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Limits of Spacetimes

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Abstract. The limits of a one-parameter family of spacetimes are defined, and the properties of such limits discussed. The definition is applied to an investigation of the Schwarzschild solution as a limit of the Reissner-Nordström solution as the charge parameter goes to zero. Two new techniques — rigidity of a geometrical structure and Killing transport — are introduced. Several applications of these two subjects, both to limits and to certain other questions in differential geometry, are discussed.

1. Introduction

One frequently hears statements concerning *the* limit of a family of solutions of EINSTEIN'S equations as some free parameter approaches a certain value. There is, however, a serious ambiguity in such statements, for they normally refer to a particular system of coordinates: by changing coordinates, one can usually obtain some quite different spacetime in the limit. The concept of a limit applied to spacetimes is, nontheless, a useful one, and so we are led to formulate some unambiguous definition of this notion. In this paper we shall define the limits of a family of spacetimes and display a simple characterization of these limits.

In Section 2 we give the definition of a limit. The main theorem of that section asserts that a knowledge of the limit "locally" determines, completely and uniquely, a corresponding global limit. As an example, our definition is applied to clarify the way in which the Reissner-Nordström solution reduces to the Schwarzschild solution as the charge parameter approaches zero.

In Section 3 we discuss those properties of spacetimes which are heriditary, i.e., which pass from a given family of spacetimes to their limits.

The two topics treated in the appendices are useful in many contexts in differential geometry other than merely questions involving limits. The appendices can be read independently of the rest of the paper. In Appendix *A* we define rigidity of a geometrical structure and prove that nonsingular metrics are rigid. That metrics are rigid while, for example,

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symplectic structures are not is reflected in the fact that groups of isometries are "small" (i.e., Lie groups) while groups of canonical transformations are "large". The notion of a limit, as defined here, is applicable quite generally to any rigid geometrical structure. Closely related to rigidity is Killing transport, which is discussed in Appendix *B*. Killing transport is used in Section 3 to establish the heriditary properties of isometry groups. A number of further applications of this type of transport are given in the appendix.

2. Limits

By a *spacetime* we understand a (connected, Hausdorff) 4-dimensional manifold with a (C^∞) metric g^{ab} of signature $(+, -, -, -)$[1]. Consider a one-parameter family of such spacetimes. (Our results are easily generalized to many parameter families of spacetimes and, possibly, to families which depend on arbitrary functions.) That is to say, for each value of a parameter $\lambda (> 0)$ we have a 4-manifold M_λ and a metric $g^{ab}(\lambda)$ on M_λ. We are interested in finding the limits of this family as $\lambda \to 0$. It might be asked at this point why we do not simply take the $g^{ab}(\lambda)$ as a 1-parameter family of metrics on a given fixed manifold M. Such a formulation would certainly simplify the problem: it amounts to a specification of when two points $p_\lambda \in M_{\lambda'}$ and $p_{\lambda'} \in M_\lambda$ $(\lambda \neq \lambda')$ are to be considered as representing "the same point" of M. It is not appropriate, however, to provide this additional information, for it always involves singling out a particular limit, while we are interested in the general problem of finding all limits and studying their properties.

We illustrate this point with the example of the Schwarzschild solution. Consider the family of metrics

$$ds^2 = \left(1 - \frac{2}{\lambda^3 r}\right) dt^2 - \left(1 - \frac{2}{\lambda^3 r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (1)$$

which depend on the single parameter $\lambda (= m^{-1/3})$. In the form (1) the metric clearly does not approach a limit as $\lambda \to 0$. Suppose, however, we apply the coordinate transformation

$$\tilde{r} = \lambda r, \ \tilde{t} = \lambda^{-1} t, \ \tilde{\varrho} = \lambda^{-1} \theta \, .$$

Then (1) becomes

$$ds^2 = \left(\lambda^2 - \frac{2}{\tilde{r}}\right) d\tilde{t}^2 - \left(\lambda^2 - \frac{2}{\tilde{r}}\right)^{-1} d\tilde{r}^2 - \tilde{r}^2 (d\tilde{\varrho}^2 + \lambda^{-2} \sin^2(\lambda \tilde{\varrho}) \, d\varphi^2) \, .$$

[1] It will be more convenient in the present discussion to introduce the contravariant rather than the covariant metric as the basic object. All our considerations are easily generalized to include any further tensor fields on the mainfold, e.g., an electromagnetic field, the velocity, pressure, and density fields of a perfect fluid, etc. However, it is essential that among the basic fields on the manifold there is at least one — such as a metric — which is rigid in the sense of Appendix *A*.

The limit as $\lambda \to 0$ now exists and gives the metric

$$ds^2 = -\frac{2}{\tilde{r}} d\tilde{t}^2 + \frac{\tilde{r}}{2} d\tilde{r}^2 - \tilde{r}^2 (d\tilde{\varrho}^2 + \tilde{\varrho}^2 d\varphi^2) .$$

This is a nonflat solution of EINSTEIN's equations discovered originally by KASNER [1] and obtained by ROBINSON as a limit of the Schwarzschild solution. On the other hand, the coordinate transformation $x = r + \lambda^{-4}$, $\varrho = \lambda^{-4}\theta$ applied to (1) yields flat space in the limit $\lambda \to 0$. Thus, one cannot speak simply of "the limit of the Schwarzschild solution as $\lambda \to 0$", for the spacetime one obtains in the limit depends on the choice of coordinates. The essential difference between the various limits above consists in different identifications of the M_λ.

How then can we express the idea that the $(M_\lambda, g^{ab}(\lambda))$ depend smoothly on λ (which we shall certainly need in order to define limits) without at the same time prejudicing the particular limit we are to obtain? Let us assume that the manifolds M_λ may be put together to make a smooth (Hausdorff) 5-dimensional manifold $\mathscr{M}$. Each M_λ is to be a 4-dimensional submanifold of $\mathscr{M}$. The parameter λ now represents a scalar field on $\mathscr{M}$, while the metric tensors $g^{ab}(\lambda)$ on the M_λ define a single tensor field $g^{\alpha\beta}$ on $\mathscr{M}$, which we assume to be smooth[2]. The signature of $g^{\alpha\beta}$ is $(0, +, -, -, -)$: in fact, the singular direction is precisely the gradient of λ, i.e., we have $g^{\alpha\beta}\nabla_\beta \lambda = 0$. (Consequently, the tensor field $g^{\alpha\beta}$ on already completely defines the surfaces M_λ.) The 5-manifold $\mathscr{M}$ contains all the information of our original collection $(\mathscr{M}_\lambda, g^{ab}(\lambda))$, but does not define a preferred correspondence between different M_λ*.

The problem of finding limits of the family $(M_\lambda, g^{ab}(\lambda))$ amounts to that of placing a suitable boundary on $\mathscr{M}$. We define a *limit space* of $\mathscr{M}$ as a 5-manifold $\mathscr{M}'$ with boundary $\partial\mathscr{M}'$, equipped with a tensor field $g'^{\alpha\beta}$, a scalar field λ', and a smooth, one-to-one mapping Ψ of $\mathscr{M}$ onto the interior of $\mathscr{M}'$ such that the following three conditions are satisfied:

1. Ψ is an *isometry*, i.e., Ψ takes $g^{\alpha\beta}$ into $g'^{\alpha\beta}$ and λ into λ'.

2. $\partial\mathscr{M}'$ is the region given by $\lambda' = 0$. We require, furthermore, that $\partial\mathscr{M}'$ be connected, Hausdorff, and nonempty.

3. $g'^{\alpha\beta}$ has signature $(0, +, -, -, -)$ on $\partial\mathscr{M}'$.

The first condition ensures that $\mathscr{M}'$ really represents $\mathscr{M}$ with a boundary attached; the second condition ensures that the boundary represents a limit as $\lambda \to 0$; and the third condition ensures that the

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[2] Latin and Greek indices represent tensor fields on 4-dimensional and 5-dimensional manifolds, respectively.

* Such a correspondence could be defined by giving a vector field on $\mathscr{M}$, nowhere vanishing and nowhere tangent to the M_λ: $p_\lambda \in M_\lambda$ and $p_{\lambda'} \in M_{\lambda'}$ are in correspondence if a trajectory of this vector field joins p_λ and $p_{\lambda'}$. However, no such vector field is in the structure of $\mathscr{M}$.

metric on the boundary is nonsingular[3]. We shall often simply identify $\mathscr{M}$ with the interior of $\mathscr{M}'$.

The above definition certainly corresponds to our intuitive idea of a limit of a collection of spacetimes. It is not, however, very useful for actually writing down limits. We next show how the limit spaces may be characterized in terms of certain structures on $\mathscr{M}$.

By a *family of frames* in $\mathscr{M}$ we mean an orthonormal tetrad $w(\lambda)$ of vectors tangent to M_λ and attached to a single point $p_\lambda \in M_\lambda$, for each $\lambda > 0$, such that the $w(\lambda)$ vary smoothly along the smooth curve in $\mathscr{M}$ defined by the points p_λ. If $\mathscr{M}'$ is any limit space of $\mathscr{M}$, we may ask whether or not a given family of frames assumes a limit, i.e., approaches a frame $w(0)$ at some point $p_0 \in \partial\mathscr{M}'$ as $\lambda \to 0$. In general, of course, the answer will be no. However, it is clearly always possible, given a limit space $\mathscr{M}'$, to find some family of frames which does have a limit in $\mathscr{M}'$.

Let $\mathscr{M}'$ be a limit space of $\mathscr{M}$, and let $w(\lambda)$ be a family of frames which assumes a limit as $\lambda \to 0$. Let us represent points in M_λ in a neighborhood of p_λ in terms of the system of normal coordinates based on $w(\lambda)$. In terms of these coordinates, the components of the metric tensor in the M_λ approach a limit as $\lambda \to 0$, and the limiting components are precisely the components of $g^{ab}(0)$ in $\partial\mathscr{M}'$ in a neighborhood of p_0. Thus, the family of frames $w(\lambda)$ uniquely defines the limit space $\mathscr{M}'$, at least in a sufficiently small neighborhood of p_0. We now have a computational technique to find all limit spaces: each $\mathscr{M}'$ is characterized by some family of frames for which the components of the metric in the corresponding normal neighborhoods approach a limit as $\lambda \to 0$.

All we have done so far is to cast the usual definition of a limit (in terms of coordinates) into a slightly different language. To obtain useful information about spacetimes, however, it is necessary to consider also the global properties of limits, and it is here that our formalism will simplify matters considerably.

Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two limit spaces of $\mathscr{M}$. We say that $\mathscr{M}_1$ is an *extension* of $\mathscr{M}_2$ if there exists a smooth mapping of $\mathscr{M}_1$ into $\mathscr{M}_2$ which preserves the metric $g^{\alpha\beta}$ and leaves invariant each point of $\mathscr{M}$. The above discussion implies that, when $\mathscr{M}_1$ is an extension of $\mathscr{M}_2$, there exists a family of frames in $\mathscr{M}$ which has a limit in both $\mathscr{M}_1$ and $\mathscr{M}_2$. It now follows from theorem A 1 (Appendix A) that, if $\mathscr{M}_1$ is an extension of $\mathscr{M}_2$ and $\mathscr{M}_2$ an extension of $\mathscr{M}_1$, then $\mathscr{M}_1 = \mathscr{M}_2$.

[3] There is a complication here with regard to Hausdorffness. The spacetimes M_λ are Hausdorff, and so is $\mathscr{M}$. However, we cannot take the limit spaces $\mathscr{M}'$ to be Hausdorff if we are to be able to deal with pathological cases. In fact, Theorem 1 is false unless we admit non-Hausdorff limit spaces.

Now let $\mathscr{M}'$ be any limit space of $\mathscr{M}$, and let $\mathscr{N}$ denote the disjoint union of all extensions of $\mathscr{M}'$ where, in this union, we identify corresponding points of $\mathscr{M}$. We now define an equivalence relation in $\mathscr{N}$. If $p_1 \in \partial \mathscr{M}_1$, $p_2 \in \partial \mathscr{M}_2$, where $\mathscr{M}_1$ and $\mathscr{M}_2$ are extensions of $\mathscr{M}'$, write $p_1 \approx p_2$ if there exists a family of frames in $\mathscr{M}$ which, in $\mathscr{M}_1$, has a limit at p_1 and, in $\mathscr{M}_2$, has a limit at p_2. From the above discussion of normal neighborhoods, we see that, whenever $p_1 \approx p_2$, there exist neighborhoods of p_1 and p_2 which are also identified. Thus, the set of equivalence classes form, in a natural way, a limit space $\bar{\mathscr{M}}$. By construction, $\bar{\mathscr{M}}$ is an extension of every extension of $\mathscr{M}$. But two limit spaces, each of which is an extension of the other, are equal, and so $\bar{\mathscr{M}}$ is unique and has no proper extension. We have outlined the proof of:

Theorem 1. *Every limit space $\mathscr{M}'$ has a unique extension $\bar{\mathscr{M}}$ such that (1) $\bar{\mathscr{M}}$ has no proper extension, and (2) $\bar{\mathscr{M}}$ is an extension of every extension of $\mathscr{M}'$. In particular, every family of frames either defines no limit space, or else defines a limit space which is "maximal" in the sense of Theorem 1.*

A simple example will serve to show the way in which useful information can be extracted from our characterization of limits. Consider the Riessner-Nordström solutions for a fixed value m_0 of the "mass", but with a variable value of the "charge" λ (Fig. 1). When $\lambda \to 0$, in the usual coordinates, we obtain the Schwarzschild solution with mass value m_0 (Fig. 2). It is obvious from Figs. 1 and 2 that something drastic is happening in the limit: the region inside the "throat" of the Reissner-Nordström solution appears to become swallowed up in the singularity in the limit, so that it does not appear in the Schwarzschild picture. Let us try to formulate (and answer) the question: Do the points between $r = r_-$ and $r = 0$ (shaded in Fig. 1) disappear or not in the limit $\lambda \to 0$?

We are here dealing with a particular limit, and so we must first choose an appropriate family of frames. In each Reissner-Nordström solution, let us choose a frame which is centered at the point p of Fig. 1, and such that two of the spacelike vectors of the tetrad point along the 2-spheres of spherical symmetry. (The frame is not, of course, uniquely determined by these conditions, but any two such frames are related by a symmetry of the spacetime.) Now consider a collection of points $q_\lambda \in M_\lambda$ such that, for each λ, q_λ lies in the shaded region in Fig. 1. It is a well-defined question to ask whether or not the curve in $\mathscr{M}$ defined by the q_λ approaches a limit in the maximal limit space defined by our frame at p. To calculate the answer, we refer each q_λ to our frame by means of a broken geodesic (c.f., Appendix A), take the limit of the numbers which define this geodesic, and ask whether the limiting numbers exist and define a broken geodesic in the Schwarzschild solution. The answer is no: the corresponding geodesic in the Schwarzschild solution always runs into the "singularity" at $r = 0$. Thus, in a well-defined sense, the throat of the Reissner-Nordström solution "squeezes up" as $\lambda \to 0$ and eventually swallows all points to the future of the horizon $r = r_-$.

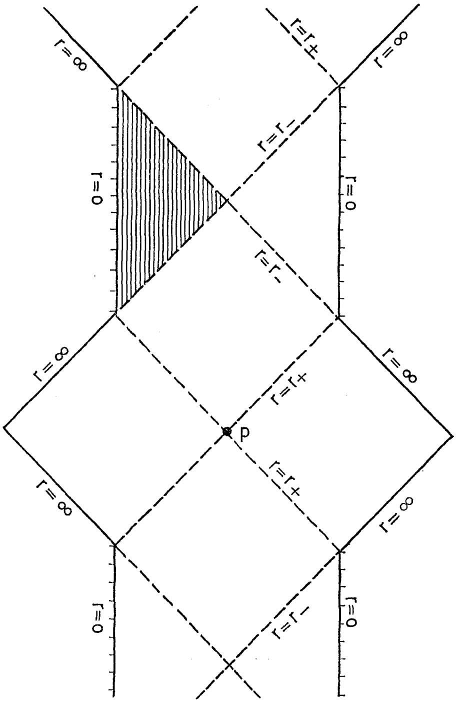

Fig. 1. The Reissner-Nordström solution. Each point in the figure represents a 2-sphere of spherical symmetry in the 4-dimensional spacetime. The radii of these 2-spheres define a scalar field r on the diagram. The horizons occur at the r-values

$$r_{\pm} = m_0 \pm (m_0^2 - \lambda^2)^{1/2}$$

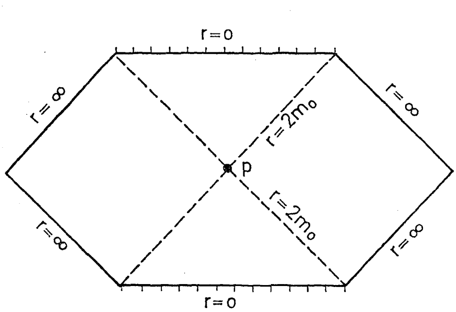

Fig. 2. The Schwarzschild solution. Each point in the figure represents a 2-sphere of spherical symmetry in the 4-dimensional spacetime

What is the fate of points below $r = r_-$? Now, however, the answer depends on the detailed behavior of the point q_λ as $\lambda \to 0$. If the point wanders too near $r = r_-$, it will not appear in the Schwarzschild solution, while if it remains well below $r = r_-$ it will appear in the limit. The precise statement of how a point must behave in the family of Reissner-Nordström solutions in order to remain in the limit is somewhat complicated, but completely well-defined.

Note that, among all possible frames in the Reissner-Nordström solution, the one we have used above is preferred in that it admits a simple description in terms of the Killing vectors. Thus, the Schwarzschild solution is, in a certain sense, the "canonical" limit of the Reissner-Nordström solution as the charge parameter goes to zero.

Finally, we ask whether it is possible to choose a family of frames which remain in the shaded region in Fig. 1 and which define a limit as $\lambda \to 0$. Such a limit would not, by what we have already shown, include the asymptotically flat regions of the spacetime. However, it is easily verified that, no matter what family of frames is erected in this region, no corresponding limit space exists.

3. Hereditary Properties

A property of spacetimes will be called *heriditary* if, whenever a family $(M_\lambda, g^{ab}(\lambda))$ of spacetimes have that property, all the limits of this family also have the property. In this section we shall classify a number of properties of spacetimes according to whether or not they are heriditary. While the answer is obvious is many cases, there are, however, a few surprises.

Suppose that there exists some tensor field, constructed from the Riemann tensor and its derivatives, which vanishes in each of the $(M_\lambda, g^{ab}(\lambda))$. Then, since $g^{\alpha\beta}$ is to be smooth on each limit space, our tensor field must also vanish on the boundary of each limit space. Einstein's source-free equations ($R_{ab} = 0$) and the condition of conformal flatness ($C_{abcd} = 0$) are of this type, and so are heriditary properties of spacetimes.

Consider next the type of the Weyl tensor. It is known [2, 3] that associated with each of the six types there is an algebraic expression in the Weyl tensor which vanishes whenever the Weyl tensor is of the corresponding type. Conversely, if one of these expressions vanishes, then the Weyl tensor is necessarily of that type or of one of its specializations. Thus, although the type of the Weyl tensor is not heriditary, properties such as "at least as specialized as type . . ." are.

Practically no topological properties of the underlying manifold are heriditary. (In fact, quite generally, no property of spacetimes is heriditary if it can be violated by merely removing some region from the manifold.) For example, neither the homology nor homotopy groups are heriditary: such groups can be either enlarged or diminished in the limit. However, the existence of spinor structure *is* heriditary. This fact follows immediately from the characterization of spinor structure in terms of neighborhoods of certain 2-spheres immersed in the spacetime [4]. Although the existence of spinor structure is heriditary, its absence is not, for this property can be destroyed by removing a suitable region (for example, all of the manifold except a small Euclidean neighborhood) from the spacetime.

Absence of closed timelike curves is heriditary. (If $\partial \mathscr{M}'$ has a closed timelike curve, then we may find a closed timelike curve in each M_λ for small λ.) The presence of closed timelike curves is not. Neither the presence of closed timelike curves is not. Neither the presence nor the absence of a Cauchy surface [5, 6], of asymptotic simplicity [7], or of a singularity (i.e., geodesic incompleteness [8]) is heriditary.

The situation with regard to Killing vectors is somewhat more complicated. Suppose we have a family $(M_\lambda, g^{ab}(\lambda))$ of spacetimes each of which has two Killing vectors. It might be thought that limits of this family need not have two Killing vectors, for, as $\lambda \to 0$, the Killing vectors in the $(M_\lambda, g^{ab}(\lambda))$ could conceivably approach each other and thus define only a single Killing vector in the limit. However, this circumstance cannot arise.

Consider a family of frames in $\mathscr{M}$. For each point $p_\lambda \in M_\lambda$ of this curve, let V_λ denote the 10-dimensional vector space consisting of all pairs (ξ^a, F^{ab}) of tensors at p_λ and in M_λ, where F^{ab} is skew. Given a Killing vector field on M_λ, its value and derivative at p_λ defines a point of V_λ, and so the set of Killing fields defines a vector subspace K_λ of V_λ. The dimension of K_λ is n, where n is the number of independent Killing fields in the $(M_\lambda, g^{ab}(\lambda))$. But the collection of all n-dimensional subspaces of a 10-dimensional vector space[4] is compact. Hence, if V_0 denotes the corresponding vector space at p_0, there must be some n-dimensional subspace K_0 of V_0 which is an accumulation space[5] of the K_λ. We will show that each element $(\xi^a, F^{ab})_0$ of K_0 defines a Killing field on $\partial \mathscr{M}'$. Choose any closed curve γ in $\partial \mathscr{M}'$, beginning and ending at p_0. We have only to prove that, under Killing transport (see Appendix B) around γ_0, $(\xi^a, F^{ab})_0$ remains unchanged. Let γ_λ be a curve in M_λ, beginning and ending at p_λ, and such that γ_λ approaches γ_0 in the limit, and let $(\xi^a, F^{ab})_\lambda \in K_\lambda$ accumulate at $(\xi^a, F^{ab})_0$. But now $\gamma_\lambda \to \gamma_0$, $(\xi^a, F^{ab})_\lambda$

[4] This space is called a *Grassmann manifold*, $G(n, 10)$.

[5] Even though the metric approaches its limit smoothly, the K_λ will not in general approach K_0 as a limit. Note, therefore, that we only require the existence of an accumulation point.

$\to (\xi^a, F^{ab})_0$, and the change, $\Delta\,(\xi^a, F^{ab})_\lambda$, in $(\xi^a, F^{ab})_\lambda$ on Killing transport about γ_λ is 0. Therefore, $\Delta\,(\xi^a, F^{ab})_0 = 0$, and so $(\xi^a, F^{ab})_0$ defines a Killing vector in $\partial \mathscr{M}'$. We conclude that if the $(M_\lambda,\ g^{ab}(\lambda))$ have an n-parameter group of motions, then each limit has at least an n-parameter group of motions.

By similar arguments we see that, for example, if each $(M_\lambda, g^{ab}(\lambda))$ has a hypersurface orthogonal Killing vector, so does each limit; if one Killing vector in each $(M_\lambda, g^{ab}(\lambda))$ commutes with all the others in an n-parameter family, then there exists a Killing vector in each limit which commutes with an n-parameter family of Killing vectors.

Similar remarks apply to conformal Killing vectors, where we must now use conformal Killing transport (Appendix B).

As an example of the above properties, let us consider limits of the Weyl solutions [9]. Each limit must be a sourcefree solution of Einstein's equations with spinor structure, no closed timelike curves, and at least two Killing vectors, one of which is hypersurface orthogonal and commutes with the other. Note that a spacetime with these properties need not, a priori, be a Weyl solution. Thus, the possibility exists that one can find wide classes of new solutions of Einstein's equations as limits of known solutions. In particular, we may call a family of solutions of Einstein's equations *closed* if it contains all its limits. For example, the plane wave solutions are closed, while the Weyl solutions are, presumably, not closed.

Appendix A. Rigidity

Let M and M' be two spacetimes, and suppose that M is isometric to a subset of M'. There may, of course, exist many different isometries. The assertion that Lorentz metrics are rigid (of order one) states that, once we specify how the tangent space of a particular point p of M is to be mapped into the tangent space of a particular point p' of M', the isometry Ψ, if there exists one at all, is uniquely determined. Thus, given the action of Ψ "to first order" at p, the requirement that Ψ be an isometry determines its behavior everywhere.

Theorem A 1. *Let M and M' be connected spacetimes, and let w be an orthonormal tetrad at a point $p \in M$ and w' at $p' \in M'$. Then there is at most one isometry of M into M' which takes w into w'.*

Proof. Let $(\eta_1^a, \eta_2^a, \ldots, \eta_n^a)$ be any collection of n nonzero vectors at p. We construct a broken geodesic as follows. Let γ_1 be the geodesic which passes through p and whose tangent vector at p is η_1^a. Choose an affine parameter τ on γ_1 such that $\tau = 0$, $\eta_1^a \nabla_a \tau = 1$ at p, and let p_1 denote the point on γ_1 unit affine distance from p. Parallel transport the $n-1$ vectors $(\eta_2^a, \eta_3^a \ldots, \eta_n^a)$ along γ_1 to p_1. Now repeat this construction with these $n-1$ vectors at p_1, and thus define a point p_2;

with $n-2$ vectors at p_2, and thus define a point p_3; etc. After the n[th] step, we obtain a point p_n. (We restrict ourselves to n-tuples of vectors at p for which, at each step in the construction, the appropriate geodesic *can* be extended unit affine length.) Since any point $q \in M$ may be joined to p by a broken geodesic, we may always choose n and $(\eta_1^a, \eta_2^a, \ldots, \eta_n^a)$ so that $p_n = q$.

Let Ψ and $\tilde{\Psi}$ be two isometries from M into M' each of which takes w into w'. Then Ψ and $\tilde{\Psi}$ have the same action on any $(\eta_1^a, \eta_2^a, \ldots, \eta_n^a)$, and so $\Psi(q)$ and $\tilde{\Psi}(q)$ are defined by broken geodesics in M' with the same set of n initial vectors. Therefore, $\Psi(q) = \tilde{\Psi}(q)$ for each point $q \in M$.

Theorem A 1, by referring points of any connected spacetime to the tangent space of a point, allows us to compare spacetimes by working to first order at a single point. It is this comparison property which was necessary to obtain Theorem 1. In contrast to Theorem A 1, manifolds without any further structure are completely non-rigid. In fact, it is well-known that, given any (connected, Hausdorff) manifold M and $2m$ points $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m$ of M, all distinct, then there exists a diffeomorphism of M onto itself which takes p_1 to q_1, p_2 to q_2, etc.

We now briefly summarize the general situation. By a *geometrical structure* we mean a general statement of the types of fields under consideration, that is, the number of connections, the numbers and valences of tensor fields (and, more generally, the types of geometrical objects [10]). For example, "a Lorentz metric", "a Lorentz metric and skew covariant tensor", and "three linearly independent vectors and a connection" are geometrical structures. By a *realization* of a geometrical structure we mean a (connected, Hausdorff) manifold equipped with fields of the type described by the geometrical structure. This distinction between a geometrical structure and its realizations is important: the notion of rigidity will apply only to the former. (That is, we say "Lorentz metrics are rigid", not "This Lorentz metric is rigid and that one is not".) Let M and M' be manifolds with realizations Φ and Φ', respectively, of a given geometrical structure. By an *isometry* of (M, Φ) into (M', Φ') we mean a diffeomorphism of M onto a subset of M' which takes Φ into Φ'.

We are now in a position to define rigidity. A geometrical structure is said to be *rigid of order n* $(n = 0, 1, 2, \ldots)$ if, given any two isometries Ψ and $\tilde{\Psi}$ of a realization (M, Φ) into a realization (M', Φ') of this geometrical structure such that the value and first n derivatives of Ψ coincide with those of $\tilde{\Psi}$ at some point of M, then $\Psi = \tilde{\Psi}$*. We illustrate this definition with the following list of rigidities.

* It follows immediately that the group of isometries of any realization of a rigid geometrical structure into itself form a Lie group. Is the converse true?

Order zero: m linearly independent vector fields on an m-manifold.

Order one: a nonsingular metric; a symmetric connection.

Order two: a conformal structure.

Not rigid to any order: m vector fields; symplectic structure.

On 5-manifolds, in particular, although "a metric $g^{\alpha\beta}$ of signature $(0, +, -, -, -)$" is not rigid, "a metric of this signature along with a family of frames" is rigid. It was for this reason that we introduced families of frames, and were thereby able to obtain unique limits of spacetimes.

Appendix B. Killing Transport

Let M be a connected spacetime with metric g_{ab}. Let ξ^a be a Killing vector field on M, and set

$$F_{ab} = \nabla_a \xi_b = F_{[ab]} .$$

We then have

$$\nabla_{[c} F_{a]b} = R_{cabd} \xi^d . \tag{B.1}$$

Rearranging the indices in (B.1), and using the fact that F_{ab} is skew, we obtain

$$\nabla_a F_{bc} = R_{bcad} \xi^d .$$

Let η^a be the tangent vector to some curve γ beginning at the point p. The above equations, contracted with η^a, yield

$$\eta^a \nabla_a \xi_b = F_{ab} \eta^a , \tag{B.2}$$

$$\eta^a \nabla_a F_{bc} = R_{bcad} \xi^d \eta^a .$$

Eqs. (B.2) give the values of (ξ^a, F_{ab}) along γ in terms of their values at p.

More generally, given any pair (ξ^a, F_{ab}) (not necessarily corresponding to some Killing vector) at p, we may always define such a pair at each point of γ via (B.2). We call this operation *Killing transport*. In general, if we apply Killing transport to some pair (ξ^a, F_{ab}) along a closed curve beginning and ending at p, then, on returning to p, the new pair (ξ'^a, F'_{ab}) will not coincide with our original pair. Suppose, however, that there exists a Killing vector on M whose value and derivative at p is (ξ^a, F_{ab}). Then, evidently, for every closed curve, we shall have $(\xi'^a, F'_{ab}) = (\xi^a, F_{ab})$. Conversely, if (ξ^a, F_{ab}) is given at p and if, for every closed curve we have $(\xi'^a, F'_{ab}) = (\xi^a, F_{ab})$, then there exists a Killing vector on M whose value and derivative at p is precisely (ξ^a, F_{ab}).

Let V denote the 10-dimensional vector space of all pairs (ξ^a, F_{ab}) at p. Each closed curve, beginning and ending at p, defines a linear transformation on V. That is, we have a "Killing holonomy group" at p. The fixed points under this group correspond precisely to the Killing fields on M. In particular, this group permits us to make the useful distinction between global Killing vectors, which are well-defined over the entire manifold, and local Killing vectors, which are defined in a neighborhood and which, when extended over the entire manifold, become many-valued. Local Killing vectors may be defined as the fixed points of the subgroup of the "Killing holonomy group" obtained by permitting only closed curves through p which may be contracted to a point.

We mention the following corollaries of the above discussion.

Corollary B 1. *If a Killing vector and its derivative both vanish at a single point, then the Killing vector vanishes everywhere.* (This corollary may be regarded as the infinitesimal statement of Theorem A 1.)

Corollary B 2. *Let M be a spacetime, and suppose that there is a Killing vector ξ^a defined on some open subset U of M. Then ξ^a and all its derivatives approach finite values on ∂U**.

Corollary B 2 provides a useful test for the extendability of a spacetime. One way to establish the nonexistence of an extension of a given spacetime is to find some scalar invariant which becomes infinite in the region across which we plan to carry out the extension. Corollary B 2 asserts that, in the construction of such invariants, it is also permissible to use scalars constructed from Killing vectors and their derivatives.

Since Killing transport is essentially tied up with the rigidity of the metric, and since conformal metrics are also rigid, we might expect to be able to define conformal Killing transport. Let ξ^a be a conformal Killing vector, and set

$$\nabla_a \xi_b = F_{ab} + \frac{1}{2} \varphi g_{ab} ,$$
$$k_a = \nabla_a \varphi ,$$

where F_{ab} is skew. Commuting derivatives as before, we find that, for any curve γ with tangent vector η^a,

$$\begin{aligned} \eta^a \nabla_a \xi_b &= \eta^a \left(F_{ab} + \frac{1}{2} g_{ab} \varphi\right) , \\ \eta^a \nabla_a \varphi &= \eta^a k_a , \\ \eta^a \nabla_a F_{bc} &= \eta^a (R_{bcad} \xi^d + k_{[b} g_{c]a}) , \\ \eta^a \nabla_a k_b &= \eta^a (\xi^d \nabla_d L_{ab} + \varphi L_{ab} + 2 R_{d(a} F_{b)}{}^d) , \end{aligned} \qquad \text{(B.3)}$$

* Of course, ξ^a will not in general be extendable to a Killing vector over all of M, e.g., the timelike Killing vector in the exterior region of the oscillating fluid ball solutions [11].

where $L_{ab} = R_{ab} - \frac{1}{6} g_{ab} R$. These equations define *conformal Killing transport* of the 4-tuple $(\xi^a, F_{ab}, \varphi, k_a)$. The same remarks concerning a holonomy group (now acting on a 15-dimensional vector space) and its fixed points apply here, too. The fact that we need specify two derivatives of ξ^a at a point in the conformal case is a reflection of the fact that conformal metrics are rigid of order two.

We mention two applications of conformal Killing transport. From Eqs. (B.3) it is immediately clear how to write down the general conformal Killing vector in flat space. Introducing Minkowskian coordinates x^a, then, when $R_{abcd} = 0$, we may successively integrate equations (B.3) beginning with the last:

$$k_a = \bar{k}_a ,$$
$$F_{ab} = \bar{k}_{[b} x_{a]} + \bar{F}_{ab} ,$$
$$\varphi = \bar{k}_a x^a + \bar{\varphi} ,$$
$$\xi_a = \frac{1}{4} \bar{k}_a (x^b x_b) - \frac{1}{2} (x^c \bar{k}_c) x_a + \bar{F}_{ba} x^b + \frac{1}{2} \bar{\varphi} x_a + \bar{\xi}_a ,$$

where $\bar{\xi}_a$, $\bar{F}_{ab}$, $\bar{\varphi}$, and $\bar{k}_a$ are constant tensors (fifteen numbers to define a conformal Killing vector).

Conformal transport also provides an elementary proof of the well-known [12] fact that a spacetime whose Weyl tensor vanishes is, locally, conformally equivalent to flat space. When $C_{abcd} = 0$, Eqs. (B.3) imply that the spacetime has, locally, fifteen conformal Killing vectors. Select one of these Killing vectors corresponding to a 4-tuple $(\xi^a, 0, 0, 0)$ at p, and then choose the conformal factor so that the norm of the corresponding conformal Killing vector is constant.

Finally, we remark that there exists an analogous projective Killing transport. The basic equations are identical with (B.2), except that F_{ab} need no longer be skew.

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