Twisted Group Algebras I

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Abstract. A generalization of the group algebra of a locally compact group is studied, by expressing the group algebra of a central group extension as the direct sum of closed *-ideals each one of which is isomorphic to such a "twisted" group algebra. In particular, the representation theory of such algebras is associated with the theory of projective representations by studying the representations of the group algebra of the group extension and the associated unitary representations of the group extension.

§ 1. Introduction

Projective representations of locally compact groups arise naturally in quantum mechanics in several ways. Groups of automorphisms of the algebra of bounded operators on a Hilbert space $\mathfrak{H}$ give rise to projective representations on $\mathfrak{H}$, and so the study of symmetry groups acting on the observables leads to a study of their projective representations. WEYL's form of the canonical commutatin relations [15] define a projective representation of the vector group $\mathbb{R}^{2n}$. Again in the application of MACKEY's theory of induced representations to the determination of the representations of semi-direct products of locally compact groups one is led to consider them from the outset as MACKEY has done in [11].

Each projective representation of a group G defines a true representation of a group extension, and as MACKEY does in [8] one can use the group algebra of the group extension in investigating the projective representation. Alternatively a twisted group algebra of the group G can be used. In this paper we give the basic definitions and results for twisted group algebras of locally compact groups and examine their relation to group algebras of group extensions. The key tool is a structure theorem which we prove in § 3. Many of the results, which are generalisations of well-known results for ordinary group algebras, follow as corollaries of the structure theorem. A fuller study of twisted group algebras together with detailed proofs of the results stated here can be found in EDWARDS [2].

The theory of twisted group algebras for finite groups is classical and references to the literature are to be found in WEYL's book [15]. KASTLER

[4] has studied twisted group algebras over vector spaces. His results are not special cases of ours since the vector spaces which he considers are not necessarily locally compact, but some of the results of Loupias and Miracle-Sole [7] on twisted group algebras for $\mathbb{R}^{2n}$ are. Montvay [12] has defined twisted convolution for an arbitrary locally compact group G with respect to a multiplier which is continuous on $G \times G$; in this paper we make no such restriction on the multiplier. Montvay pointed out that in the case which he considered many of the results on group algebras together with their proofs generalise straightforwardly to twisted group algebras.

§ 2. The Twisted Group Algebra

Let G be a separable locally compact group with unit element e and let m be a left-invariant Haar measure on G; let δ be the modular function on G and let $L_1(G)$, $L_2(G)$ be the spaces of equivalence classes of complex-valued m-measurable functions on G which are respectively absolutely integrable, absolutely square-integrable. $L_1(G)$ is a Banach space with respect to the norm $\| \cdot \|_1$ defined for each element ψ by

$$\| \psi \|_1 = \int_G | \psi(g) | \, dm(g),$$

and $L_2(G)$ is a Hilbert space with respect to the inner product $\langle \,.\,,\,. \rangle$ defined for each pair ψ_1, ψ_2 of elements by

$$\langle \psi_1, \psi_2 \rangle = \int_G \psi_1(g) \, \overline{\psi_2(g)} \, dm(g).$$

The corresponding norm is denoted by $\| \cdot \|_2$. (For the properties of $L_1(G)$ and $L_2(G)$ see Chapter IV of [3].)

Let T denote the group of complex numbers of unit modulus. A Borel function ω from $G \times G$ to T such that for all g, g', g'' in G,

(i) $\omega(g, e) = \omega(e, g) = 1$,

(ii) $\omega(g', g'') \, \omega(g, g'g'') = \omega(g, g') \, \omega(gg', g'')$

is said to be a multiplier on G. A multiplier ω is said to be trivial if there exists a Borel function ϱ from G to T such that $\varrho(e) = 1$ and for each pair g, g' of elements of G

$$\omega(g, g') = \varrho(g) \, \varrho(g') \, \varrho(gg')^{-1}$$

(for details see [9, 10]). For each multiplier ω and each pair ψ_1, ψ_2 of elements of $L_1(G)$ define the function $\psi_1 \omega \psi_2$, called the twisted convolution of ψ_1 and ψ_2, by

$$(\psi_1 \omega \psi_2)(g) = \int_G \psi_1(h) \, \psi_2(h^{-1}g) \, \omega(h, h^{-1}g) \, dm(h).$$

Since ω is a bounded Borel function on $G \times G$ it follows that $g \mapsto \omega$ $(h, h^{-1}g)$ and $h \mapsto \omega(h, h^{-1}g)$ are bounded Borel functions on G and so the integral is well-defined and defines an m-measurable function on G. A straightforward computation shows that the function $\psi_1 \omega \psi_2$ defines an element of $L_1(G)$ such that

$$\|\psi_1 \omega \psi_2\|_1 \leq \|\psi_1\|_1 \|\psi_2\|_1 \,.$$

For each multiplier ω on G and each element ψ of $L_1(G)$ let ψ^ω be defined by

$$\psi^\omega(g) = \overline{\omega(g, g^{-1})} \, \delta(g^{-1}) \, \overline{\psi(g^{-1})} \,.$$

Then it is clear that since ω is Borel and δ is continuous the function ψ^ω is m-measurable. A further computation shows that the map $\psi \mapsto \psi^\omega$ is an involution:

$$\|\psi^\omega\|_1 = \|\psi\|_1, \qquad (\psi^\omega)^\omega = \psi \,.$$

Furthermore, the identities

$$(\psi_1 \omega \psi_2)^\omega = (\psi_2^\omega) \, \omega \, (\psi_1^\omega) \,,$$

$$(\psi_1 \omega \psi_2) \, \omega \, \psi_3 = \psi_1 \omega \, (\psi_2 \omega \psi_3) \,,$$

follow from routine computations. Let $(L_1(G), \omega)$ denote the Banach space $L_1(G)$ equipped with the multiplication $\psi_1, \psi_2 \mapsto \psi_1 \omega \psi_2$ and with the involution $\psi \mapsto \psi^\omega$. Summarising, we have

Theorem 1. $(L_1(G), \omega)$ *is a Banach $*$-algebra.*

We call $(L_1(G), \omega)$ the twisted group algebra over G corresponding to the multiplier ω. In the special case in which ω is identically one, twisted convolution becomes ordinary convolution and then we write $\psi_1 * \psi_2$ for $\psi_1 \omega \psi_2$. Similarly the involution $\psi \mapsto \psi^\omega$ coincides with the usual involution and we write ψ^* for ψ^ω; then $(L_1(G), *)$ is the group algebra of G.

§ 3. The Structure Theorem

Let A be a separable compact abelian group written additively with unit element 0 and let n be normalised Haar measure on A. A Borel function f from $G \times G$ to A such that for all g, g', g'' in G

i) $f(g, e) = f(e, g) = 0$,

ii) $f(g', g'') + f(g, g'g'') = f(g, g') + f(gg', g'')$,

is said to be a 2-cocycle of G over A. The set of 2-cocycles of G over A forms an abelian group $Z^2(G, A)$. An element f of $Z^2(G, A)$ is said to be a 2-coboundary of G over A if there exists a Borel function h from G to A such that $h(e) = 0$, and for each pair g, g' of elements of G,

$$f(g, g') = h(g) + h(g') - h(gg') \,.$$

The set of 2-coboundaries of G over A forms a subgroup $B^2(G, A)$ of $Z^2(G, A)$.

For each element f of $Z^2(G, A)$ we may define a multiplication on $A \times G$ by

$$(a, g)(a', g') = (a + a' + f(g, g'), gg') .$$

Endowed with this multiplication the set $A \times G$ becomes a group which we denote by G^f, called the central extension of A by G defined by f. Giving G^f the Borel structure of $A \times G$, the product measure $n \times m$ is a left-invariant measure on G^f and there is a unique locally compact topology on G^f under which it is a separable locally compact group whose associated Borel structure is that just described [10]. The modular function Δ on G^f is given for each element (a, g) of G^f by $\Delta(a, g) = \delta(g)$ [2]. Note that the set of multipliers on G is $Z^2(G, T)$ and the set of trivial multipliers is $B^2(G, T)$.

Let $A^{\wedge}$ denote the dual of A; since A is a compact abelian group $A^{\wedge}$ is a discrete abelian group, and since A is separable $A^{\wedge}$ is countable (see Chapter 1 of [14]). For each element α of $A^{\wedge}$ and each element f of $Z^2(G, A)$ the composed function $\alpha \circ f$ defined by

$$(\alpha \circ f)(g, g') = \alpha(f(g, g'))$$

is a multiplier on G. In particular, since $T^{\wedge}$ can be identified with $\mathbb{Z}$, the additive group of integers, for each multiplier ω and each integer r the function $g, g' \mapsto (\omega(g, g'))^r$ is a multiplier on G.

The main theorem concerns the relation between the group algebra $(L_1(G^f), *)$ and the twisted group algebra $(L_1(G), \alpha \circ f)$. Let α be an element of $A^{\wedge}$; for each element Ψ of $L_1(G^f)$ define the function $\alpha(\Psi)$ on G by

$$\alpha(\Psi)(g) = \int_A \Psi(a, g) \alpha(a) \, dn(a) .$$

Fubini's theorem shows that $\alpha(\Psi)$ is m-measurable and a routine calculation shows that it defines an element of $L_1(G)$ such that

$$\|\alpha(\Psi)\|_1 \leqq \|\Psi\|_1 .$$

Further, if ψ is an element of $L_1(G)$, the function $\bar{\alpha} \otimes \psi$ defined by

$$(\bar{\alpha} \otimes \psi)(a, g) = \overline{\alpha(a)} \, \psi(g)$$

defines an element of $L_1(G^f)$ such that $\alpha(\bar{\alpha} \otimes \psi) = \psi$. It follows that the map $\Psi \mapsto \alpha(\Psi)$ of $L_1(G^f)$ into $L_1(G)$ is surjective. We have

Lemma 3.1. *The map* $\Psi \mapsto \alpha(\Psi)$ *is a norm non-increasing* ∗*-homomorphism from* $(L_1(G^f), *)$ *onto* $(L_1(G), \alpha \circ f)$.

Proof. We have seen that it is a norm non-increasing map from $L_1(G^f)$ onto $L_1(G)$ and a routine computation shows it to be a ∗-homomorphism.

Lemma 3.2. *The map* $\psi \mapsto \bar{\alpha} \otimes \psi$ *is an isometric* ∗*-isomorphism, from* $(L_1(G), \alpha \circ f)$ *onto a closed two-sided ideal in* $(L_1(G^f), *)$.

Proof. A simple computation shows that it is a map of $L_1(G)$ into $L_1(G^f)$ which is an isometric $*$-homomorphism. It remains to prove that the image is a closed two-sided ideal. First we show that it is closed. Let $\{\psi_r\}$ be a sequence of elements of $L_1(G)$ such that $\{\bar{\alpha} \otimes \psi_r\}$ converges to an element Ψ of $L_1(G^f)$. Then since the map $\Psi \mapsto \alpha(\Psi)$ is norm non-increasing it follows that $\{\psi_r\}$ converges to $\alpha(\Psi)$. Therefore,

$$\|\Psi - \bar{\alpha} \otimes \alpha(\Psi)\|_1 \leq \|\Psi - \bar{\alpha} \otimes \psi_r\|_1 + \|\bar{\alpha} \otimes \psi_r - \bar{\alpha} \otimes \alpha(\Psi)\|_1$$

$$= \|\Psi - \bar{\alpha} \otimes \psi_r\|_1 + \|\psi_r - \alpha(\Psi)\|_1 .$$

Taking limits we see that $\Psi = \bar{\alpha} \otimes \alpha(\Psi)$ so that the image of the map $\psi \mapsto \bar{\alpha} \otimes \psi$ is closed. Routine calculations show that for each element Ψ of $L_1(G^f)$ and each element ψ of $L_1(G)$ we have, writing $\omega = \alpha \circ f$,

$$\Psi * (\bar{\alpha} \otimes \psi) = \bar{\alpha} \otimes (\alpha(\Psi) \omega \psi) ,$$

$$(\bar{\alpha} \otimes \psi) * \Psi = \bar{\alpha} \otimes (\psi \omega \alpha(\Psi))$$

which shows that the image set is a two-sided ideal.

For each element Ψ of $L_1(G^f)$ and each element α of $A\hat{\ }$ let $P_\alpha \Psi = \bar{\alpha} \otimes \alpha(\Psi)$. In terms of P_α the conclusions of Lemmas 3.1 and 3.2 may be rephrased: P_α is a norm non-increasing $*$-homomorphism from $(L_1(G^f), *)$ onto a closed two-sided ideal which is isometrically $*$-isomorphic to $(L_1(G), \alpha \circ f)$. Furthermore, the orthogonality of distinct characters of A implies that

$$P_\alpha P_\beta = P_\beta P_\alpha = \delta_{\alpha\beta} P_\alpha$$

and since the characters of A generate $L_1(A)$ topologically we have

$$\sum_{\alpha \in A\hat{\ }} P_\alpha = 1 .$$

We have proved

Theorem 2. $(L_1(G^f), *)$ *is the direct sum over* $A\hat{\ }$ *of closed two-sided ideals* $\{I_\alpha\}$, *where* I_α *is isometrically* $*$-*isomorphic to the twisted group algebra* $(L_1(G), \alpha \circ f)$.

In § 4 we prove a result about the regular representation of G^f; for this we shall require an easily proved analogue of Theorem 2 which we state as

Theorem 2′. *The Hilbert space* $L_2(G^f)$ *is the direct sum of closed subspaces each one of which can be identified with* $L_2(G)$ *being the image of an orthogonal projection* $\Psi \mapsto \bar{\alpha} \otimes \alpha(\Psi)$.

§ 4. Projective Representations

In the theory of unitary representations it is sometimes convenient to make use of the one-to-one correspondence which exists between

essential *-representations of $(L_1(G), *)$ and continuous unitary represen-
tations of G (see [13] for example). In this section we prove the corres-
ponding theorem for essential *-representations of twisted group algebras
and projective representations of groups.

Let $\mathfrak{H}$ be a separable Hilbert space; let $U(\mathfrak{H})$ denote the group of
unitary operators on $\mathfrak{H}$. A projective representation of a locally compact
group G on $\mathfrak{H}$ is a map $\pi: G \mapsto U(\mathfrak{H})$ such that

i) $\pi(e) = 1$,

ii) the map $g \mapsto \langle \pi(g) x, y \rangle$ is Borel for each pair x, y of vectors
in $\mathfrak{H}$,

iii) for each pair g, g' of elements of G,

$$\pi(g) \pi(g') \pi(gg')^{-1} = \omega(g, g') \cdot 1$$

where $\omega(g, g')$ is a complex number of modulus unity.

It follows from the associativity of group multiplication that ω
satisfies

$$\omega(g', g'') \omega(g, g'g'') = \omega(g, g') \omega(gg', g'')$$

and it is easy to check that ω is a multiplier in the sense of § 2. Conver-
sely, every multiplier in the sense of § 2 is a multiplier for some projective
representation as we see below. We shall refer to a projective represen-
tation π having multiplier ω.

A projective representation π of G on $\mathfrak{H}$ is said to be irreducible if
the only closed subspaces of $\mathfrak{H}$ invariant under the set $\pi(G)$ of unitary
operators are $\{0\}$ and $\mathfrak{H}$. Two projective representations π and π' on $\mathfrak{H}$
and $\mathfrak{H}'$ respectively are said to be unitarily equivalent if there exists an
isomorphism $T: \mathfrak{H} \to \mathfrak{H}'$ satisfying

$$T \pi(g) = \pi'(g) T$$

for all g in G. Clearly two unitarily equivalent projective representations
have the same multiplier.

Let f be an element of $Z^2(G, A)$ and let G^f be the central extension
of A by G defined by f. Let α be an element of $A^\wedge$ and let π_α be a pro-
jective representation of G on $\mathfrak{H}_\alpha$ with multiplier $\alpha \circ f$. Then the map
$(a, g) \mapsto \alpha(a) \pi_\alpha(g)$ from G^f to $U(\mathfrak{H}_\alpha)$ is a weakly Borel unitary represen-
tation of G^f on $\mathfrak{H}_\alpha$ and so, by 22.20 of [3], is a continuous unitary repre-
sentation. We shall denote it by $\alpha \otimes \pi_\alpha$:

$$(\alpha \otimes \pi_\alpha)(a, g) = \alpha(a) \pi_\alpha(g) .$$

Conversely, let u be a continuous unitary representation of G^f on $\mathfrak{H}$.
The restriction of u to A is a continuous unitary representation of a
compact abelian group and so

$$u(a, e) = \bigoplus_{\alpha \in A^\wedge} \alpha(a) E_\alpha$$

where $\alpha \mapsto E_\alpha$ is a projection-valued measure on $A\,\hat{}$ and E_α is defined for each pair x, y of vectors in by

$$\langle E_\alpha x, y \rangle = \int_A \overline{\alpha(a)} \, \langle u(a, e) \, x, y \rangle \, dn(a) \,. \tag{4.1}$$

Since A lies in the centre of G^f, for each element (a, g) of G^f and each element a' of A,

$$u(a', e) \, u(a, g) = u(a, g) \, u(a', e)$$

so that E_α commutes with $u(a, g)$ for each element (a, g) of G^f. It follows that u_α, defined for each element (a, g) of G^f by

$$u_\alpha(a, g) = u(a, g) \, E_\alpha \,,$$

is a continuous unitary representation of G^f on $\mathfrak{H}_\alpha = E_\alpha \mathfrak{H}$, and that

$$u = \bigoplus_{\alpha \in A\hat{}} u_\alpha \,. \tag{4.2}$$

Let π_α denote the restriction of u_α to G. Then, π_α is a projective representation of G on $\mathfrak{H}_\alpha$ with multiplier $\alpha \circ f$ and

$$u_\alpha = \alpha \otimes \pi_\alpha \,.$$

Furthermore, SCHUR's lemma shows that u is irreducible if and only if there is only one non-vanishing term in (4.2). Summarising, we have

Lemma 4.1. (i) *Let π_α be a projective representation of G on $\mathfrak{H}_\alpha$ with multiplier $\alpha \circ f$. Then $\alpha \otimes \pi_\alpha$ is a continuous unitary representation of G^f on $\mathfrak{H}_\alpha$.*

(ii) *Let u be a continuous unitary representation of G^f on $\mathfrak{H}$. Then, there exists a projective representation π_α of G with multiplier $\alpha \circ f$, defined on $\mathfrak{H}_\alpha = E_\alpha \mathfrak{H}$ such that*

$$u = \bigoplus_{\alpha \in A\hat{}} \alpha \otimes \pi_\alpha \,. \tag{4.3}$$

(iii) *A continuous unitary representation u of G^f on $\mathfrak{H}$ is irreducible if and only if $u = \alpha \otimes \pi_\alpha$ for some element α of $A\,\hat{}$ and some irreducible projective representation π_α of G with multiplier $\alpha \circ f$.*

Further, let π_α, π'_α be unitarily equivalent projective representations of G with multiplier $\alpha \circ f$ on $\mathfrak{H}_\alpha, \mathfrak{H}'_\alpha$ respectively. Then, clearly $\alpha \otimes \pi_\alpha$, $\alpha \otimes \pi'_\alpha$ are unitarily equivalent continuous unitary representations of G^f. Conversely, let u, u' be unitarily equivalent continuous unitary representations of G^f on $\mathfrak{H}, \mathfrak{H}'$ respectively. Let $\alpha \mapsto E_\alpha$, $\alpha \mapsto E'_\alpha$ be the corresponding projection-valued measures on $A\,\hat{}$ and let

$$u = \bigoplus_{\alpha \in A\hat{}} \alpha \otimes \pi_\alpha, \; u' = \bigoplus_{\alpha \in A\hat{}} \alpha \otimes \pi'_\alpha \,.$$

There exists an isomorphism $T : \mathfrak{H} \to \mathfrak{H}'$ such that for each element (a, g) of G^f, $T u(a, g) = u'(a, g) \, T$. Then, it is clear that for each element α of $A\,\hat{}$, $T_\alpha = T E_\alpha = E'_\alpha T$ is an isomorphism of $E_\alpha \mathfrak{H}$ and $E'_\alpha \mathfrak{H}'$ such that for each element g of G, $T_\alpha \pi_\alpha(g) = \pi'_\alpha(g) \, T_\alpha$. Summarising, we have

Lemma 4.2. (i) *Let* π_α, π'_α *be projective representations of* G *on* $\mathfrak{H}_\alpha$, $\mathfrak{H}'_\alpha$ *respectively, both having the same multiplier* $\alpha \circ f$. *Then* π_α, π'_α *are unitarily equivalent if and only if the continuous unitary representations* $\alpha \otimes \pi_\alpha$, $\alpha \otimes \pi'_\alpha$ *of* G^f *are unitarily equivalent.*

(ii) *Let* u, u' *be continuous unitary representations of* G^f *on* $\mathfrak{H}$, $\mathfrak{H}'$ *respectively; let*

$$u = \bigoplus_{\alpha \in A^\smallfrown} \alpha \otimes \pi_\alpha, \, u' = \bigoplus_{\alpha \in A^\smallfrown} \alpha \otimes \pi'_\alpha$$

be their corresponding decompositions according to Lemma 4.1 (ii). *Then* u, u' *are unitarily equivalent if and only if for each element* α *of* $A^\smallfrown$, *the projective representations* π_α, π'_α *of* G *are unitarily equivalent.*

There is a related decomposition for essential $*$-representations of $(L_1(G^f), *)$. Let Π_α be an essential $*$-representation of $(L_1(G), \alpha \circ f)$ on the separable Hilbert space $\mathfrak{H}_\alpha$. Let $\Pi_\alpha \circ \alpha$ be the mapping defined for each element Ψ of $L_1(G^f)$ by

$$(\Pi_\alpha \circ \alpha)\,(\Psi) = \Pi_\alpha(\alpha(\Psi))\,.$$

Then, $\Pi_\alpha \circ \alpha$ is an essential $*$-representation of $(L_1(G^f), *)$ since α is a $*$-homomorphism of $(L_1(G^f), *)$ onto $(L_1(G), \alpha \circ f)$. Conversely, let U be an essential $*$-representation of $(L_1(G^f), *)$ on the separable Hilbert space $\mathfrak{H}$. Let $\{\Xi_i : i \in \Lambda\}$ be an approximate identity for $(L_1(G^f), *)$. Then,

$$U(P_\alpha \Xi_i)\,U(\Psi) = U(P_\alpha(\Xi_i) * \Psi)$$

$$= U(P_\alpha(\Xi_i * \Psi))$$

by Theorem 2. Hence the set $\{U(P_\alpha(\Xi_i)) : i \in \Lambda\}$ has a limit F_α in the strong operator topology such that for each element Ψ of $L_1(G^f)$

$$F_\alpha U(\Psi) = U(P_\alpha \Psi)\,. \tag{4.4}$$

A similar argument shows that

$$U(\Psi)\,F_\alpha = U(P_\alpha \Psi)\,. \tag{4.5}$$

From Theorem 2 it is clear that $\alpha \mapsto F_\alpha$ is a projection-valued measure on $A^\smallfrown$. Let U_α be the restriction of U to $\mathfrak{H}_\alpha = F_\alpha \mathfrak{H}$. Then (4.4) and (4.5) show that U_α is an essential $*$-representation of $(L_1(G^f), *)$ on $\mathfrak{H}_\alpha$ such that

$$U = \bigoplus_{\alpha \in A^\smallfrown} U_\alpha \tag{4.6}$$

and that

$$U_\alpha = \Pi_\alpha \circ \alpha$$

where Π_α is the essential $*$-representation of $(L_1(G), \alpha \circ f)$ defined by

$$\Pi_\alpha(\psi) = U_\alpha(\bar{\alpha} \otimes \psi)\,.$$

Furthermore, SCHUR's Lemma shows that U is irreducible if and only if there is only one non-vanishing term in (4.6). Summarising, we have

Lemma 4.3. (i) *Let Π_α be an essential $*$-representation of $(L_1(G), \alpha \circ f)$ on $\mathfrak{H}_\alpha$. Then $\Pi_\alpha \circ \alpha$ is an essential $*$-representation of $(L_1(G^f), *)$ on $\mathfrak{H}_\alpha$.*

(ii) *Let U be an essential $*$-representation of $(L_1(G^f), *)$ on $\mathfrak{H}$. Then, there exists a projection-valued measure $\alpha \mapsto F_\alpha$ on $A\,\hat{}$ and for each element α of $A\,\hat{}$ an essential $*$-representation Π_α of $(L_1(G), \alpha \circ f)$ on $\mathfrak{H}_\alpha = F_\alpha \mathfrak{H}$ such that*

$$U = \bigoplus_{\alpha \in A\,\hat{}} \Pi_\alpha \circ \alpha \ .$$

(iii) *An essential $*$-representation U of $(L_1(G^f), *)$ on $\mathfrak{H}$ is irreducible if and only if $U = \Pi_\alpha \circ \alpha$ for some element α of $A\,\hat{}$ and some irreducible essential $*$-representation Π_α of $(L_1(G), \alpha \circ f)$.*

Further, let Π_α, Π_α' be unitarily equivalent essential $*$-representations of $(L_1(G), \alpha \circ f)$ on $\mathfrak{H}_\alpha, \mathfrak{H}_\alpha'$ respectively. Then, clearly $\Pi_\alpha \circ \alpha, \Pi_\alpha' \circ \alpha$ are unitarily equivalent essential $*$-representations of $(L_1(G^f), *)$. Conversely, let U, U' be essential $*$-representations of $(L_1(G^f), *)$ on $\mathfrak{H}, \mathfrak{H}'$ respectively. Let $\alpha \mapsto F_\alpha$, $\alpha \mapsto F_\alpha'$ be the corresponding projection-valued measures on $A\,\hat{}$ and let

$$U = \bigoplus_{\alpha \in A\,\hat{}} \Pi_\alpha \circ \alpha, \ U' = \bigoplus_{\alpha \in A\,\hat{}} \Pi_\alpha' \circ \alpha \ .$$

Then, there exists an isomorphism $T : \mathfrak{H} \to \mathfrak{H}'$ such that for each element Ψ of $(L_1(G^f), *)$, $T U(\Psi) = U'(\Psi) T$. Then, it is clear that for each element α of $A\,\hat{}$, $T_\alpha = T F_\alpha = F_\alpha' T$ is an isomorphism of $F_\alpha \mathfrak{H}$ and $F_\alpha' \mathfrak{H}'$ such that for each element ψ of $L_1(G)$, $T_\alpha \Pi_\alpha(\psi) = \Pi_\alpha'(\psi) T_\alpha$. Summarising, we have

Lemma 4.4. (i) *Let Π_α, Π_α' be essential $*$-representations of $(L_1(G), \alpha \circ f)$ on $\mathfrak{H}_\alpha, \mathfrak{H}_\alpha'$ respectively. Then, Π_α, Π_α' are unitarily equivalent if and only if the essential $*$-representations $\Pi_\alpha \circ \alpha, \Pi_\alpha' \circ \alpha$ of $(L_1(G^f), *)$ are unitarily equivalent.*

(ii) *Let U, U' be essential $*$-representations of $(L_1(G^f), *)$ on $\mathfrak{H}, \mathfrak{H}'$ respectively.; let*

$$U = \bigoplus_{\alpha \in A\,\hat{}} \Pi_\alpha \circ \alpha, \ U' = \bigoplus_{\alpha \in A\,\hat{}} \Pi_\alpha' \circ \alpha$$

be their corresponding decompositions according to Lemma 4.3 (ii). Then, U, U' are unitarily equivalent if and only if for each element α of $A\,\hat{}$, the essential $*$-representations Π_α, Π_α' of $(L_1(G), \alpha \circ f)$ are unitarily equivalent.

There is a one-to-one correspondence between continuous unitary representations of an arbitrary locally compact group and essential $*$-representations of its group algebra. This correspondence preserves irreducibility and unitary equivalence (see for example NAIMARK [13],

p. 376). Applying this result to a continuous unitary representation u of G^f on $\mathfrak{H}$, the corresponding essential $*$-representation U of $(L_1(G^f), *)$ is defined for each element Ψ of $L_1(G^f)$ and each pair x, y of elements of $\mathfrak{H}$ by

$$\langle U(\Psi) x, y \rangle = \int_{G^f} \Psi(a, g) \langle u(a, g) x, y \rangle \, dn(a) \, dm(g).$$

Moreover, for each element (a, g) of G^f,

$$u(a, g) = \lim U(_{(a, g)} \varXi_i)$$

where $\{ \varXi_i : i \in \varLambda \}$ is an approximate identity for $(L_1(G^f), *)$, and for each element Ψ of $L_1(G^f)$, $_{(a, g)} \Psi$ denotes the left translate of Ψ by $(a, g) : {}_{(a, g)} \Psi(a', g') = \Psi((a, g)^{-1}(a', g'))$ Applying these results to the representations u, U of G^f and $(L_1(G^f), *)$ defined in Lemmas 4.1 and 4.3 respectively, a straightforward calculation shows that the projection-valued measures $\alpha \mapsto E_\alpha$, $\alpha \mapsto F_\alpha$ are identical. It is then clear from the results of this section that the following generalization of the above results holds:

Theorem 3. *There exists a one-to-one correspondence between essential $*$-representations $\varPi$ of $(L_1(G), \alpha \circ f)$ and projective representations π of G with multiplier $\alpha \circ f$ given for each element ψ of $L_1(G)$ and each pair x, y of vectors in $\mathfrak{H}$, the separable Hilbert space on which $\varPi$ and π are defined, by*

$$\langle \varPi(\psi) x, y \rangle = \int_G \psi(g) \langle \pi(g) x, y \rangle \, dm(g).$$

This correspondence preserves irreducibility and unitary equivalence.

As an application, consider the left-regular representation l of G^f on $L_2(G^f)$:

$$(l(a, g) \varPhi)(a', g') = \varPhi((a, g)^{-1}(a', g'))$$

By Lemma 4.1 (ii),

$$l = \bigoplus_{\alpha \in \varLambda^\wedge} \alpha \otimes \lambda_\alpha$$

where λ_α is the projective representation of G with multiplier $\omega = \alpha \circ f$ on $L_2(G)$ given by

$$(\lambda_\alpha(g) \, \phi)(g') = \omega(g, g^{-1}g') \, \phi(g^{-1}g').$$

λ_α is called the twisted left-regular representation of G with multiplier ω. Similarly let L be the left-regular representation of $(L_1(G^f), *)$ on $L_2(G^f)$:

$$L(\Psi) \varPhi = \Psi * \varPhi.$$

By Lemma 4.3 (ii),

$$L = \bigoplus_{\alpha \in \varLambda^\wedge} \varLambda_\alpha \circ \alpha$$

where $\varLambda_\alpha$ is the essential $*$-representation of $(L_1(G), \omega)$ on $L_2(G)$ given by

$$\varLambda_\alpha(\psi) \, \phi = \psi \omega \phi.$$

§ 5. Properties of Twisted Group Algebras

In this section the results of § 3,4 are applied to an arbitrary twisted group algebra $(L_1(G), \omega)$ by replacing A by T. The following results are immediate consequences of Theorem 2.

Theorem 4. $(L_1(G), \omega)$ *possesses an approximate identity.*

Theorem 5. $(L_1(G), \omega)$ *has an identity if and only if G is discrete.*

For each element g of G and each element ψ of $L_1(G)$ let $\overset{\omega}{\underset{g}{}} \psi$ be the function defined for each element g' of G by

$$\left(\overset{\omega}{\underset{g}{}} \psi \right)(g') = \omega(g, g^{-1}g')\, \psi(g^{-1}g') \,.$$

The map $\psi \mapsto \overset{\omega}{\underset{g}{}} \psi$ is a linear isometry on $L_1(G)$. $\overset{\omega}{\underset{g}{}} \psi$ is said to be the twisted left-translate of ψ by g with multiplier ω. An application of Theorem 2 shows

Theorem 6. *A closed subset S of $L_1(G)$ is a left-ideal in $(L_1(G), \omega)$ if and only if it is closed under all twisted left-translations.*

The proof of the following result is best obtained by direct methods. It follows closely the proof given in Chapter IV of LOOMIS [6] for $(L_1(G), *)$ and depends on a result of CALABI [1], first used in this form by KLEPPNER [5], which states that ω is trivial if G^ω is abelian.

Theorem 7. $(L_1(G), \omega)$ *is commutative if and only if G is abelian and ω is trivial.*

The remarks following Theorem 3 show that $(L_1(G), \omega)$ possesses a faithful essential *-representation. This fact gives rise to

Theorem 8. $(L_1(G), \omega)$ *is an A^*-algebra.*

Let π be a mapping from G to the group $U(\mathfrak{H})$ of unitary operators on the separable Hilbert space $\mathfrak{H}$ such that

   (i) $\pi(e) = 1$

   (ii) the mapping $g \to \langle \pi(g)\, x, y \rangle$ is m-measurable for each pair x, y of elements of $\mathfrak{H}$,

   (iii) $\pi(g)\, \pi(g')\, \pi(gg')^{-1} = \omega(g, g') \cdot 1$ for each pair g, g' of elements of G. Then π is a weakly m-measurable projective representation of G on $\mathfrak{H}$ with multiplier ω. It follows that the mapping $(t, g) \mapsto t\pi(g)$ on G^ω is an $n \times m$-measurable unitary representation of G^ω on $\mathfrak{H}$. It follows from 22.20 of [3] that the mapping is weakly continuous. Hence the mapping π is weakly Borel. We have therefore proved

Theorem 9. *Every weakly m-measurable projective representation of G on separable Hilbert space is a projective representation.*

A further immediate consequence of Theorem 3 is the following

Theorem 10. *The set of irreducible projective representations of G with multiplier ω forms a complete family.*

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