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Direction-Dependent Singularities for A^4-Coupling*

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Abstract. For the model of A^4-coupling it is shown that in perturbation theory a direction-dependent term is required for formulating the local field equation in limit form.

1. Introduction

In a recent paper [1] (henceforth quoted as I) a finite form of the local field equation was proposed for the model of A^4-coupling and studied in renormalized perturbation theory. The purpose of this note is to discuss certain direction-dependent singularities of the propagator which were not taken into account in I. It will be shown that these singularities lead to an additional term in the field equation of the field operator as was conjectured by K. WILSON on the basis of dimensional arguments [2]. The modified form of the field equation is[1]

$$-(\Box + m^2)\, A(x) = \lambda \lim_{\xi \to 0} j(x\,\xi) \tag{1}$$

with

$$j(x\,\xi) = \frac{:A(x+\xi)\,A(x)\,A(x-\xi): + \sigma^{\mu\nu}(\xi)\,\partial_\mu\,\partial_\nu\,A(x) - \alpha(\xi)\,A(x)}{g(\xi)}$$

$$\sigma^{\mu\nu}(\xi) = \frac{\xi^\mu\,\xi^\nu}{\xi^2}\,\sigma(\xi^2)$$

and

$$:A(x_1)\,A(x_2)\,A(x_3): = A(x_1)\,A(x_2)\,A(x_3) - \langle A(x_1)\,A(x_2)\rangle_0\,A(x_3) - \text{cycl. perm.}$$

for spacelike distances $(x_i - x_j)^2 < 0\,(i \neq j)$.

Unless otherwise noted the notation of I will be used throughout the present paper.

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[1] Throughout this paper $\lim\limits_{\xi \to 0}$ will denote the spacelike limit with $\xi^2 < 0$ and $\xi^\mu/\sqrt{-\xi^2}$ bounded.

2. Relation for Proper Self-Energy Parts

We start from the algebraic identity (5.38) of I

$$\overline{R}_S = -\frac{6i\lambda}{(2\pi)^4} F_S R_{\hat{S}} - \sum_{\gamma\in T(S,W)} \overline{R}_{S/\gamma} F^0_\gamma R^0_{\hat{\gamma}} \tag{2}$$

which holds for any proper self-energy part S. The integrand R_S of the renormalized integral associated with S is given by

$$R_S(pk\varepsilon) = (1 - t^2_p)\, \overline{R}_S(pk\varepsilon)\,.$$

The even part of a function f of the four vector p will be denoted by

$$f^+(p) = \frac{1}{2}(f(p) + f(-p))\,. \tag{3}$$

Taking the even part of (2) and applying the operator $(1 - t^2_p)$ we obtain

$$R^+_S(pk\varepsilon) + \sum_{\gamma\in T(S,W)} R^+_{S/\gamma}(pk\varepsilon)\, F^0_\gamma(k\varepsilon)\, R_{\hat{\gamma}}(k\varepsilon)$$
$$= -\frac{6i\lambda}{(2\pi)^4}\left\{E^+_S(pk\varepsilon) - E^+_S(0k\varepsilon) - \sum \frac{p^\mu p^\nu}{2} E^+_{S\mu\nu}(0k\varepsilon)\right\} \tag{4}$$
$$E_S(pk\varepsilon) = F_S(pk\varepsilon)\, R_S(pk\varepsilon)\,.$$

The contribution from the self-energy part S to the propagator is given by

$$J_S(p^2) = \lim_{\varepsilon\to+0} J_S(p\varepsilon)\,,$$
$$J_S(p\varepsilon) = \int dk\, R_S(pk\varepsilon)\,.$$

Since J_S depends on p^2 only we have

$$J_S(p^2) = \lim_{\varepsilon\to+0} J_S(p\varepsilon)\,,$$
$$J_S(p\varepsilon) = \int dk\, R^+_S(pk\varepsilon)\,. \tag{5}$$

Multiplying (4) by $\exp(-i\xi(k_1 - k_2))$ and integrating over the internal momenta we obtain

$$J^+_S(p\varepsilon) + \sum_{i=0}^{3} \sum_{\gamma\in C''_i(S)} J^+_{S/\gamma}(p\varepsilon)\, X''_{i\gamma\nu}(\xi\varepsilon)$$
$$= -\frac{6i\lambda}{(2\pi)^4}\left\{\mathscr{E}^+_S(\xi p\varepsilon) - \mathscr{E}^+_S(\xi 0\varepsilon) - \frac{p^\mu p^\nu}{2}\mathscr{E}^+_{S\mu\nu}(\xi 0\varepsilon)\right\} \tag{6}$$
$$+ Y_{S\nu}(\xi p\varepsilon) + \sum_{i=1}^{2} \sum_{\gamma\in C'_i(S)} Y'_{i\gamma\nu}(\xi p\varepsilon) + \sum_{i=1}^{2} \sum_{\gamma\in C''_i(S)} Y''_{i\gamma\nu}(\xi p\varepsilon)\,.$$

The sets C'_i, C''_i were defined in I, p. 183. In deriving (6) it was used that $J_{S/\gamma} = 0$ for $\gamma \in C''_i(S)$. The quantities X', X'', Y, Y', Y'', $\mathscr{E}^+$ are given by Eq. (5.42) of I, but with E^+_S, R^+_S, $R^+_{S/\gamma}$ as defined by (3).

Summing (6) over all proper self-energy parts S one gets

$$\gamma(\xi\varepsilon)\sum_{S\in\mathscr{K}}\frac{1}{\mathscr{S}(S)}J_S^{+}(p\varepsilon)$$
$$= -i\lambda\left\{Q(\xi p\varepsilon) - Q(\xi 0\varepsilon) - \frac{p^\mu p^\nu}{2}Q_{\mu\nu}(\xi 0\varepsilon)\right\} + i\lambda q(\xi p\varepsilon)\,. \tag{7}$$

Here γ and Q are given by the expansions (5.23–25) of I[2]. Using (5) and

$$\Pi^*(p^2) = \sum_{S\in\mathscr{K}}\frac{1}{\mathscr{S}(S)}J_S(p^2) + A(\lambda) + B(\lambda)(p^2 - m^2) \tag{8}$$

(Eq. (5.31) of I) we obtain for Q in the limit $\varepsilon \to +0$

$$Q(\xi p) = \frac{i}{\lambda}\gamma(\xi)\{\Pi^*(p^2) - A(\lambda) - B(\lambda)(p^2 - m^2)\}$$
$$+ Q(\xi 0) + \frac{p^\mu p^\nu}{2}Q_{\mu\nu}(\xi 0) + q(\xi p)\,, \tag{9}$$
$$\lim_{\xi\to 0} q(\xi p) = 0\,.$$

Here

$$\gamma(\xi) = \gamma(\xi^2)\,, \qquad Q(\xi 0) = a(\xi^2)$$

are functions of ξ^2 only. $Q_{\mu\nu}$ is of the general form

$$\frac{1}{2}Q_{\mu\nu}(\xi 0) = g_{\mu\nu}\,\varrho(\xi^2) + \frac{\xi_\mu\,\xi_\nu}{\xi^2}\sigma(\xi^2)\,. \tag{10}$$

The renormalization functions α, β will now be defined differently from I. First we introduce the function

$$T(\xi, p^2) = Q(\xi, p) - p^2\frac{\xi_0^2}{\xi^2}\sigma(\xi^2)\,, \qquad p = \left(\sqrt{p^2}, 0, 0, 0\right) \tag{11}$$

for $p^2 > 0$. α and β are then defined by

$$\alpha(\xi) = T(\xi, m^2)\,, \qquad \beta(\xi) = \lambda\left.\frac{\partial T(\xi, p^2)}{\partial p^2}\right|_{p^2=m^2}. \tag{12}$$

With these definitions we form the expression

$$Q(\xi p) - \alpha(\xi) - (p^2 - m^2)\frac{\beta(\xi)}{\lambda} = \frac{i}{\lambda}\gamma(\xi^2)\,\Pi^*(p^2) + \frac{(p\,\xi)^2}{\xi^2}\sigma(\xi^2)$$
$$+ q(\xi p) - r(\xi, m^2) - (p^2 - m^2)\left.\frac{\partial r(\xi, p^2)}{\partial p^2}\right|_{p^2=m^2} \tag{13}$$

where

$$r(\xi, p^2) = q(\xi p)\,, \qquad p = \left(\sqrt{p^2}, 0, 0, 0\right) \quad \text{for} \quad p^2 > 0\,.$$

Solving (13) for Π^* we get

$$\gamma(\xi)\,\Pi^*(p^2) = -i\lambda Q(\xi p) + i\lambda\alpha'(\xi p) + i(p^2 - m^2)\,\beta(\xi) \tag{14}$$

[2] The first term on the right hand side of Eq. (5.24) should read A_r with $A_r = 0$ for $r \neq 3$ and

$$A_3 = \frac{1}{(2\pi)^4}\sum_{(a,b)} e^{-i(p_a - p_b)\xi}\,.$$

The sum extends over all ordered pairs a, b of integers 1, 2, 3 with $a \neq b$.

with

$$\alpha'(\xi p) = \alpha(\xi) + \frac{(p\xi)^2}{\xi^2}\sigma(\xi^2) + o(\xi p)\,,$$
$$o(\xi p) = q(\xi p) - r(\xi, m^2) - (p^2 - m^2)\frac{\partial r}{\partial p^2}(\xi, m^2)\,, \qquad (15)$$
$$\lim_{\xi\to 0} o(\xi p) = 0\,.$$

3. Field Equation for the Propagator and Field Operator

Multiplying (15) by $\hat{\Delta}'_F$ we obtain

$$\gamma(\xi)\,\Pi^*(p^2)\,\hat{\Delta}'_F(p)$$
$$= -i\lambda G(\xi p) + i\lambda\alpha'(\xi p)\,\hat{\Delta}'_F(p) + i(p^2 - m^2)\,\beta(\xi)\,\hat{\Delta}'_F(p)\,. \qquad (16)$$

Inserting

$$\Pi^*\,\hat{\Delta}'_F = -i(p^2 - m^2)\,\hat{\Delta}'_F - 1$$

into (16) yields

$$(\beta(\xi) + \gamma(\xi))\,(p^2 - m^2)\,\hat{\Delta}'_F = \lambda G(\xi p) - \lambda\alpha'(\xi p)\,\hat{\Delta}'_F + i\gamma(\xi)\,. \qquad (17)$$

Dividing by $\beta + \gamma$ and taking the limit $\xi \to 0$ the equation

$$(p^2 - m^2)\,\hat{\Delta}'_F = \lim_{\xi\to 0}\frac{\lambda G(\xi p) - \lambda\alpha'(\xi p)\,\hat{\Delta}'_F + i\gamma(\xi)}{\beta(\xi) + \gamma(\xi)} \qquad (18)$$

follows. In coordinate space this becomes the field equation of the propagator

$$-(\Box + m^2)\,\langle T A(x)\,A(y)\rangle_0 = \lim_{\xi\to 0}\frac{\langle N(\xi x y)\rangle_0}{\beta(\xi) + \gamma(\xi)} \qquad (19)$$

$$N(\xi x y) = \lambda T(:A(x+\xi)\,A(x)\,A(x-\xi):\,A(y)\rangle_0 - \lambda\alpha(\xi)\,\langle T A(x)\,A(y)\rangle_0$$
$$+ \lambda\frac{\xi^\mu\,\xi^\nu}{\xi^2}\sigma(\xi)\,\partial_\mu\,\partial_\nu\langle T A(x)\,A(y)\rangle_0 + i\gamma(\xi)\,\delta(x-y)\,.$$

In order to derive the field equation of an arbitrary τ-function we multiply (17) by $\Lambda(p\,p_1\ldots p_r)$.
Using relation (5.1) and (2.15) of I we get

$$(\beta(\xi) + \gamma(\xi))\,(p^2 - m^2)\,\hat{\eta}(p p_1\ldots p_r)$$
$$= \lambda G(\xi p p_1\ldots p_r) - \lambda\alpha'(\xi p)\,\hat{\eta}(p p_1\ldots p_r) + i\,\varphi(p p_1\ldots p_r) \qquad (20)$$

where

$$r \geqq 3$$
$$\varphi = \omega(\xi p p_1\ldots p_r)\,\hat{\Delta}'_F(p_1)\ldots\hat{\Delta}'_F(p_r)\,.$$

This implies for the τ-functions

$$(\beta(\xi) + \gamma(\xi))\,(p^2 - m^2)\,\tilde{\tau}(p\,p_1\ldots p_r\!:k_1\ldots k_n\!:)$$
$$= \lambda H(\xi\,p\,p_1\ldots p_r\!:\!k_1\ldots k_n\!:) - \lambda\alpha'(\xi p)\,\tilde{\tau}(p\,p_1\ldots p_r\!:\!k_1\ldots k_n\!:) \qquad (21)$$
$$+ i\gamma(\xi)\sum_{j=1}^{r}\delta(p + p_j)\,\tilde{\tau}(p_1\ldots p_{j-1}\,p_{j+1}\ldots p_r\!:\!k_1\ldots k_n\!:)$$
$$+ \psi(\xi\,p\,p_1\ldots p_r\!:\!k_1\ldots k_n\!:)$$

where

$$H(\xi\, p\, p_1 \ldots p_r : k_1 \ldots k_n :)$$
$$= \frac{1}{(2\pi)^4} \int dl_1 \ldots dl_3\, \delta(\Sigma l_j - p)\, e^{-i(l_1-l_2)\xi} \tag{22}$$
$$\cdot\, \tilde{\tau}(: l_1 \ldots l_3 : p\, p_1 \ldots p_r : k_1 \ldots k_n :)\,.$$

For $r = 0$ the third term of the right hand side of (21) is missing, for $r = 1$ it is $i\gamma(\xi)\,\delta(p + p_1)\,\tilde{\tau}(: k_1 \ldots k_n :)$.

Setting in (21) $n = 0$, dividing by $\beta + \gamma$ and taking the limit $\xi \to 0$ we obtain the following set of field equations for the time ordered functions

$$(p^2 - m^2)\,\tilde{\tau}(p\, p_1 \ldots p_r) = \lambda \lim_{\xi \to 0} \frac{\tilde{n}(\xi\, p\, p_1 \ldots p_r)}{\beta(\xi) + \gamma(\xi)} \tag{23}$$

with

$$\tilde{n}(\xi\, p\, p_1 \ldots p_r) = H(\xi\, p\, p_1 \ldots p_r) - \left(\alpha(\xi) + \frac{(p\,\xi)^2}{\xi^2}\right) \tilde{\tau}(p\, p_1 \ldots p_r)$$
$$+\, i\gamma(\xi) \sum_{j=1}^{r} \delta(p + p_j)\, \tilde{\tau}(p_1 \ldots p_{j-1}\, p_{j+1} \ldots p_r)\,. \tag{24}$$

Field equations for the time ordered products $TA(x_1) \ldots A(x_r)$ follow by putting the momenta k_j in (22) on the mass shell and taking the Fourier transform with respect to k_j. The resulting set of equations are Eq. (5.57–58) of I with the operator product N defined by

$$N(\xi\, x\, x_1 \ldots x_r) = T(: x + \xi, x, x - \xi : x_1 \ldots x_r) - \alpha(\xi)\, T(x\, x_1 \ldots x_r)$$
$$+ \frac{\xi^\mu \xi^\nu}{\xi^2}\, \sigma(\xi)\, \frac{\partial^2}{\partial x^\mu \partial x^\nu}\, T(x\, x_1 \ldots x_r) \tag{25}$$
$$+\, i\gamma(\xi) \sum_{j=1}^{r} \delta(x - x_j)\, T(x_1 \ldots x_{j-1}\, x_{j+1} \ldots x_r)\,.$$

The field Eq. (1) of the field operator $A(x)$ is the special case of I, (5.57–58) and (25) with $r = 0$ and $g = \beta + \gamma$.

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