) \approx H^1(\mathscr{G}, \mathscr{S}, V_\Phi), \\ H^2(\mathscr{G}, V_\Phi) &\approx \boldsymbol{F} \otimes H^2(\mathscr{G}, \mathscr{S}, V_\Phi) \approx H^2(\mathscr{G}, \mathscr{S}, V_\Phi). \end{aligned} \tag{II.4}$$

For these exceptional cases the existence of an isomorphism between the cohomology spaces and the relative cohomology spaces can also be proved directly [9].

II.2. The $\mathscr{P}$-Modules $H^i(\mathscr{P}, K_\Phi)$ $(i = 0, 1, 2)$

Now let $\mathscr{P}$ be the Poincaré algebra with the subalgebras $\mathscr{L}$ (Lorentz algebra) and $\mathscr{T}$ (translation algebra). We shall discuss the cohomology spaces $H^i(\mathscr{P}, K_\Phi)$ $(i = 0, 1, 2)$ of $\mathscr{P}$ with values in the $\mathscr{P}$-module K_Φ.

The abelian Lie algebra structure on K_Φ is tacitly understood throughout. Of course K_Φ can be considered as a semisimple $\mathscr{L}$-module associated with the restriction $\Phi|\mathscr{L}$ of Φ to $\mathscr{L}$.

a) $H^0(\mathscr{P}, K_\Phi)$: The relevant vector spaces are:

$$\begin{aligned} &C^0(\mathscr{P}, K_\Phi) = K_\Phi;\; B^0(\mathscr{P}, K_\Phi) = \{0\}, \\ &Z^0(\mathscr{P}, K_\Phi) = \{k | k \in K_\Phi;\, \Phi(p)\, k = 0 \;\forall\; p \in \mathscr{P}\} = K_\Phi^{\mathscr{P}}, \\ &H^0(\mathscr{P}, K_\Phi) \approx Z^0(\mathscr{P}, K_\Phi) = K_\Phi^{\mathscr{P}}. \end{aligned} \tag{II.5}$$

In the Lie algebra $\mathscr{E}(\mathscr{P}, \mathscr{K})$ the elements of $K_\Phi^{\mathscr{P}}$ make up the center $\mathscr{C}(\mathscr{E})$. Therefore if $H^0(\mathscr{P}, K_\Phi) = \{0\}$, we have only extensions $(\mathscr{E}, \tau)$ of $\mathscr{P}$ by K_Φ with $\mathscr{C}(\mathscr{E}) = \{0\}$ and conversely.

b) $H^1(\mathscr{P}, K_\Phi)$:

$$\begin{aligned} C^1(\mathscr{P}, \mathscr{L}, K_\Phi) = \{f_1 | f_1 \in \boldsymbol{L}(P, K);\, f_1(l) = 0 \;\forall\; l \in \mathscr{L}; \\ \Phi(l)\, f_1(t) = f_1([l, t]) \;\forall\; (l \in \mathscr{L};\, t \in \mathscr{T})\}. \end{aligned} \tag{II.6}$$

If $f_1 \neq 0$ we have the induced structure of simple $\mathscr{L}$-module $K_{\mathfrak{H}\{1/2, 1/2\}}$ on the set $\operatorname{Im} f_1 = \{f_1(t) \,|\, t \in \mathscr{T}\} \subseteq K_\Phi$.

$$\begin{aligned} Z^1(\mathscr{P}, \mathscr{L}, K_\Phi) = \{f_1 \,|\, f_1 \in C^1(\mathscr{P}, \mathscr{L}, K_\Phi);\, \Phi(t_1)\, f_1(t_2) \\ = \Phi(t_2)\, f_1(t_1) \;\forall\; t_1, t_2 \in \mathscr{T}\} \end{aligned} \tag{II.7}$$

and

$$B^1(\mathscr{P}, \mathscr{L}, K_\Phi) = \{f_1 | f_1(t) = \Phi(t)\, f_0;\, f_1(l) = \Phi(l)\, f_0 = 0 \\ \forall\, (t \in \mathscr{T};\, l \in \mathscr{L};\, f_0 \in K_\Phi^{\mathscr{L}})\}\,. \tag{II.8}$$

By the theorem of HOCHSCHILD-SERRE we can choose a representative element $f_1 \in Z^1(\mathscr{P}, \mathscr{L}, K_\Phi)$ in every cohomology class of $Z^1(\mathscr{P}, K_\Phi)$.

Proposition 1. *If the semisimple $\mathscr{L}$-module K_Φ has no simple components $K_{\mathfrak{D}\{1/2,1/2\}}$, then $H^1(\mathscr{P}, K_\Phi) = \{0\}$.*

c) $H^2(\mathscr{P}, K_\Phi)$:

$$C^2(\mathscr{P}, \mathscr{L}, K_\Phi) = \{f_2 | f_2 \in A_2(P, K);\, f_2(l_1, l_2) = f_2(l, t) = 0 \\ \forall\, (l, l_1, l_2 \in \mathscr{L};\, t \in \mathscr{T});\, \Phi(l)\, f_2(t_1, t_2) = f_2([l, t_1], t_2) \\ + f_2(t_1, [l, t_2])\, \forall\, (t_1, t_2 \in \mathscr{T};\, l \in \mathscr{L})\}\,. \tag{II.9}$$

On the set $\mathrm{Im} f_2 = \{f_2(t_1, t_2) | t_1, t_2 \in \mathscr{T}\} \subseteq K_\Phi$ we have the induced structure of simple $\mathscr{L}$-module $K_{\mathfrak{D}\{1,0\}}$ if $f_2 \neq 0$.

$$Z^2(\mathscr{P}, \mathscr{L}, K_\Phi) = \{f_2 | f_2 \in C^2(\mathscr{P}, \mathscr{L}, K_\Phi);\, \Phi(t_1)\, f_2(t_2, t_3) \\ + \Phi(t_2)\, f_2(t_3, t_1) + \Phi(t_3)\, f_2(t_1, t_2) = 0\, \forall\, t_1, t_2, t_3 \in \mathscr{T}\}\,. \tag{II.10}$$

We consider in $\mathscr{T}$ the standard basis $\{t_0, t_1, t_2, t_3\}$. The condition $f_2 \in Z^2(\mathscr{P}, \mathscr{L}, K_\Phi)$ reads:

$$\Phi(t_\mu)\, f_2(t_\nu, t_\sigma) + \Phi(t_\nu)\, f_2(t_\sigma, t_\mu) + \Phi(t_\sigma)\, f_2(t_\mu, t_\nu) = 0\; \forall\, \mu, \nu, \sigma \in \{0, 1, 2, 3\}.$$

Let us define $f_{2\mu\nu} = f_2(t_\mu, t_\nu)$ and $\Phi_\sigma = \Phi(t_\sigma)$. The vectors $\Phi_\sigma f_{2\mu\nu}$ belong to a $\mathscr{L}$-module $K_{\mathfrak{D}\{1/2,1/2\} \otimes \mathfrak{D}\{1,0\}} \approx K_{\mathfrak{D}\{3/2,1/2\}} \oplus 2K_{\mathfrak{D}\{1/2,1/2\}}$. The two simple $\mathscr{L}$-modules $K_{\mathfrak{D}\{1/2,1/2\}}$ are generated by the vectors $t'_\varrho = \varepsilon_{\varrho\sigma\mu\nu}\, \Phi^\sigma\, f_2^{\mu\nu}$ and $t''_\varrho = \Phi^\mu f_{2\mu\varrho}$. Therefore: $f_2 \in Z^2(\mathscr{P}, \mathscr{L}, K_\Phi)$, provided that $t'_\varrho = 0\; \forall\, \varrho \in \{0, 1, 2, 3\}$, that is if the $\mathscr{L}$-module generated by $\{t'_\varrho\}$ is identically zero.

The theorem of HOCHSCHILD-SERRE states that every cohomology class of $Z^2(\mathscr{P}, K_\Phi)$ contains a representative element $f_2 \in Z^2(\mathscr{P}, \mathscr{L}, K_\Phi)$.

Proposition 2. *If the semisimple $\mathscr{L}$-module K_Φ does not contain any simple component $K_{\mathfrak{D}\{1,0\}}$, then $H^2(\mathscr{P}, K_\Phi) = \{0\}$.*

Corollary. *All extensions $(\mathscr{E}, \tau)$ of $\mathscr{P}$ by K_Φ are inessential, provided that the semisimple $\mathscr{L}$-module K_Φ does not contain simple components $K_{\mathfrak{D}\{1,0\}}$.*

II.3. Essential and Inessential Extensions

We are going to discuss the extensions of $\mathscr{P}$ with an abelian kernel $\mathscr{K}$. Given a representation $\Phi: \mathscr{P} \to \mathrm{End}_{\mathbf{R}}(K)$ which defines the structure of a $\mathscr{P}$-module K_Φ on K and given a representative element of an equivalence class of $Z^2(\mathscr{P}, K_\Phi)$, we have a representative element of the corresponding class of equivalent extensions of $\mathscr{P}$ by K_Φ.

There is an extension with factor set $f_2 \in Z^2(\mathcal{P}, \mathcal{L}, K_\Phi)$, constructed according to (I.12), in every equivalence class. $(\mathcal{E}, \tau)_{\Phi, f_2}$ will denote the extension $(\mathcal{E}, \tau)$ of $\mathcal{P}$ by K_Φ considered with the factor set $f_2 \in Z^2(\mathcal{P}, \mathcal{L}, K_\Phi)$ throughout the paper.

We consider an extension $(\mathcal{E}, \tau)_{\Phi, f_2}$. If $f_2 \neq 0$, a simple $\mathcal{L}$-module $K_{\mathfrak{D}\{1,0\}} \subseteqq K_\Phi$ is associated with f_2.

We call such a simple $\mathcal{L}$-module the *fundamental $\mathcal{L}$-module $K(f_2)$ of $(\mathcal{E}, \tau)_{\Phi, f_2}$*. The $\mathcal{L}$-module $K(f_2) = \{0\}$ is associated with $f_2 = 0$.

If $K(f_2) = \{0\}$ the extensions of the class $\{(\mathcal{E}, \tau)_{\Phi, f_2}\}$ are inessential.

If $K(f_2) \neq \{0\}$ and $C^1(\mathcal{P}, \mathcal{L}, K_\Phi) = Z^1(\mathcal{P}, \mathcal{L}, K_\Phi)$ are satisfied, then the extensions of the class $\{(\mathcal{E}, \tau)_{\Phi, f_2}\}$ are essential (i.e. not inessential). This is clearly the case if $K(f_2) \neq \{0\}$ and $C^1(\mathcal{P}, \mathcal{L}, K_\Phi) = \{0\}$, i.e. if K_Φ does not contain simple components $K_{\mathfrak{D}\{1/2, 1/2\}}$.

Proposition 3. *The extensions of the class $\{(\mathcal{E}, \tau)_{\Phi, f_2}\}$ with $K(f_2) \neq \{0\}$ are inessential if and only if there exists a $f_1 \in C^1(\mathcal{P}, \mathcal{L}, K_\Phi)$ and a pair of elements $t_1, t_2 \in \mathcal{T}$, such that:*

$$(\delta_1 f_1)(t_1, t_2) \in K(f_2); \; (\delta_1 f_1)(t_1, t_2) \neq 0 . \tag{II.11}$$

Proof. The necessity of this condition is obvious. The condition is also sufficient: if (II.11) is satisfied we have $(\delta_1 f_1)(t_1, t_2) = k \in K(f_2)$ and the application to k of the endomorphisms $\Phi(l) \; \forall \, l \in \mathcal{L}$ gives the vectors $(\delta_1 f_1)(t, t') \; \forall \, t, t' \in \mathcal{T}$. Thus $\mathrm{Im}(\delta_1 f_1) = K(f_2)$.

Let A be the linear transformation defined by $A(\delta_1 f_1)(t, t') = f_2(t, t')$ $\forall \, t, t' \in \mathcal{T}$ and $\Phi(l) | K(f_2)$ the restriction of $\Phi(l)$ to $K(f_2)$. Then $A(\Phi(l) | K(f_2)) A^{-1} = \Phi(l) | K(f_2) \; \forall \, l \in \mathcal{L}$ and by SCHUR's Lemma $A = \lambda I$. If we choose $f_1' = \lambda f_1$, then $\delta_1 f_1' = f_2$. This proves the proposition.

III. The Structure of $\mathcal{E}(\mathcal{P}, \mathcal{K})$

III.1. A Levi Decomposition of $\mathcal{E}(\mathcal{P}, \mathcal{K})$

By LEVI's theorem [7, 8] every Lie algebra $\mathcal{G}$ has a decomposition $\mathcal{G} = \mathcal{S} \oplus \mathcal{R}$ where $\mathcal{S}$ and $\mathcal{R}$ denote a Levi subalgebra and the radical of $\mathcal{G}$ respectively. We have the following structure theorem.

Theorem 2. *Let $(\mathcal{E}, \tau)$ be an abelian extension of $\mathcal{P}$. The Lie algebra $\mathcal{E}$ then contains a subalgebra isomorphic to $\mathcal{L}$ by τ.*

Proof. The Lie algebra structure on $E \approx P \oplus K$ is defined by a bilinear alternating map $\alpha' : E \times E \to E$ such that:

$$\begin{aligned} \alpha'((p_1, 0), (p_2, 0)) &= (\alpha(p_1, p_2), f_2(p_1, p_2)) \;\; \forall \, p_1, p_2 \in P , \\ \alpha'((0, k_1), (0, k_2)) &= (0, 0) \qquad \forall \, k_1, k_2 \in K , \\ \alpha'((p, 0), (0, k)) &= (0, \Phi(p) k)) \; \forall \, (p \in P; k \in K) , \end{aligned} \tag{III.1}$$

where $\alpha : P \times P \to P$ is the bilinear alternating map which defines the Lie product of $\mathcal{P}$. Φ and f_2 are respectively the representation and the factor set determining the extension $(\mathscr{E}, \tau)$. The Lie algebra $\mathscr{E}$ being isomorphic to all Lie algebras of the class of equivalent extensions $\{(\mathscr{E}, \tau)\}$, we can choose $f_2 \in Z^2(\mathscr{P}, \mathscr{L}, K_\Phi)$ and the corresponding extension $(\mathscr{E}', \tau')_{\Phi, f_2}$ in $\{(\mathscr{E}, \tau)\}$. If we identify $\mathscr{E}'$ and $\mathscr{E}$, then $\alpha'((l_1, 0), (l_2, 0)) = (\alpha(l_1, l_2), 0)$ $\forall\, l_1, l_2 \in L$. We consider the monomorphism $\psi: L \rightarrowtail E$ such that $\psi(l) = (l, 0)$ $\forall\, l \in L$. ψ is a Lie algebra monomorphism as can easily be verified. Therefore: $\mathscr{L} \approx \operatorname{Im}\psi$ and $\tau | \operatorname{Im}\psi = \psi^{-1}$.

Every extension of the Lorentz algebra (abelian or not) is inessential because of the simplicity of $\mathscr{L}$. The following theorem is a generalization of the trivial statement that every inessential extension of the Poincaré algebra $\mathscr{P}$ is also an extension of $\mathscr{L}$.

Theorem 3. *Let $(\mathscr{E}, \tau)$ be an abelian extension of $\mathscr{P}$ by $\mathscr{K}$. Then there exists an inessential extension $(\mathscr{E}, \tau')$ of $\mathscr{L}$ such that τ' factors uniquely through τ. $(\mathscr{E}, \tau')$ is an abelian extension of $\mathscr{L}$ if and only if $(\mathscr{E}, \tau)$ is an inessential abelian extension of $\mathscr{P}$ where $\mathscr{E} \approx \mathscr{L} \mathbin{+\!\!\!\!\supset} (\mathscr{T} \oplus \mathscr{K})$.*

Proof. We use the notation of the proof of Theorem 2. By this theorem there exists a monomorphism $\psi: \mathscr{L} \rightarrowtail \mathscr{E}$. $\operatorname{Im}\psi$ has a supplementary Lie algebra in $\mathscr{E}$ which is an ideal. This is easy to verify: we see that α' induces the structure of a Lie algebra on the subspace $R = T' \oplus K$, $R \subset E$ and $T' \approx T$ by τ, making it an ideal of $\mathscr{E}$. $\mathscr{R}$ contains the abelian ideal $\mathscr{K}$, but we have in $\mathscr{E}$ no Lie algebra structure on $T' \approx T$ if $f_2 \neq 0$, since $\alpha'((t_1, 0), (t_2, 0)) = (0, f_2(t_1, t_2))$ $\forall\, t_1, t_2 \in T$. We consider $\tau'': \mathscr{P} \twoheadrightarrow \mathscr{L}$. Then $\tau' = \tau'' \circ \tau$ is an epimorphism $\tau': \mathscr{E} \twoheadrightarrow \mathscr{L}$ and $(\mathscr{E}, \tau')$ is an inessential extension of $\mathscr{L}$. The Lie algebra $\mathscr{R}$ is abelian if and only if $\alpha'((t_1, 0), (t_2, 0)) = (0, 0)$ $\forall\, t_1, t_2 \in T$ and $\mathscr{R} \approx \mathscr{T} \oplus \mathscr{K}$.

From the proof of Theorem 3 we infer that every abelian extension $(\mathscr{E}, \tau)$ of $\mathscr{P}$ by $\mathscr{K}$ is such that $\mathscr{E} = \operatorname{Im}\psi \mathbin{+\!\!\!\!\supset} \mathscr{R}$ where the abelian ideal $\mathscr{K}$ belongs to $\mathscr{R}$, $R \approx T \oplus K$,

$$[(t_1, 0), (t_2, 0)] = (0, f_2(t_1, t_2))\ \forall\, t_1, t_2 \in T \quad \text{and} \quad f_2 \in Z^2(\mathscr{P}, \mathscr{L}, K_\Phi)\,.$$

Therefore $\mathscr{E} = \mathscr{L}' \mathbin{+\!\!\!\!\supset} \mathscr{R}$ where $\mathscr{L}' \approx \mathscr{L}$ by τ. This is a Levi decomposition of $\mathscr{E}$ since the ideal $\mathscr{R}$ is solvable, with $D^1\mathscr{R} = [\mathscr{R}, \mathscr{R}] \subseteq \mathscr{K}$ and $D^2\mathscr{R} = [D^1\mathscr{R}, D^1\mathscr{R}] = \{0\}$.

Corollary. *If $(\mathscr{E}, \tau)$ is an abelian extension of $\mathscr{P}$, then $\mathscr{E} = \mathscr{L}' \mathbin{+\!\!\!\!\supset} \mathscr{R}$, where $\mathscr{L}' \approx \mathscr{L}$ by τ, is a Levi decomposition of $\mathscr{E}$. The radical $\mathscr{R}$ is such that $D^2\mathscr{R} = \{0\}$.*

$\mathscr{R}$ is the biggest nilpotent ideal of $\mathscr{E}$:

1. If K_Φ does not contain any simple trivial $\mathscr{L}$-module $K_{\mathfrak{D}\{0,0\}} \neq \{0\}$, then $\mathscr{R}$ is the nil-radical of $\mathscr{E}$, since $[\mathscr{E}, \mathscr{R}] = \mathscr{R}$. Hence $\mathscr{R}$ is the intersection of the kernels of all the finite irreducible representations of $\mathscr{E}$. In this case each simple $\mathscr{E}$-module is also a simple $\mathscr{L}$-module and conversely. $\mathscr{R}$ is of course the biggest nilpotent ideal of $\mathscr{E}$.

2. If K_Φ contains a simple $\mathscr{L}$-module $K_{\mathfrak{D}\{0,0\}} \neq \{0\}$, we consider the descending central series $\{C^i\mathscr{R}\}$ of ideals of $\mathscr{R}$. $C^1\mathscr{R} = [\mathscr{R}, \mathscr{R}] \subseteq \mathscr{K}$, $C^2\mathscr{R} = [\mathscr{R}, C^1\mathscr{R}] \subseteq \Phi(\mathscr{T})\,\mathscr{K}, \ldots, C^i\mathscr{R} = [\mathscr{R}, C^{i-1}\mathscr{R}] \subseteq \Phi^{i-1}(\mathscr{T})\,\mathscr{K}, \ldots$ where $\Phi^i(\mathscr{T})\ \mathscr{K} = \overline{\{\Phi^{i_1}(t_1)\ \Phi^{i_2}(t_2) \ldots \Phi^{i_j}(t_j)\ k | t_1, t_2, \ldots, t_j \in \mathscr{T}; i_1 + i_2 + \cdots + i_j = i; k \in \mathscr{K}\}}$. $\overline{\{\ \}}$ means the natural abelian Lie algebra spanned by $\{\ \}$. There exists $n \in \boldsymbol{N}^+$ such that $\Phi^n(\mathscr{T}) = \{0\}$, since $\mathscr{T}$ is the nil-radic al of $\mathscr{P}$. Therefore $C^{n+1}\mathscr{R} = \{0\}$ and $\mathscr{R}$ is the biggest nilpotent ideal of $\mathscr{E}$.

Clearly, if $(\mathscr{E}, \tau)$ is an extension of the Poincaré algebra by K_Φ, $\mathscr{E}$ contains a subalgebra isomorphic to $\mathscr{P}$ in the following cases:

1. $(\mathscr{E}, \tau)$ is an inessential extension of $\mathscr{P}$ by K_Φ,
2. the semisimple $\mathscr{L}$-module K_Φ has a simple component $K_{\mathfrak{D}\{1/2,1/2\}}$.

III.2. Extensions of $\mathscr{P}$ by Simple $\mathscr{P}$-modules K_Φ

The structure K_Φ of simple $\mathscr{P}$-module is induced on the vector space K by the irreducible representation $\Phi: \mathscr{P} \to \mathrm{End}_{\boldsymbol{R}}(K)$. The simple $\mathscr{P}$-module K_Φ has also a simple $\mathscr{L}$-module structure $K_{\mathfrak{D}\{j_1,j_2\}}$. This follows from the fact that $\mathscr{T}$ is the nil-radical of $\mathscr{P}$. It follows from the Corollary to Proposition 2 that among the extensions by the simple $\mathscr{P}$-modules $K_\Phi = K_{\mathfrak{D}\{j_1,j_2\}}$, only those by $K_{\mathfrak{D}\{1,0\}}$ can be essential. Therefore:

$$\mathrm{Ext}(\mathscr{P}, K_{\mathfrak{D}\{j_1,j_2\}}) = \{0\}\ \forall\ \{j_1, j_2\} \neq \{1, 0\}\,. \tag{III.2}$$

Or, equivalently, the only essential extensions of $\mathscr{P}$ by a finite irreducible representation, are those by an abelian Lie algebra of dimension 6. As a consequence of SCHUR's Lemma we have

$$\dim \mathrm{Ext}(\mathscr{P}, K_{\mathfrak{D}\{1,0\}}) = \dim H^2(\mathscr{P}, K_{\mathfrak{D}\{1,0\}}) = 1\,. \tag{III.3}$$

Definition 1. *The essential extensions of $\mathscr{P}$ by a simple $\mathscr{P}$-module $K_{\mathfrak{D}\{1,0\}}$ are referred to as minimal essential extensions.*

Let $(\mathscr{E}_1, \tau_1)$ be a minimal essential extension. $\mathscr{E}_1 = \mathscr{L}'_1 \oplus \mathscr{R}_1$, with $\mathscr{L}'_1 \approx \mathscr{L}$ by τ_1, is a Lie algebra of dimension 16 whose radical satisfies: $D^1\mathscr{R}_1 = C^1\mathscr{R}_1 = \mathscr{K}$, $D^2\mathscr{R}_1 = C^2\mathscr{R}_1 = \{0\}$ $(K_\Phi = K_{\mathfrak{D}\{1,0\}} = K(f_2))$.

Let us introduce in $\mathscr{E}_1$ the basis $\{l'_{\mu\nu}\}_{\mu,\nu \in I}$, $\{t'_\varrho\}_{\varrho \in I}$, $\{k_{\sigma\tau}\}_{\sigma,\tau \in I}$, $I = \{0,1,2,3\}$, such that:

$\{l'_{\mu\nu}\}$ generates a subalgebra $\mathscr{L}'_1 \approx \mathscr{L}$ by τ_1,

$\{t'_\varrho\}$ generates a vector subspace T' isomorphic to T by τ_1,

$\{k_{\sigma\tau}\}$ generates the abelian subalgebra $\mathscr{K}$,

and with: $[t'_\sigma, t'_\tau] = f_2(t_\sigma, t_\tau) = k_{\sigma\tau}$.

These subspaces and Lie algebras are linked in $\mathscr{E}_1$ by: $\mathrm{ad}_{T'}\mathscr{L}'_1 \approx \mathrm{ad}_T\mathscr{L}$, $\mathrm{ad}_K\mathscr{L}'_1 \approx \mathrm{ad}_L\mathscr{L}$, and $\mathrm{ad}_{T'}\mathscr{K} = \{0\}$ (see also [1]).

If K_Φ is a semisimple $\mathscr{P}$-module, i.e. if the representation Φ is completely reducible, only the extensions by K_Φ with simple components $K_{\mathfrak{D}\{1,0\}}$ can give essential extensions. Then we have: $\mathscr{E} \approx \mathscr{E}_1 \dotplus \mathscr{K}'$, with $K_\Phi = K'_\Phi \oplus K(f_2)$, provided that $(\mathscr{E}, \tau)_{\Phi, f_2}$ is an essential extension of $\mathscr{P}$ by K_Φ semisimple.

Let Φ be the trivial representation of $\mathscr{P}$ in K, i.e. $\Phi = 0$. There exists one and only one equivalence class $\{(\mathscr{E}, \tau)_{0, f_2}\}$ of extensions of $\mathscr{P}$ by K_0 (the class of the central extensions of $\mathscr{P}$ by $\mathscr{K}$) since $f_2 = 0 \; \forall \, f_2 \in Z^2(\mathscr{P}, \mathscr{L}, K_0)$. Moreover any central extension of $\mathscr{P}$ by $\mathscr{K}$ is obviously trivial. This is an exceptional case of a result of MICHEL [2] and GALINDO [1].

IV. The Irreducible and Truly Irreducible Abelian Extensions of $\mathscr{P}$

IV.1. Irreducibility and True Irreducibility of Abelian Extensions

An abelian extension $(\mathscr{E}, \varrho)$ of $\mathscr{B}$ can contain an extension $(\mathscr{E}', \varrho')$ in a sense to be specified later on.

Definition 2. [4] *Let $(\mathscr{E}, \varrho)$ be an abelian extension of $\mathscr{B}$. We call $(\mathscr{E}, \varrho)$ irreducible if there is no proper subalgebra $\mathscr{E}' \subset \mathscr{E}$ such that $\varrho'(\mathscr{E}') = \varrho|\mathscr{E}'(\mathscr{E}') = \mathscr{B}$.*

When an abelian extension $(\mathscr{E}, \varrho)$ is irreducible (reducible), all extensions of the equivalence class $\{(\mathscr{E}, \varrho)\}$ are irreducible (reducible). $(\mathscr{E}, \varrho) \supset (\mathscr{E}', \varrho')$ means that $(\mathscr{E}, \varrho)$ is reducible and $\mathscr{E} \supset \mathscr{E}', \varrho|\mathscr{E}' = \varrho'$. $(\mathscr{E}, \varrho) = (\mathscr{E}', \varrho')$ means $\mathscr{E} = \mathscr{E}', \varrho = \varrho'$. It is clear that if $(\mathscr{E}, \varrho) \supseteqq (\mathscr{E}', \varrho')$ and $(\mathscr{E}', \varrho') \supseteqq (\mathscr{E}'', \varrho'')$, then $(\mathscr{E}, \varrho) \supseteqq (\mathscr{E}'', \varrho'')$. We say that $(\mathscr{E}, \varrho)$ contains $(\mathscr{E}', \varrho')$ if $(\mathscr{E}, \varrho) \supseteqq (\mathscr{E}', \varrho')$.

An abelian extension $(\mathscr{E}, \varrho)$ can contain several extensions $(\mathscr{E}^{(i)}, \varrho^{(i)})$. Thus we get sequences of abelian extensions:

$$(\mathscr{E}, \varrho) \supset (\mathscr{E}', \varrho') \supset \cdots \supset (\mathscr{E}^{(j)}, \varrho^{(j)}) \supset \cdots$$

Every sequence ends with a lower irreducible extension $(\mathscr{E}^{(n)}, \varrho^{(n)})$ which is uniquely determined by the following theorem:

Theorem 4. *Every abelian extension $(\mathscr{E}, \varrho)$ of $\mathscr{B}$ contains one and only one irreducible extension $(\mathscr{E}', \varrho')$.*

Proof. It is obvious that there exists one irreducible extension $(\mathscr{E}', \varrho')$ which is contained in $(\mathscr{E}, \varrho)$. We suppose that there exists another irreducible extension $(\mathscr{E}'', \varrho'') \neq (\mathscr{E}', \varrho')$ contained in $(\mathscr{E}, \varrho)$ and we consider the Lie subalgebra $\mathscr{E}' \cap \mathscr{E}''$ of $\mathscr{E}$. Let $\varrho' \cap \varrho'' = \varrho|\mathscr{E}' \cap \mathscr{E}''$, then $(\varrho' \cap \varrho'')(\mathscr{E}' \cap \mathscr{E}'') = \varrho'|\mathscr{E}' \cap \mathscr{E}''(\mathscr{E}' \cap \mathscr{E}'') = \mathscr{B}$ and $(\mathscr{E}' \cap \mathscr{E}'', \varrho' \cap \varrho'')$ is an abelian extension of $\mathscr{B}$ such that $(\mathscr{E}' \cap \mathscr{E}'', \varrho' \cap \varrho'') \subset (\mathscr{E}', \varrho')$. Hence the extension $(\mathscr{E}', \varrho')$ is not irreducible in contradiction with the assumption.

We can now say that an abelian extension $(\mathscr{E}, \varrho)$ reduces to $(\mathscr{E}', \varrho')$ if $(\mathscr{E}, \varrho)$ contains the irreducible extension $(\mathscr{E}', \varrho')$.

Let $\mathscr{A}^* \subset \mathscr{A}$ be a proper ideal of $\mathscr{E}(\mathscr{B}, \mathscr{A})$. The abelian extension $(\mathscr{E}, \varrho)$ of $\mathscr{B}$ by $\mathscr{A}$ induces an extension $(\mathscr{E}/\mathscr{A}^*, \varrho_q)$ of $\mathscr{B}$ by $\mathscr{A}/\mathscr{A}^*$. ϱ_q denotes the epimorphism $\mathscr{E}/\mathscr{A}^* \twoheadrightarrow \mathscr{B}$ obtained by passing to the quotient and the extension $(\mathscr{E}/\mathscr{A}^*, \varrho_q)$ is of course abelian.

It is easy to see that if $(\mathscr{E}/\mathscr{A}^*, \varrho_q)$, with $\mathscr{A}^* \subset \mathscr{A}$ proper ideal of $\mathscr{E}$, is reducible, then also $(\mathscr{E}, \varrho)$ is reducible.

Theorem 5. *(Irreducibility criterion)* [4]. *The abelian extension* $(\mathscr{E}, \varrho)$ *of* $\mathscr{B}$ by $\mathscr{A}$ *is irreducible if and only if the induced extensions* $(\mathscr{E}/\mathscr{A}^*, \varrho_q)$ *of* $\mathscr{B}$ by $\mathscr{A}/\mathscr{A}^*$ *are essential for every proper ideal* $\mathscr{A}^* \subset \mathscr{A}$ *of* $\mathscr{E}$.

Proof. Necessity: Let $(\mathscr{E}, \varrho)$ be irreducible and $\mathscr{A}^* \subset \mathscr{A}$ be a proper ideal of $\mathscr{E}$. Then $(\mathscr{E}/\mathscr{A}^*, \varrho_q)$ is an abelian extension of $\mathscr{B}$ by $\mathscr{A}/\mathscr{A}^*$. We prove that this extension is essential. If $(\mathscr{E}/\mathscr{A}^*, \varrho_q)$ is inessential, $\mathscr{E}/\mathscr{A}^* = \mathscr{B}' \oplus \mathscr{A}/\mathscr{A}^*$ and $\varrho_q|\mathscr{B}'$ is an isomorphism $\mathscr{B}' \approx \mathscr{B}$. Hence $\mathscr{E}$ is such that we have the structure of a Lie algebra on a subspace $E' \subset E$, with $E' = B'' \oplus A^*$ and $B'' \approx B$ by ϱ. Thus $(\mathscr{E}, \varrho)$ is reducible in contradiction with the assumption.

Sufficiency: Let $(\mathscr{E}, \varrho)$ be such that the induced extensions $(\mathscr{E}/\mathscr{A}^*, \varrho_q)$ are essential for every proper ideal $\mathscr{A}^* \subset \mathscr{A}$ of $\mathscr{E}$. If $(\mathscr{E}, \varrho)$ is reducible, then there exists a subalgebra $\mathscr{E}' \subset \mathscr{E}$ such that $\varrho|\mathscr{E}'(\mathscr{E}') = \mathscr{B}$. We consider the subalgebra $\mathscr{A}' = \mathscr{A} \cap \mathscr{E}'$, which is also an ideal of $\mathscr{E}$. Hence $\mathscr{E}/\mathscr{A}' = \mathscr{B}' \oplus \mathscr{A}/\mathscr{A}'$ with $\mathscr{B}' \approx \mathscr{B}$ by ϱ_q, and $(\mathscr{E}/\mathscr{A}', \varrho_q)$ is an inessential extension of $\mathscr{B}$ by $\mathscr{A}/\mathscr{A}'$. It follows that necessarily $(\mathscr{E}, \varrho)$ is an irreducible extension of $\mathscr{B}$ by $\mathscr{A}$.

Definition 3. *By a truly irreducible extension we mean an abelian extension of* $\mathscr{B}$ *by* $\mathscr{A}$ *such that no proper ideal* $\mathscr{A}^* \subset \mathscr{A}$ *of* $\mathscr{E}(\mathscr{B}, \mathscr{A})$ *exists.*

It follows immediately from this definition that only an irreducible abelian extension can be truly irreducible. The minimal essential extensions of $\mathscr{P}$ are truly irreducible as well as the truly trivial extension $(\mathscr{P}, I)$, with $I =$ identity map.

The following theorem is a direct consequence of the Chevalley-Eilenberg's construction of abelian extensions by a representation.

Theorem 6. *An extension* $(\mathscr{E}, \tau)$ *of* $\mathscr{P}$ *by* K_Φ *is truly irreducible if and only if* K_Φ *is a simple* $\mathscr{P}$*-module or* $K_\Phi = \{0\}$.

The considerations of Section III.2 can now be stated as follows:

Corollary. *An abelian extension* $(\mathscr{E}, \tau)$ *of* $\mathscr{P}$ *is truly irreducible if and only if it is minimal essential or truly trivial.*

This corollary gives a characterization of the minimal essential extensions of $\mathscr{P}$ in terms of truly irreducible extensions.

Another useful concept is the following:

Definition 4. *We say that the abelian extensions* $(\mathscr{E}, \varrho)$ *of* $\mathscr{B}$ *by* $\mathscr{A}$ *and* $(\mathscr{E}', \varrho')$ *of* $\mathscr{B}$ *by* $\mathscr{A}'$ *are of the same type if* $\mathscr{E} \approx \mathscr{E}'$.

This definition induces an equivalence relation on the set $\operatorname{Ext}_a \mathscr{B}$ of the abelian extensions of $\mathscr{B}$. The extensions of the same equivalence class of Ext $(\mathscr{B}, A_\Phi)$ are all of the same type, but we also have extensions of the same type belonging to different classes of Ext $(\mathscr{B}, A_\Phi)$ or belonging to two different spaces Ext $(\mathscr{B}, A_\Phi^{(i)})$ $i = 1, 2$.

Let us concentrate again on the extensions $(\mathscr{E}, \tau)$ of $\mathscr{P}$ by K_Φ. We consider the extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ in the equivalence class $\{(\mathscr{E}, \tau)\}$.

We say that $K(f_2)$ is *maximal in* K_Φ if $K(f_2) \neq K_\Phi$ and no $\mathscr{P}$-submodule K'_Φ of K_Φ exists such that $K(f_2) \subsetneqq K'_\Phi \subset K_\Phi$. Then:

Theorem 7. *The extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ of $\mathscr{P}$ by K_Φ is irreducible if and only if $K(f_2) = K_\Phi$ or $K(f_2)$ maximal in K_Φ.*

Proof. The condition is necessary: If $K(f_2)$ is not maximal in K_Φ there exists a $\mathscr{P}$-module K'_Φ such that $K(f_2) \subsetneqq K'_\Phi \subset K_\Phi$ or $K(f_2) = K_\Phi$. In the first case the extension $(\mathscr{E}/\mathscr{K}', \tau_q)$ is inessential and not truly trivial. Therefore $(\mathscr{E}, \tau)_{\Phi, f_2}$ is reducible. The irreducibility of $(\mathscr{E}, \tau)_{\Phi, f_2}$ requires $K(f_2) = K_\Phi$ or $K(f_2)$ maximal in K_Φ.

Sufficiency of the condition: If $K(f_2) = K_\Phi$, then we have a minimal essential extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ or the truly trivial extension $(\mathscr{P}, I)$, both irreducible. If $K(f_2)$ is maximal in K_Φ then $(\mathscr{E}/\mathscr{K}', \tau_q)$ is essential for every proper ideal $\mathscr{K}' \subset \mathscr{K}$ of $\mathscr{E}$, since $K(f_2) \subsetneqq K'_\Phi$ is excluded.

Corollary. *If $K(f_2) \neq K_\Phi$ and $\Phi(t)\, k = 0\ \forall (t \in \mathscr{T};\, k \in K(f_2))$ the extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ of $\mathscr{P}$ by K_Φ is reducible.*

The theorem has to be understood in the following way: the necessary and sufficient condition for an irreducible extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ of $\mathscr{P}$ by K_Φ is

$$\sum_{i=0}^{n-1} \Phi^i(\mathscr{T})\, K(f_2) = K_\Phi \tag{IV.1}$$

where $\Phi^i(\mathscr{T})\, K(f_2) = \overline{\{\Phi^{i_1}(t_1)\, \Phi^{i_2}(t_2) \ldots \Phi^{i_j}(t_j)\, k \mid t_1, t_2, \ldots, t_j \in \mathscr{T};\, i_1 + i_2 + \cdots + i_j = i;\quad k \in K(f_2)\}}$ $\forall\, i \in \boldsymbol{N}^+$; $\Phi^0(\mathscr{T})\, K(f_2) = K(f_2)$ and $\Phi^n(\mathscr{T})\, K(f_2) = \{0\}$. $\overline{\{\ \}}$ means the natural vector space spanned by $\{\ \}$.

We consider an extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ of $\mathscr{P}$ by K_Φ, such that the $\mathscr{L}$-module K_Φ contains simple components $K_{\mathfrak{S}\{j_1, j_2\}}$ with $(j_1 + j_2)$ half integer. This means that the representation $\Phi | \mathscr{L}$ contains spinorial irreducible subrepresentations. By Theorem 7 we can infer that the extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ is reducible. In the study of irreducible extensions of $\mathscr{P}$ we have therefore to consider extensions by $\mathscr{L}$-modules K_Φ with only tensorial simple components.

IV.2. Examples

a) We consider the extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ of $\mathscr{P}$ by $K_\Phi = K(f_2) \oplus K_{\mathfrak{S}\{3/2, 1/2\}}$ ($\mathscr{L}$-module decomposition), where $f_2 \neq 0$. The radical $\mathscr{R}$ of $\mathscr{E}$ is then also the nil-radical and we can apply the following corollaries to ENGEL'S theorem:

Proposition 4. *Let $\mathscr{E}$ be a Lie algebra and let $\mathscr{L}, \mathscr{R}$ and $\mathscr{K}$ be Lie subalgebras of $\mathscr{E}$. Suppose that $\mathscr{R}$ is a nilpotent ideal of $\mathscr{E}$ and $\mathscr{K}$ an ideal of $\mathscr{R}$. Assume furthermore that $\mathscr{K}$ carries the structure of a simple $\mathscr{L}$-module $K_{\mathrm{ad}\,\mathscr{L}}$. Then $[r, k] = 0\ \forall\ (r \in \mathscr{R};\ k \in \mathscr{K})$, and $\mathscr{K}$ is abelian.*

Proof. We consider the representation $\mathrm{ad}_K\mathscr{R}$ of $\mathscr{R}$. There exists a vector $k \neq 0$, $k \in K_{\mathrm{ad}\,\mathscr{L}}$, such that $(\mathrm{ad}_K r)\, k = 0\ \forall\ r \in \mathscr{R}$ (Engel's theorem). Therefore in $\mathscr{E}: 0 = [l, [r, k]] = [r, [l, k]] + [[l, r], k]\ \forall\ (r \in \mathscr{R}; l \in \mathscr{L})$ and $[[l, r], k] = (\mathrm{ad}_K [l, r])\ k = 0$ since $[l, r] \in \mathscr{R}$. Then $[r, [l, k]] = (\mathrm{ad}_K r)\,(\mathrm{ad}_K l)\ k = 0\ \forall\ (r \in \mathscr{R};\ l \in \mathscr{L})$. By the simplicity of the $\mathscr{L}$-module $K_{\mathrm{ad}\,\mathscr{L}}$ there exists for any $k' \in K_{\mathrm{ad}\,\mathscr{L}}$ a $l' \in \mathscr{L}$ such that $(\mathrm{ad}_K l')\ k = k'$. So we obtain the quoted result.

Proposition 5. *Let $\mathscr{L}, \mathscr{R}, \mathscr{K}$ be Lie subalgebras of the Lie algebra $\mathscr{E}$. Suppose that $\mathscr{R}$ is a nilpotent ideal of $\mathscr{E}$ and $\mathscr{K}$ an ideal of $\mathscr{R}$. Assume furthermore that $\mathscr{K}$ carries the structure $K_{\mathrm{ad}\,\mathscr{L}}$ of a semisimple $\mathscr{L}$-module. There exists an abelian ideal $\mathscr{K}' \subseteq \mathscr{K}$ of $\mathscr{R}$ with the induced structure $K'_{\mathrm{ad}\,\mathscr{L}}$ of a simple $\mathscr{L}$-module such that $[r, k'] = 0\ \forall\ (r \in \mathscr{R};\ k' \in \mathscr{K}')$.*

Proof. We consider $\mathrm{ad}_K\mathscr{R}$ and we apply Engel's theorem as in the proof of Proposition 4. The simple $\mathscr{L}$-module $K'_{\mathrm{ad}\,\mathscr{L}}$ is constructed by applying the endomorphisms $\mathrm{ad}_K l\ \forall\ l \in \mathscr{L}$ to a vector $k \in K_{\mathrm{ad}\,\mathscr{L}}$, $k \neq 0$ such that $(\mathrm{ad}_K r)\ k = 0\ \forall\ r \in \mathscr{R}$. $\mathscr{K}'$ is then an abelian ideal of $\mathscr{R}$ and $[r, k'] = 0\ \forall\ (r \in \mathscr{R};\ k' \in \mathscr{K}')$.

We return to our example and we apply the foregoing propositions. We have the following possibilities:

1. $\Phi(\mathscr{T})\, K(f_2) = \{0\}$: The extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ reduces to a minimal essential extension $(\mathscr{E}_1, \tau_1)_{\Phi, f_2}$ (Exceptional case: $\Phi(\mathscr{T})\, K_\Phi = \{0\}$).

2. $\Phi(\mathscr{T})\, K(f_2) = K_{\mathfrak{D}\{3/2, 1/2\}}$ and $\Phi(\mathscr{T})\, K_{\mathfrak{D}\{3/2, 1/2\}} = \{0\}$: The extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ is irreducible. We denote by $(\mathscr{E}^{[\frac{1}{2}]}, \tau^{[\frac{1}{2}]})_{\Phi, f_2}$ this irreducible extension. Then $\dim \mathscr{E}^{[\frac{1}{2}]} = 32$. The existence of a pair (k', t') where $t' \in \mathscr{T}$, $k' \in K(f_2)$ such that $\Phi(t')\, k' \neq 0$ already implies this result.

b) Let $f_2 \neq 0$ and $K_\Phi = K(f_2) \oplus K_{\mathfrak{D}\{1/2, 1/2\}}$ ($\mathscr{L}$-module decomposition). We have the same possibilities as in case a). The only difference is that in case b) 2. we obtain another irreducible extension $(\mathscr{E}^{[\frac{2}{2}]}, \tau^{[\frac{2}{2}]})_{\Phi, f_2}$ with $\dim \mathscr{E}^{[\frac{2}{2}]} = 20$. In this case $K_{\mathfrak{D}\{1/2, 1/2\}}$ is generated by the vectors $\{t''_\varrho\}$ (see Section II.2).

IV.3. Classification of Types of Irreducible Abelian Extensions of the Poincaré Algebra

We consider an extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ of $\mathscr{P}$ by K_Φ and the descending central series $\{C^i\mathscr{R}\}$ of nilpotent ideals of the radical $\mathscr{R}$ of $\mathscr{E}$. The subalgebras $C^i\mathscr{R}$ are nilpotent ideals of $\mathscr{E}$ and therefore, for $i \in \boldsymbol{N}^+$, $\mathscr{P}$-

submodules of K_Φ too. We have the following possibilities:

1. $C^1\mathscr{R} = \{0\}$: the extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ is inessential. It is irreducible (and truly irreducible) if and only if $\mathscr{K} = \{0\}$, i.e. if the extension is the truly trivial extension $(\mathscr{P}, I)$ [of type (0)].

2. $C^1\mathscr{R} \neq \{0\}$, $C^1\mathscr{R} \subset \mathscr{K}$: by passing to the quotient we have the inessential extension $(\mathscr{E}/C^1\mathscr{R}, \tau_q)$, since $\mathscr{R}/C^1\mathscr{R}$ is abelian. The extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ is therefore reducible.

Theorem 8. *Let $\mathscr{R}$ be the radical of $\mathscr{E}$ in the extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ of $\mathscr{P}$ by K_Φ. Then $(\mathscr{E}, \tau)_{\Phi, f_2}$ is irreducible if and only if $C^1\mathscr{R} = \mathscr{K}$.*

Proof. The necessity of $C^1\mathscr{R} = \mathscr{K}$ follows from the foregoing considerations. Let $C^1\mathscr{R} = \mathscr{K}$. Then $C^2\mathscr{R} = \Phi(\mathscr{T})\,\mathscr{K}$ is such that $\Phi(\mathscr{T})\,K_\Phi \oplus K(f_2) = K_\Phi$ ($\mathscr{L}$-module decomposition). If $K(f_2) = \{0\}$ the nilpotency of $\mathscr{R}$ requires $\mathscr{K} = \{0\}$, i.e. the irreducibility of the extension. If $K(f_2) \neq \{0\}$ we remark that:

$$K_\Phi = \Phi^j(\mathscr{T})\,K_\Phi + \sum_{i=0}^{j-1} \Phi^i(\mathscr{T})\,K(f_2)\ \forall\, j \in \boldsymbol{N}^+. \qquad \text{(IV.2)}$$

(IV.2) is easily proved by induction on j. Since $\mathscr{R}$ is nilpotent, there exists an $n \in \boldsymbol{N}^+$ such that $\Phi^n(\mathscr{T})\,K_\Phi = \{0\}$ and thus $K_\Phi = \sum_{i=0}^{n-1} \Phi^i(\mathscr{T})\,K(f_2)$.

By Theorem 7 the extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ is irreducible.

3. $C^1\mathscr{R} = \mathscr{K} \neq \{0\}$: this brings about $C^2\mathscr{R} = \Phi(\mathscr{T})\,\mathscr{K} \subset \mathscr{K}$. Then we have:

a) $C^2\mathscr{R} = \{0\}$: in this case $K_\Phi = K(f_2)$ which gives a minimal essential extension [of type (1)].

We can now give another proof of the statement of the Corollary to Theorem 6 that only the minimal essential and truly trivial extensions of $\mathscr{P}$ are truly irreducible. Let $(\mathscr{E}, \tau)$ be a truly irreducible abelian extension of $\mathscr{P}$ and consider $(\mathscr{E}, \tau)_{\Phi, f_2}$. If $C^1\mathscr{R} = \{0\}$, then $\mathscr{K} = \{0\}$ and $(\mathscr{E}, \tau)$ is truly trivial. If $C^1\mathscr{R} \neq \{0\}$, we have to require $C^1\mathscr{R} = \mathscr{K}$. Let $C^2\mathscr{R} \neq \{0\}$, then $C^2\mathscr{R}$ is a proper ideal contained in $\mathscr{K}$, in contradiction with the assumption. Therefore we must have $C^2\mathscr{R} = \{0\}$ and $(\mathscr{E}, \tau)$ is minimal essential.

b) $C^2\mathscr{R} \neq \{0\}$: From the $\mathscr{L}$-module decomposition $K_\Phi = K(f_2) \oplus \Phi(\mathscr{T})\,K_\Phi$ we infer $C^2\mathscr{R} \cap \mathscr{K}(f_2) = \{0\}$. We consider the induced extension $(\mathscr{E}/C^2\mathscr{R}, \tau_q)$ of $\mathscr{P}$ by $\mathscr{K}/C^2\mathscr{R}$. $(\mathscr{E}/C^2\mathscr{R}, \tau_q)$ is irreducible and minimal essential. We consider now $C^3\mathscr{R}$. If $C^3\mathscr{R} = \{0\}$ we obtain $(\mathscr{E}, \tau)_{\Phi, f_2} = (\mathscr{E}^{[\frac{i}{2}]}, \tau^{[\frac{i}{2}]})_{\Phi, f_2}$, where the index i characterises the different types of irreducible abelian extensions (2, [i]) with $C^3\mathscr{R} = \{0\}$:

1. the extension $(\mathscr{E}^{[\frac{1}{2}]}, \tau^{[\frac{1}{2}]})_{\Phi, f_2}$ with $\Phi(\mathscr{T})\,K(f_2) = K_{\mathfrak{D}\{3/2, 1/2\}}$ is of type (2, [1]);

2. the extension $(\mathscr{E}^{[2]}_{2}, \tau^{[2]}_{2})_{\Phi, f_2}$ with $\Phi(\mathscr{T})\, K(f_2) = K_{\mathfrak{S}\{1/2, 1/2\}}$ is of type $(2, [2])$;

3. the extension $(\mathscr{E}^{[3]}_{2}, \tau^{[3]}_{2})_{\Phi, f_2}$ such that $\Phi(\mathscr{T})\, K(f_2) = K_{\mathfrak{S}\{3/2, 1/2\}} \oplus K_{\mathfrak{S}\{1/2, 1/2\}}$ is of type $(2, [3])$.

Let $C^3\mathscr{R} \neq \{0\}$. A straightforward consequence of $C^1\mathscr{R} = \mathscr{K}$ is $C^3\mathscr{R} \cap \mathscr{K}(f_2) = \{0\}$. We consider the induced extension $(\mathscr{E}/C^3\mathscr{R}, \tau_q)$. The irreducibility of $(\mathscr{E}/C^3\mathscr{R}, \tau_q)$ implies that it is of type $(2, [i])$.

The above particular remarks can be stated more generally. We consider the set $\mathfrak{E}_n$, with $n \in \boldsymbol{N}$, of all abelian extensions $(\mathscr{E}, \tau)$ of $\mathscr{P}$ with the radical $\mathscr{R}$ of $\mathscr{E}$ satisfying $C^{n+1}\mathscr{R} = \{0\}$ and $C^j\mathscr{R} \neq \{0\}\ \forall\, (j \leq n; j \in \boldsymbol{N})$. $(\mathscr{E}, \tau)_{\Phi, f_2}$ is an irreducible extension of $\mathscr{P}$ by K_Φ belonging to $\mathfrak{E}_n$ with $n \in \boldsymbol{N}$, only if:

$$\mathscr{K}(f_2) \cap C^1\mathscr{R} = \mathscr{K}(f_2)$$

and (IV.3)

$$\mathscr{K}(f_2) \cap C^j\mathscr{R} = \{0\} \qquad \forall\, (j \neq 1; j \in \boldsymbol{N}^+)\,.$$

The following proposition is a straightforward consequence of the nilpotency of $\mathscr{R}$:

Proposition 6. *Let $(\mathscr{E}, \tau)$ be an extension of $\mathscr{P}$ by K_Φ and let K^* be a simple $\mathscr{L}$-submodule of K_Φ. Then:*

$$\Phi^2(\mathscr{T})\, K^* \cap K^* = \{0\}\,. \tag{IV.4}$$

Corollary. *We consider the extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ of $\mathscr{P}$ by K_Φ. Then:*

$$\Phi^2(\mathscr{T})\, K(f_2) \cap K(f_2) = \{0\}\,.$$

Let I_n be an index set of the types of irreducible abelian extensions of $\mathscr{P}$ belonging to $\mathfrak{E}_n$. We say that the irreducible abelian extension $(\mathscr{E}^{[i]}_{n}, \tau^{[i]}_{n})$ of $\mathscr{P}$ belonging to $\mathfrak{E}_n$ is of type $(n, [i])$ if $i \in I_n$.

Theorem 9. *We consider the extension $(\mathscr{E}, \tau)_{\Phi, f_2}$ of $\mathscr{P}$ by K_Φ. If $n \in \boldsymbol{N}^+$, $(\mathscr{E}, \tau)_{\Phi, f_2}$ is irreducible of type $(n, [i])$ if and only if:*

$$\Phi^n(\mathscr{T})\, K(f_2) = \{0\};\ \Phi^j(\mathscr{T})\, K(f_2) \neq \{0\}\ \forall\, (j < n; j \in \boldsymbol{N})$$

and (IV.5)

$$\sum_{j=0}^{n-1} \Phi^j(\mathscr{T})\, K(f_2) = K_\Phi\,.$$

If $n = 0$ the necessary and sufficient condition for the irreducibility is $\mathscr{K} = \{0\}$.

Proof. The necessity of the requirements (IV.5) or $\mathscr{K} = \{0\}$ follows from the definition of $\mathfrak{E}_n$ and Theorem 7. $\mathscr{K} = \{0\}$ is obviously also sufficient if $n = 0$. The conditions (IV.5) imply $C^1\mathscr{R} = \mathscr{K}$, hence the irreducibility of $(\mathscr{E}, \tau)_{\Phi, f_2}$ by Theorem 8, as well as $(\mathscr{E}, \tau)_{\Phi, f_2} \in \mathfrak{E}_n$, $n \in \boldsymbol{N}^+$.

Let $\mathfrak{F}$ be the set of all types of irreducible abelian extensions of $\mathscr{P}$. We can then consider families of types $\mathfrak{F}_n = \{(n, [i]) \mid i \in I_n\}$ $n \in \boldsymbol{N}$, such that $\mathfrak{F} = \bigcup_{n \in \boldsymbol{N}} \mathfrak{F}_n$. In particular we have the family $\mathfrak{F}_0$ of the truly trivial extension $(\mathscr{P}, I)$ and the family $\mathfrak{F}_1$ of the minimal essential extensions, both containing only one element. $\mathfrak{F}_2$ contains 3 elements as can easily be proved.

It is possible to construct from an extension of type (1) a representative element of the classes of type $(n, [i])\ \forall\ (n \in \boldsymbol{N}^+; i \in I_n)$ by induction. It is sufficient to consider condition (IV.4) and to recall that, by Theorem 9, $K_\Phi = \sum_{j=0}^{n-1} \Phi^j(\mathscr{T})\, K(f_2)$ where $\Phi(t_1)\,\Phi(t_2) = \Phi(t_2)\,\Phi(t_1)\ \forall\ t_1, t_2 \in \mathscr{T}$.

The minimal essential extensions, or extensions of type (1), play an important role in the set of all abelian extensions of the Poincaré algebra $\mathscr{P}$. They are the starting point for constructing any irreducible essential abelian extension of $\mathscr{P}$. They are also the only extensions of $\mathscr{P}$ by K_Φ (besides the truly trivial extension) with the property that $\mathscr{K}$ contains no proper subalgebra which is an ideal of $\mathscr{E}$.

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Appendix

Abelian Extensions of $\mathscr{P}_C$

Let $\mathscr{P}_C$ be the complexification of $\mathscr{P}$ and consider Lie algebras over $\boldsymbol{C}$. The finite irreducible representations of $\mathscr{L}_C$ are no longer $\mathfrak{D}^{\{j_1, j_2\}}$, but the well known $\mathfrak{D}^{(j_1, j_2)}$ such that $\mathfrak{D}^{\{j_1, j_2\}} = \mathfrak{D}^{(j_1, j_2)} \oplus \mathfrak{D}^{(j_2, j_1)}$ if $j_1 > j_2$; $\mathfrak{D}^{\{j, j\}} = \mathfrak{D}^{(j, j)}$.

For the abelian extensions of $\mathscr{P}_C$ we have results analogous to the abelian extensions of $\mathscr{P}$. Any theorem for the abelian extensions of $\mathscr{P}$ can be easily translated into a corresponding theorem for extensions of $\mathscr{P}_C$.

In particular: $\mathfrak{D}^{\{1,0\}} = \mathfrak{D}^{(1,0)} \oplus \mathfrak{D}^{(0,1)}$ and as a consequence there exist two types (1, [1]) and (1, [2]) of minimal essential extensions. If $(\mathscr{E}^{[1]}_{1}, \tau^{[1]}_{1})_{\Phi, f_2}$ and $(\mathscr{E}^{[2]}_{1}, \tau^{[2]}_{1})_{\Phi, f_2}$ are respectively of type (1, [1]) and (1, [2]), then $\dim \mathscr{E}^{[2]}_{1} = \dim \mathscr{E}^{[1]}_{1} = 13$. The two corresponding fundamental $\mathscr{L}$-modules are respectively $K_{\mathfrak{D}^{(1,0)}}$ and $K_{\mathfrak{D}^{(0,1)}}$. The $(2; \mathscr{P}, K_\Phi)$-cocycle condition (II.10) transforms into the conditions:

$$\Phi(\mathscr{T})\, K_{\mathfrak{D}^{(1,0)}} = K_{\mathfrak{D}^{(3/2,1/2)}}\,;\ \Phi(\mathscr{T})\, K_{\mathfrak{D}^{(0,1)}} = K_{\mathfrak{D}^{(1/2,3/2)}}\,.$$

The family $\mathfrak{F}_1$ now contains 3 elements, since besides the types (1,[1]) and (1, [2]) we also have (1, [3]) corresponding to the minimal essential extensions of $\mathscr{P}$. If $(\mathscr{E}^{[3]}_{1}, \tau^{[3]}_{1})_{\Phi, f_2}$ is of type (1, [3]), then $\dim \mathscr{E}^{[3]}_{1} = 16$ and the fundamental $\mathscr{L}$-module is $K_{\mathfrak{D}^{(1,0)} \oplus \mathfrak{D}^{(0,1)}}$.

References

1. Galindo, A.: J. Math. Phys. 8, 768—774 (1967).
2. Michel, L.: Lectures on "Group Extensions and Quantum Mechanics" at the Istanbul Summer School 1962. London: Gordon and Breach, 1965. Nucl. Phys. **57**, 356—386 (1964).
3. Macaulay, R. A.: Trans. Am. Math. Soc. **95**, 530—553 (1960).
4. Chevalley, C., and S. Eilenberg: Trans. Am. Math. Soc. **63**, 85—124 (1948).
5. Cartan, H., and S. Eilenberg: Homological algebra. Princeton: Princeton Univ. Press 1956.
6. Hochschild, G., and J. P. Serre: Ann. Math. **57**, 591—603 (1953).
7. Bourbaki, N.: Groupes et algèbres de Lie, Ch. 1, ASI 1285. Paris: Hermann 1960.
8. Jacobson, N.: Lie algebras. New York: Interscience Publ. Inc. 1962.
9. Leger, G. F.: Proc. Am. Math. Soc. 8, 1010—1020 (1957).

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