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On the Reduction of n-fold Tensor Product Representations of Noncompact Groups

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Abstract. The reduction of n-fold tensor products of induced unitary representations of noncompact groups into irreducible constituents is shown. Clebsch-Gordon coefficients are then calculated. The technique is applied to the n-fold tensor products of the positive mass representations of the Poincaré group.

Introduction

Noncompact groups have been used in elementary particle physics in a number of ways. The classic use of a noncompact group was Wigner's analysis of the irreducible representations of the Poincaré group; more recently the Lorentz group has been used in connection with Regge poles and current algebras. Higher symmetry groups containing, for example, the Lorentz group and some internal symmetry group are also currently under investigation.

The problem which we wish to discuss in this paper is the reduction of n-fold tensor products of representations of a noncompact group G into its irreducible constituents. Letting $[\chi]$ label an irreducible representation of G and $\{x\}$ denote a set of eigenvalues of a complete set of commuting observables chosen from G, we wish to find the coefficients which reduce the tensor product $|[\chi_1]\, x_1\rangle\, |[\chi_2]\, x_2\rangle \ldots |[\chi_n]\, x_n\rangle$ into irreducible basis elements $|[\chi]\, x; \eta\rangle$ where η denotes the degenerary parameters which label irreducible subspaces of the tensor product space with the same $[\chi]$.

The reduction of n-fold tensor products is of course not a new problem. The most obvious way to carry out the reduction is to reduce the tensor product of two representations and then apply the two fold reduction n times in a stepwise fashion. For the Poincaré group this reduction has been carried out by MACFARLANE [1] using the two fold reduction given by JACOB and WICK [2]. The big disadvantage of this stepwise reduction technique is its assymmetry. For compact groups such as $SO(3)$ it necessitates introducing $3\,J$, $6\,J$, etc. symbols which connect different stepwise reductions.

What is actually needed is a reduction process which is symmetric in as many variables as possible. Goldberg [3] and Werle [4] have begun such a program for the Poincaré group while Levy-Leblond [5] and Lurcat [6] have utilized the special properties of the Galilean group to effect a symmetric reduction.

Roffman [7] has given a completely symmetric method for carrying out n-fold tensor product decompositions if the two fold tensor product decomposition is known. What we wish to show in this paper is that for groups having unitary irreducible representations which can be written as induced representations, a symmetric n-fold tensor product reduction can be carried out with no knowledge of the two fold tensor product reduction needed. In Section I a general discussion of the technique used to obtain Clebsch-Gordan coefficients will be given, while in Section II the technique will be applied to the Poincaré group. In the process of carrying out the reduction all of the invariants which parameterize the irreducible subspaces are automatically found and since the reduction is symmetric, a highly symmetric choice of degenerary labels can be made.

I. General Technique for Obtaining Clebsch-Gordan Coefficients

In this section Mackey's theory of induced representations [8] will be used to obtain Clebsch-Gordan coefficients resulting from the decomposition of tensor product representations of a group G. An induced representation of G is built from a representation $\mathscr{H}$ of a subgroup $\mathfrak{H}$ of G and defined to be $U(g_0)\, f(g) = f(gg_0)$ [8] where g_0 is an element of G and $f(g)$ are functions which map g, an element of G, into the vector space $V(\mathscr{H})$ upon which $\mathscr{H}$ acts. The set of functions $f(g)$ is restricted to those satisfying the condition $f(hg) = \mathscr{H}(h)\, f(g)$ for all h in $\mathfrak{H}$ and g in G. This restricted set of functions forms a vector space

$$\hat{V}(\mathscr{H}) = \{f \mid f(g) \in V(\mathscr{H}),\, f(hg) = \mathscr{H}(h)\, f(g)\ \forall\, h \in \mathfrak{H} \text{ and } g \in G\}\,. \quad (1)$$

The induced representations act on the vector space $\hat{V}(\mathscr{H})$. We shall consider only unitary induced representations which implies that $\hat{V}(\mathscr{H})$ is actually a Hilbert space [8].

Now G can be decomposed into right cosets

$$G = \bigcup_c \mathfrak{H} g_c \quad (2)$$

where the elements g_c of G label right cosets. With this decomposition the functions $f(g)$ may be thought of as functions over right cosets $f(g_c)$, since the set of functions $f(g)$ are restricted to those functions satisfying the condition $f(hg) = \mathscr{H}(h)\, f(g)$.

Consider now induced representations on n Hilbert spaces $\hat{V}(\mathscr{H}_1)$, $\hat{V}(\mathscr{H}_2), \ldots, \hat{V}(\mathscr{H}_n)$ with norms

$$\|f(g_{c_i})\|^2 = \int dg_{c_i} |f(g_{c_i})|^2 < \infty\,. \quad (3)$$

The tensor product space $\hat{V}(\mathscr{H}_1) \times \hat{V}(\mathscr{H}_2) \times \cdots \times \hat{V}(\mathscr{H}_n)$ is the Hilbert space of functions $F(g_{c_1}, g_{c_2}, \ldots, g_{c_n})$ with norm

$$\|F(g_{c_1}, g_{c_2}, \ldots, g_{c_n})\|^2 = \int dg_{c_1}\, dg_{c_2}, \ldots, dg_{c_n}\, |F(g_{c_1}, \ldots, g_{c_n})|^2 < \infty \tag{4}$$

and the induced representation on this space is

$$U(g_0)\, F(g_{c_1}, g_{c_2}, \ldots, g_{c_n}) = F(g_{c_1}g_0, g_{c_2}g_0, \ldots, g_{c_n}g_0)\,. \tag{5}$$

The tensor product space is reducible, and in fact the problem is to reduce this space into a direct integral of irreducible subspaces. The tensor product functions will be decomposed in two steps. Mackey has shown that a partial decomposition of the tensor product space can be obtained through a double coset decomposition, since induced representations defined on the tensor product spaces are equivalent to induced representations defined on subspaces labelled by double cosets.

Let $\mathfrak{H}'$ and $\mathfrak{H}''$ be subgroups of G. By a double coset is meant the subset of G consisting of all elements of the form $\mathfrak{H}' g \mathfrak{H}''$ for fixed g in G. A set of elements g_D in G can be found such that the union of a set of double cosets covers G:

$$G = \bigcup_D \mathfrak{H}' g_D \mathfrak{H}''\,. \tag{6}$$

In order to define the subgroup which induces representations on tensor product functions it is convenient to consider the outer product group of G, the set of ordered pairs of elements $\{(g_1, g_2, \ldots, g_n)\} = (G_1, G_2, \ldots, G_n)$ with $g_1, g_2, \ldots, g_n$ arbitrary elements of G. Then the subgroup inducing representations on the tensor product space [Eq. (4)] is $(\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n)$ with $\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n$ subgroups of G [9]; a right coset decomposition of $(G_1, G_2, \ldots, G_n)$ with respect to $(\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n)$ is

$$(G_1, G_2, \ldots, G_n) = \bigcup_{c_1 c_2 \ldots c_n} (\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n)\, (g_{c_1}, g_{c_2}, \ldots, g_{c_n})\,. \tag{7}$$

If the outer product group is restricted to the diagonal subgroup $(G, G, \ldots, G)$, the ordered set of elements $\{(g, g, \ldots, g)\}$, then the representations induced by the subgroup $(\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n)$ can be reduced into a direct integral of induced representations which act on subspaces of the tensor product space (4) labelled by double cosets.

The double coset decomposition of $(G_1, G_2, \ldots, G_n)$ is then

$$\begin{aligned}(G_1, G_2, \ldots, G_n) = \bigcup_{D_1 D_1 \ldots D_n} & (\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n) \\ & \cdot (g_{D_1}, g_{D_2}, \ldots, g_{D_n})\, (G, G, \ldots, G)\end{aligned} \tag{8}$$

with $(g_{D_1}, \ldots, g_{D_n})$ the elements labelling the double cosets; a direct integral decomposition of $F(g_{c_1}, \ldots, g_{c_n})$ using the double cosets is

$$\|F(g_{c_1}, \ldots, g_{c_n})\|^2 = \int dD_1\, dD_2 \ldots dD_n\, \|f_{D_1 \ldots D_n}(G/\mathfrak{H}_D)\|^2 \tag{9}$$

where $f_{D_1 \ldots D_n}(G/\mathfrak{H}_D)$ are square integrable functions in the subspaces $D_1, \ldots, D_n$. $\mathfrak{H}_D$ is the subgroup which induces representations on these subspaces and is defined as [8]

$$\mathfrak{H}_D = (g_{D_1}, \ldots, g_{D_n})^{-1}(\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n)(g_{D_1}, \ldots, g_{D_n}) \cap (G, G \ldots G). \quad (10)$$

The induced representations on these subspaces are generally reducible; the final reduction into irreducible subspaces cannot be further carried out because the method used is dependent on G.

Assume however that the decomposition of $F(g_{c_1}, \ldots, g_{c_n})$ into irreducible subspaces can be carried out and that $f_{[\chi, \eta]}(g_c)$ is an element of the irreducible subspaces with norm

$$\|f_{[\chi,\eta]}(g_c)\|^2 = \int dg_c\, |f_{[\chi,\eta]}(g_c)|^2 < \infty. \quad (11)$$

Here χ denotes an irreducible representation of G while η is a set of degeneracy parameters which label mutually orthogonal subspaces carrying the same irreducible representation labels. The direct integral decomposition of $F(g_{c_1}, \ldots, g_{c_n})$ over $[\chi, \eta]$ with weight function $d[\chi, \eta]$ is

$$\|F(g_{c_1}, \ldots, g_{c_n})\|^2 = \int d[\chi, \eta]\, \|f_{[\chi,\eta]}(g_c)\|^2. \quad (12)$$

Now $f_{[\chi,\eta]}$ is an element of the Hilbert space (11) so the inner product of $f_{[\chi,\eta]}(g_c)$ and $f'_{[\chi]}$ is well defined:

$$\begin{aligned}(f'_{[\chi]}(g_c), f_{[\chi,\eta]}(g_c)) &= \int dg_c\, f'^{*}_{[\chi]}(g_c)\, f_{[\chi,\eta]}(g_c) \\ &= \int dg_c\, f'^{*}_{[\chi]}(g_c)\, [SF(g_{c_1}, \ldots, g_{c_n})]\end{aligned} \quad (13)$$

where S is the operator which carries $F(g_{c_1}, \ldots, g_{c_n})$ to the irreducible subspaces from the tensor product space $\hat{V}(\mathscr{H}_1) \times \cdots \hat{V}(\mathscr{H}_n)$.

In order to define Clebsch-Gordan coefficients we want to make a connection between the functions $f_{[\chi]}(g_c)$ and basis elements $|[\chi]\, x\rangle$, since Clebsch-Gordan coefficients are the coefficients which reduce the tensor product of basis elements $|[\chi_1]\, x_1\rangle\, |[\chi_2]\, x_2\rangle \ldots |[\chi_n]\, x_n\rangle$; x_i denotes eigenvalues of a complete set of commuting observables chosen from G. Now the non-square integrable functions over cosets

$$D^{[\chi]}_{x'x}(g_c) = \langle [\chi]\, x'|\, U(g_c)\, |[\chi]\, x\rangle \quad (14)$$

can be considered as a specific realization of the basis element $|[\chi], x\rangle$, since

$$U(g)\, D^{[\chi]}_{x'x}(g_c) = D^{[\chi]}_{x'x}(g_c g). \quad (15)$$

Hence, we choose

$$f_{[\chi]}(g_c) \equiv D^{[\chi]}_{x'x}(g_c) \quad (16)$$

and

$$F(g_{c_1}, \ldots, g_{c_n}) \equiv D^{[\chi_1]}_{x'_1 x_1}(g_{c_1}) \ldots D^{[\chi_n]}_{x'_n x_n}(g_{c_n}). \quad (17)$$

Eq. (13) can then be written as

$$\begin{aligned}\langle [\chi]\, x; \eta\, |\, [\chi_1]\, x_1; \ldots; [\chi_n]\, x_n\rangle &= N \int dg_c\, D^{[\chi]*}_{x'x}(g_c) \\ &\cdot \left[S D^{[\chi_1]}_{x'_1 x_1}(g_{c_1}) \ldots D^{[\chi_n]}_{x'_n x_n}(g_{c_n})\right]\end{aligned} \quad (18)$$

with N a normalization factor. Eq. (18) may be used to calculate the Clebsch-Gordan coefficients provided S is known and $D^{[x]}_{x'x}(g_c)$ can be calculated.

II. Clebsch-Gordan Coefficients for the Positive Mass Representation of the Poincare Group

In this section we calculate the Clebsch-Gordan coefficients for the positive mass representations of the Poincaré group P. The covering group of P will be used throughout this section; however we will not discuss its properties, since they have been discussed in detail elsewhere [10].

One can easily check that the matrix form of the covering group of P can be written as

$$P = \begin{pmatrix} \Lambda & H(a)\,\Lambda^{-1+} \\ 0 & \Lambda^{-1+} \end{pmatrix} \tag{19}$$

where Λ is an element of $SL(2, C)$ and

$$H(a) = \begin{pmatrix} a_t + a_z & a_x - i a_y \\ a_x + i a_y & a_t - a_z \end{pmatrix}. \tag{20}$$

A right coset decomposition of P with respect to the inducing subgroup

$$\mathfrak{H} = \begin{pmatrix} SU(2) & H(a)\,SU(2) \\ 0 & SU(2) \end{pmatrix} \tag{21}$$

can be written:

$$P = \bigcup_c \begin{pmatrix} SU(2) & H(a)\,SU(2) \\ 0 & SU(2) \end{pmatrix} \begin{pmatrix} \Lambda_c & 0 \\ 0 & \Lambda_c^{-1+} \end{pmatrix} \tag{22}$$

where $\Lambda = SU(2)\,\Lambda_c$ and Λ_c is any choice of right cosets of Λ with respect to $SU(2)$.

Then, square integrable functions over right cosets, denoted by $\varphi_j(\Lambda_c)$, have norm

$$\|\varphi_j(\Lambda_c)\|^2 = \sum_{j=-s}^{+s} \int d\Lambda_c\, |\varphi_j(\Lambda_c)|^2 < \infty\,. \tag{23}$$

The summation over j comes from the fact that the inducing subgroup has $2s + 1$ dimensional representations.

The induced unitary irreducible positive mass representations of P are

$$\begin{aligned} U(\Lambda_0, a_0)\,\varphi_j(\Lambda_c) &\equiv \varphi_j\left[\begin{pmatrix} \Lambda_c & 0 \\ 0 & \Lambda_c^{-1+} \end{pmatrix} \begin{pmatrix} \Lambda_0 & H(a_0)\,\Lambda_0^{-1+} \\ 0 & \Lambda_0^{-1+} \end{pmatrix}\right] \\ &= e^{i\hat{p}\cdot\Lambda_c a_0} \sum_{j'=-s}^{s} D^{[s]}_{jj'}(\widetilde{SU}(2))\,\varphi_{j'}(\Lambda_{c'}) \end{aligned} \tag{24}$$

with $\widetilde{SU}(2)$ and $\Lambda_{c'}$ defined by $\Lambda_c\Lambda_0 = \widetilde{SU}(2)\,\Lambda_{c'}$, and $\hat{p} = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

A nonnormalizable basis element of P with mass M, spin s, momentum $\boldsymbol{p}$, and spin component σ is denoted $|[Ms]\boldsymbol{p}\sigma\rangle$. The transformation properties of $|[Ms]\boldsymbol{p}\sigma\rangle$ under an arbitrary Poincaré transformation are

$$U(\Lambda_0 a_0)\,|[Ms]\boldsymbol{p}\sigma\rangle = e^{i\Lambda_0 p\cdot a_0} \sum_{\sigma'=-s}^{+s} D^{[s]}_{\sigma'\sigma}(\boldsymbol{p},\Lambda_0)\,|[Ms]\Lambda_0\boldsymbol{p},\sigma'\rangle \tag{25}$$

where the $SU(2)$ rotation $(\boldsymbol{p},\Lambda_0)$ is defined as the rotation $O^{-1}(\Lambda_0\boldsymbol{p})\cdot\Lambda_0 O(\boldsymbol{p})$. $O(\boldsymbol{p})$ is the Lorentz transformation from the rest frame of the particle with momentum $\hat{p}$ to the frame where its momentum is $p = (E, \boldsymbol{p})$. In the coset decomposition given in Eq. (22) $O(\boldsymbol{p})$ can be written as

$$O(\boldsymbol{p}) \equiv \Lambda_c(\boldsymbol{p}) \tag{26}$$

with

$$H(p) = \begin{pmatrix} E + p_z & p_x - ip_y \\ p_x + ip_y & E - p_z \end{pmatrix} = \Lambda_c H(\hat{p}) \Lambda_c^+ . \tag{27}$$

Thus, the rotation $(\boldsymbol{p},\Lambda_0)$ by our definition is

$$(\boldsymbol{p},\Lambda_0) \equiv \Lambda_c^{-1}(\Lambda_0\boldsymbol{p})\,\Lambda_0\,\Lambda_c(\boldsymbol{p}) . \tag{28}$$

The D functions can now be calculated using Eq. (14) so that

$$\begin{aligned} D^{[Ms]}_{\boldsymbol{p}'j,\boldsymbol{p}\sigma}(\Lambda_c) &\equiv \langle [Ms]\boldsymbol{p}', j|\, U(\Lambda_c)\,|[Ms]\boldsymbol{p}\sigma\rangle \\ &= E'\,\delta^3(\boldsymbol{p}' - \Lambda_c\boldsymbol{p})\,\delta_{j\sigma} . \end{aligned} \tag{29}$$

Further it is not hard to check [10] that $D^{[Ms]}_{\hat{p}j,\boldsymbol{p}\sigma}(\Lambda_c)$ can be thought of as a non square integrable function with the same transformation properties as the function $\varphi_j(\Lambda_c)$ in Eq. (24). Thus, $D^{[Ms]}_{\hat{p}j,\boldsymbol{p}\sigma}(\Lambda_c)$ can be thought of as a concrete realization of $|[M,s]\boldsymbol{p}\sigma\rangle$ and simultaneously as a function $\varphi_j(\Lambda_c)$ on which the Mackey induced representation theory can be applied.

Now we consider the tensor product space of positive mass representations of P which consists of square integrable functions

$$\varphi_{\alpha_1\ldots\alpha_n}(\Lambda_{c_1},\ldots,\Lambda_{c_n})$$

with norm

$$\begin{aligned} \|\varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(\Lambda_{c_1},\ldots,\Lambda_{c_n})\|^2 = \sum_{\alpha_1=-s_1}^{s_1} \cdots \sum_{\alpha_n=-s_n}^{s_n} \\ \int d\Lambda_{c_1}\ldots\int d\Lambda_{c_n}\,|\varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(\Lambda_{c_1},\ldots,\Lambda_{c_n})|^2 < \infty . \end{aligned} \tag{30}$$

In order to decompose the tensor product space it is necessary to consider the double coset decomposition of the outer product group $(P_1, P_2, \ldots, P_n)$ with respect to the subgroup $(\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n)$ and $(P, P, \ldots, P)$. Now $\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n$ are all the same subgroup, since we are considering only tensor products of positive mass representations.

The double coset decomposition of the outer product group is

$$(P_1, P_2, \ldots, P_n) = \bigcup_{D_1 \ldots D_n} (\mathfrak{H}_1, \ldots, \mathfrak{H}_n)\,(\Lambda_{D_1}, \ldots, \Lambda_{D_n})\,(P, \ldots, P)$$

$$\left[\begin{pmatrix} \Lambda_1 & H(a_1)\,\Lambda_1^{-1+} \\ 0 & \Lambda_1^{-1+} \end{pmatrix}, \ldots, \begin{pmatrix} \Lambda_n & H(a_n)\,\Lambda_n^{-1+} \\ 0 & \Lambda_n^{-1+} \end{pmatrix}\right]$$

$$= \bigcup_{D_1 \ldots D_n} \left[\begin{pmatrix} SU(2)_1 & H(a_1)\,SU(2)_1 \\ 0 & SU(2)_1 \end{pmatrix}, \ldots, \begin{pmatrix} SU(2)_n & H(a_n)\,SU(2)_n \\ 0 & SU(2)_n \end{pmatrix}\right]$$

$$\cdot \left[\begin{pmatrix} \Lambda_{D_1} & 0 \\ 0 & \Lambda_{D_1}^{-1+} \end{pmatrix} \begin{pmatrix} \Lambda_{D_2} & 0 \\ 0 & \Lambda_{D_2}^{-1+} \end{pmatrix}, \ldots, \begin{pmatrix} \Lambda_{D_n} & 0 \\ 0 & \Lambda_{D_n}^{-1+} \end{pmatrix}\right] \tag{31}$$

$$\cdot \left[\begin{pmatrix} \Lambda & H(a)\,\Lambda^{-1+} \\ 0 & \Lambda^{-1+} \end{pmatrix}, \ldots, \begin{pmatrix} \Lambda & H(a)\,\Lambda^{-1+} \\ 0 & \Lambda^{-1+} \end{pmatrix}\right]$$

A convenient choice of double cosets [11] is

$$\left[\begin{pmatrix} \Lambda_{D_1} & 0 \\ 0 & \Lambda_{D_1}^{-1+} \end{pmatrix}, \ldots, \begin{pmatrix} \Lambda_{D_{n-1}} & 0 \\ 0 & \Lambda_{D_{n-1}}^{-1+} \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right]$$

where

$$\begin{aligned} \Lambda_{D_i} &= \begin{pmatrix} D_i & 0 \\ Q_i & 1/D_i \end{pmatrix} \quad i = 1, 2, \ldots, n-3 \\ \Lambda_{D_{n-2}} &= \begin{pmatrix} D_{n-2} & 0 \\ |Q_{n-2}| & 1/D_{n-2} \end{pmatrix} \\ \Lambda_{D_{n-1}} &= \begin{pmatrix} D_{n-1} & 0 \\ 0 & 1/D_{n-1} \end{pmatrix} \\ \Lambda_{D_n} &= I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{32}$$

The parameters $D_1, D_2, \ldots, D_{n-1}$ are real positive numbers and the parameters $Q_1, \ldots, Q_{n-2}$ are complex numbers. The choice of double cosets is fixed in the sense that for each element in the outer product group a unique element of the double coset must be defined. A unique association will exist if the inducing subgroup elements are divided out of the diagonal subgroup. Eq. (10) defines the inducing subgroup which induces representations in the subspaces of the tensor product space labelled by the double cosets. Using this definition and our choice of double cosets $\mathfrak{H}_D$ can be calculated:

$$\begin{aligned} \mathfrak{H}_D &= (\Lambda_{D_1}, \ldots, \Lambda_{D_n})^{-1}(\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n)\,(\Lambda_{D_1}, \ldots, \Lambda_{D_n}) \cap (P, \ldots, P) \\ &= \left[\begin{pmatrix} I & H(a) \\ 0 & I \end{pmatrix}, \ldots, \begin{pmatrix} I & H(a) \\ 0 & I \end{pmatrix}\right]. \end{aligned} \tag{33}$$

Eq. (33) is valid for $n \geqq 3$. For the case, $n = 2$,

$$\mathfrak{H}_D = \left[\begin{pmatrix} U(1) & H(a)\,U(1) \\ 0 & U(1) \end{pmatrix}, \begin{pmatrix} U(1) & H(a)\,U(1) \\ 0 & U(1) \end{pmatrix}\right], \tag{34}$$

where $U(1)$ is the one dimensional unitary group [12]. Dividing the inducing subgroup $\mathfrak{H}_D$ out of the diagonal subgroup, the double coset decomposition of $(P_1, P_2, \ldots, P_n)$ is:

$$\left[\begin{pmatrix} \Lambda_1 & H(a_1)\Lambda_1^{-1+} \\ 0 & \Lambda_1^{-1+} \end{pmatrix}, \ldots, \begin{pmatrix} \Lambda_n & H(a_n)\Lambda_n^{-1+} \\ 0 & \Lambda_n^{-1+} \end{pmatrix}\right]$$
$$= \bigcup_{D_1\ldots D_n} \left[\begin{pmatrix} SU(2)_1 & H(a_1)SU(2)_1 \\ 0 & SU(2)_1 \end{pmatrix}, \ldots, \begin{pmatrix} SU(2)_n & H(a_n)SU(2)_n \\ 0 & SU(2)_n \end{pmatrix}\right] \tag{35}$$
$$\cdot \left[\begin{pmatrix} \Lambda_{D_1} & 0 \\ 0 & \Lambda_{D_1}^{-1+} \end{pmatrix}, \ldots, \begin{pmatrix} \Lambda_{D_n} & 0 \\ 0 & \Lambda_{D_n}^{-1+} \end{pmatrix}\right] \left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1+} \end{pmatrix}, \ldots, \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1+} \end{pmatrix}\right]$$

A right coset decomposition of P with respect to $\mathfrak{H}_D$ is:

$$\begin{pmatrix} \Lambda & H(a)\Lambda^{-1+} \\ 0 & \Lambda^{-1+} \end{pmatrix} = \bigcup_{\Lambda} \begin{pmatrix} I & H(a) \\ 0 & I \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1+} \end{pmatrix} \tag{36}$$

so that square integrable functions in the subspaces of the tensor product space labelled by $[\alpha_i D_i Q_i] \equiv \{\alpha_1, \ldots, \alpha_n, D_1, \ldots, D_n, Q_1, \ldots, Q_n\}$ are defined as

$$F_{[\alpha_i D_i Q_i]}(\Lambda) = \varphi_{\alpha_1 \ldots \alpha}(\Lambda_{c_1}, \Lambda_{c_2}, \ldots, \Lambda_{c_n}) \,. \tag{37}$$

From Eq. (37) it follows that under an arbitrary element of P the functions $F_{[\alpha_i D_i Q_i]}(\Lambda)$ transform as

$$U(\Lambda_0 a_0)\, F_{[\alpha_i D_i Q_i]}(\Lambda) = e^{i p_D \cdot \Lambda a_0} F_{[\alpha_i D_i Q_i]}(\Lambda \Lambda_0) \tag{38}$$

with $p_D \equiv \Lambda_{D_1}^{-1} \hat{p}_1 + \Lambda_{D_2}^{-1} \hat{p}_2 + \cdots + \Lambda_{D_n}^{-1} \hat{p}_n$, and thus the subscripts $[\alpha_i D_i Q_i]$ label the degenerate subspaces since they remain invariant under the transformation.

Hence, the first step of the tensor product decomposition is completed. The direct integral decomposition of $\varphi_{\alpha_1 \ldots \alpha_n}(\Lambda_{c_1}, \ldots, \Lambda_{c_n})$ into $F_{[\alpha_i D_i Q_i]}(\Lambda)$ is:

$$\|\varphi_{\alpha_1 \ldots \alpha_n}(\Lambda_{c_1}, \ldots, \Lambda_{c_n})\|^2 = \sum_{\alpha_1 \ldots \alpha_n} \int dQ_1\, dD_1 \ldots \tag{39}$$
$$\int dQ_{n-2}\, dD_{n-1}\, \|F_{[\alpha_i D_i Q_i]}(\Lambda)\|^2 \,.$$

Here the measure $dQ_i\, dD_i$ has not been calculated because it is not needed [8].

In order to complete the direct integral decomposition of the functions $\varphi_{\alpha_1 \ldots \alpha_n}(\Lambda_{c_1} \ldots \Lambda_{c_n})$ into irreducible subspaces, the functions $F_{[\alpha_i D_i Q_i]}(\Lambda)$ must be expanded in a series of irreducible functions of P. Since the irreducible functions of P are functions only over Λ_c and $\Lambda = SU(2)\,\Lambda_c$, it is clear that $F_{[\alpha_i D_i Q_i]}(\Lambda)$ must be expanded in the orthonormal Wigner functions $D^{[s]}_{j'j}(SU(2))$.

However a difficulty appears since p_D in Eq. (38) is not of the form of $\hat{p}$ in Eq. (24). If we wish to induce with the representation $e^{ip_D \cdot a}$, then we must find a set of transformations, denoted by $SU(2)_D$, such that

$SU(2)_D\, H(p_D)\, SU(2)_D^+ = H(p_D)$. Now $H(p_D) = \Lambda_c(p_D)\, H(\hat{p})\, \Lambda_c^+(p_D)$

$$\left(\text{where} \quad \hat{p} = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad M^2 = p_D^2\right) \quad \text{and} \quad H(\hat{p}) = SU(2)\, H(\hat{p})\, SU(2)^+$$

so that

$$\Lambda_c^{-1}(p_D)\, SU(2)_D\, \Lambda_c(p_D) = SU(2)\,. \tag{40}$$

Hence the expansion in terms of the Wigner functions is actually written as $D^s_{j'j}(\Lambda_c^{-1}(p_D)\, SU(2)_D\, \Lambda_c(p_D))$ where $\Lambda_c^{-1}(p_D)\, SU(2)_D\, \Lambda_c(p_D)$ is an $SU(2)$ element as seen from Eq. (40).

A right coset decomposition of P with respect to the inducing subgroup $\mathfrak{H} = \begin{pmatrix} SU(2)_D & H(a)\, SU(2)_D^{-1+} \\ 0 & SU(2)_D^{-1+} \end{pmatrix}$ can be written as

$$P = \bigcup_c \begin{pmatrix} SU(2)_D & H(a)\, SU(2)_D^{-1+} \\ 0 & SU(2)_D^{-1+} \end{pmatrix} \begin{pmatrix} \Lambda_c & 0 \\ 0 & \Lambda_c^{-1+} \end{pmatrix} \tag{41}$$

with $\Lambda = SU(2)_D\, \Lambda_c$ so that we can proceed with the expansion of $F_{[\alpha_i D_i Q_i]}(\Lambda) \equiv F_{[\alpha_i D_i Q_i]}(SU(2)_D\, \Lambda_c)$. Let

$$\begin{aligned} &F_{[\alpha_i D_i Q_i]}(SU(2)_D\, \Lambda_c) \\ &= \sum_{s=0}^{\infty} \sum_{j=-s}^{s} \sum_{K=-s}^{s} \sqrt{\frac{2s+1}{4\pi}}\, D^{[s]}_{K,j}(\Lambda_c^{-1}(p_D)\, SU(2)_D\, \Lambda_c(p_D)) \\ &\cdot \Phi_{j\,[\alpha_i D_i Q_i s K]}(\Lambda_c) \end{aligned} \tag{42}$$

with

$$\begin{aligned} \Phi_{j\,[\alpha_i D_i Q_i s K]}(\Lambda_c) = \sqrt{\frac{2s+1}{4\pi}} \int d\,SU(2)_D\, D^{[s]*}_{K,j}(\Lambda_c^{-1}(p_D)\, SU(2)_D\, \Lambda_c(p_D)) \\ \cdot F_{[\alpha_i D_i Q_i]}(SU(2)_D\, \Lambda_c)\,. \end{aligned} \tag{43}$$

In order to prove that the function $\Phi_{j\,[\alpha_i D_i Q_i s K]}(\Lambda_c)$ is an element of an irreducible function space of P, we must check its transformation properties under an arbitrary Poincaré transformation:

$$\begin{aligned} U(\Lambda_0, a_0)\, \Phi_{j\,[\alpha_i D_i Q_i s K]}(\Lambda_c) &= \sqrt{\frac{2s+1}{4\pi}} \int d\,SU(2)_D \\ &\cdot D^{[s]*}_{K,j}(\Lambda_c^{-1}(p_D)\, SU(2)_D\, \Lambda_c(p_D))\, e^{ip_D\cdot\Lambda a_0}\, F_{[\alpha_i D_i Q_i]}(\Lambda\Lambda_0) \\ &= e^{ip_D\cdot\Lambda_c a_0} \sum_{j'=-s}^{+s} D^{[s]}_{jj'}(\Lambda_c^{-1}(p_D)\, S\widetilde{U}(2)\, \Lambda_c(p_D))\, \Phi_{j'\,[\alpha_i D_i Q_i s K]}(\Lambda_{c'})\,. \end{aligned} \tag{44}$$

We have used the fact that $D^{[s]*}_{K,j}(R) = \sum_{j'} D^{[s]*}_{Kj'}(R')\, D^{[s]}_{jj'}(R'')$ where $R = R'R''^{-1}$ and have defined $\Lambda_{c'}$ and $S\widetilde{U}(2)$ by $\Lambda_c\Lambda_o = S\widetilde{U}(2)\, \Lambda_{c'}$. Now Eq. (44) agrees with Eq. (24) which means $\Phi_{j\,[\alpha_i D_i Q_i s K]}(\Lambda_c)$ is an element of an irreducible function space of P; the subscripts $[\alpha_i D_i Q_i s k]$ label the degenerate subspaces since they have not been transformed.

Now that the complete direct integral (and sum) decomposition of the tensor product space into irreducible subspaces is known, we can use Eq. (18) and (29) to calculate the Clebsch-Gordan coefficients. We choose $\Phi_i'(\Lambda_c) = D^{[Ms]}_{\hat{p}i,p\sigma}(\Lambda_c) = \delta^3(\boldsymbol{p}_D - \Lambda_c \boldsymbol{p})\, D^s_{i\sigma}(\boldsymbol{p}, \Lambda_c)$ and

$$F_{[\alpha_i D_i Q_i]}(\Lambda) = D^{[M_1 s_1]}_{\hat{p}_1 \alpha_1 \boldsymbol{p}_1 \sigma_1}(\Lambda_{D_1}\Lambda)\, D^{[M_2 s_2]}_{\hat{p}_2 \alpha_2 \boldsymbol{p}_2 \sigma_2}(\Lambda_{D_2}\Lambda) \ldots D^{[M_n s_n]}_{\hat{p}_n \alpha_n \boldsymbol{p}_n \sigma_n}(\Lambda_{D_n}\Lambda)$$
$$= \prod_{i=1}^{n} \delta^3(\hat{p}_i - \Lambda_{D_i}\Lambda \boldsymbol{p}_i)\, D^{[s_i]}_{\alpha_i \sigma_i}(\boldsymbol{p}_i, \Lambda_{D_i}\Lambda)$$

so that the Clebsch-Gordan coefficients are:

$$\langle [Ms]\,\boldsymbol{p}\sigma;\, \alpha_i D_i Q_i K \,|\, [M_1 s_1]\,\boldsymbol{p}_1\sigma_1; \ldots; [M_n s_n]\,\boldsymbol{p}_n \sigma_n\rangle$$
$$= N \sqrt{\frac{2s+1}{4\pi}} \sum_{i=-s}^{+s} \int d\Lambda_c\, d\, SU(2)_D\, D^{s\,*}_{i\sigma}(\boldsymbol{p}, \Lambda_c) \tag{45}$$
$$\cdot\, \delta^3(\boldsymbol{p}_D - \Lambda \boldsymbol{p})\, D^{[s]\,*}_{K,i}(\Lambda_c^{-1}(p_D)\, SU(2)_D\, \Lambda_c(p_D))$$
$$\cdot \prod_{i=1}^{n} \delta^3(\hat{p}_i - \Lambda_{D_i}\, SU(2)_D\, \Lambda_c \boldsymbol{p}_i)\, D^{[s_i]}_{\alpha_i \sigma_i}(\boldsymbol{p}_i, \Lambda_{D_i}\, SU(2)_D\, \Lambda_c)\,.$$

The delta function normalization factors have been absorbed in the normalization constant N.

The Clebsch-Gordan coefficients tell us how n single particle basis states $|[M_i s_i]\,\boldsymbol{p}_i \sigma_i\rangle$ form a single particle basis state $|[Ms]\,\boldsymbol{p}\sigma;\, \eta\rangle$ where η is a set of $4n - 6$ degeneracy parameters;

$$|[M_i s_i]\,\boldsymbol{p}_i\sigma_i\rangle \ldots |[M_n s_n]\,\boldsymbol{p}_n\sigma_n\rangle$$
$$= \sum_{\substack{\alpha_1 \alpha_2 \ldots \alpha_n \\ s, K, \sigma}} \int dD_1 \int dQ_1 \ldots \int dQ_{n-2} \int dD_{n-1} \int \frac{d^3 p}{E} \tag{46}$$
$$\cdot\, \langle [Ms]\,\boldsymbol{p}\sigma;\, \alpha_i D_i Q_i K \,|\, [M_1 s_1]\,\boldsymbol{p}_1\sigma_1; \ldots; [M_n s_n]\,\boldsymbol{p}_n\sigma_n\rangle$$
$$\cdot\, |[Ms]\,\boldsymbol{p}\sigma;\, \alpha_i D_i Q_i K\rangle\,.$$

The $3n + 3$ integrations in Eq. (46) are performed using the $n + 1$ three dimensional delta functions. An investigation of the delta functions leads to the following conclusions:

(1) $\delta^3(\boldsymbol{p}_D - \Lambda_c \boldsymbol{p})$ is equivalent to $\delta^3\left(\sum_i \boldsymbol{p}_i - \boldsymbol{p}\right)$, i.e. conservation of momentum.

(2) $p^2 = M^2$ with p defined as the total four momentum of the n particle system.

(3) The $3n - 6$ continuous degeneracy parameters $D_i, i = 1, 2 \ldots n-1$, $Q_j, j = 1, 2, \ldots n - 3$, and $|Q_{n-2}|$ are fixed by scalars constructed from the momenta p_i, but the set of scalars which fix the parameters D_i, Q_j is not unique.

Proof (1). The n delta functions imply

$$\Lambda_c \boldsymbol{p}_i = SU(2)_D^{-1} \Lambda_{D_i}^{-1} \boldsymbol{p}_i$$

so that

$$\sum_i \Lambda_c \boldsymbol{p}_i = SU(2)_D^{-1} \sum_i \Lambda_{D_i}^{-1} \boldsymbol{p}_i = SU(2)_D^{-1} \boldsymbol{p}_D = \boldsymbol{p}_D \,.$$

Therefore,

$$\delta^3(\boldsymbol{p}_D - \Lambda_c \boldsymbol{p}) = \delta^3\left(\Lambda_c\left(\sum_i \boldsymbol{p}_i - \boldsymbol{p}\right)\right)$$

which implies conservation of momenta.

Proof (2). $M^2 = |H(p)| = \left|H\left(\sum_i \Lambda_{D_i}^{-1} \hat{p}_i\right)\right|$

$$= \begin{vmatrix} \dfrac{M_1}{D_1^2} + \dfrac{M_2}{D_2^2} + \cdots + \dfrac{M_{n-1}}{D_{n-1}^2} + M_n\,, & -\dfrac{Q_1^*}{D_1} M_1 - \dfrac{Q_2^*}{D_2} M_2 - \cdots - \dfrac{Q_{n-2}^*}{D_{n-2}} M_{n-2} \\ -\dfrac{Q_1}{D_1} M_1 - \cdots - \dfrac{Q_{n-2}}{D_{n-2}} M_{n-2}\,, & M_1 D_1^2 + M_2 D_2^2 + \cdots + M_{n-1} D_{n-1}^2 M_n \end{vmatrix}$$

$$= \left[\frac{M_1}{D_1^2} + \cdots M_n\right]\left[M_1 D_1^2 + \cdots + M_n\right] - \left[-\frac{Q_1^*}{D_1} M_1 - \cdots - \frac{Q_{n-2}^*}{D_{n-2}} M_{n-2}\right] \cdot \left[-\frac{Q_1}{D_1} M_1 - \cdots - \frac{Q_{n-2}}{D_{n-2}} M_{n-2}\right]. \tag{47}$$

Also

$$p^2 = (p_1 + p_2 + \cdots + p_n)^2 = \sum_{i=1}^{n} M_i^2 + 2 \sum_{\substack{i,j \\ i<j}} p_i \cdot p_j \,. \tag{48}$$

The scalars $p_i \cdot p_j$ can be calculated using the delta functions.

$$\begin{aligned} 2p_i \cdot p_j &= |H(p_i + p_j)| - |H(p_i)| \cdot - |H(p_j)| \\ &= |\Lambda_{D_i}^{-1} H(\hat{p}_i) \Lambda_{D_i}^{-1+} + \Lambda_{D_j}^{-1} H(\hat{p}_j) \Lambda_{D_j}^{-1+}| - M_i^2 - M_j^2 \\ &= M_i M_j \left[\frac{D_i^2}{D_j^2} + \frac{D_j^2}{D_i^2} + \frac{|Q_i|^2}{D_j^2} + \frac{|Q_j|^2}{D_i^2} - \frac{(Q_i^* Q_j + Q_i Q_j^*)}{D_i D_j}\right]. \end{aligned} \tag{49}$$

With the substitution of Eq. (49) into Eq. (48) we see that Eq. (48) and Eq. (47) are equal.

Proof (3). Eq. (49) gives us $\dfrac{n(n-1)}{2}$ $p_i \cdot p_j$ scalars in terms of the degeneracy parameters. A set of $n-3$ pseudoscalars denoted by $\varepsilon_{\alpha\beta\gamma\delta}\, p_n^\alpha\, p_{n-1}^\beta\, p_{n-2}^\gamma\, p_j^\delta$ can also be written in terms of the degeneracy parameters as

$$\varepsilon_{\alpha\beta\gamma\delta}\, p_n^\alpha\, p_{n-1}^\beta\, p_{n-2}^\gamma\, p_i^\delta = \frac{M_n M_{n-1} M_{n-2} M_i}{4}\left[\frac{1}{D_{n-1}^2} - D_{n-1}^2\right]\frac{|Q_{n-2}|}{D_{n-1}}\,\frac{\operatorname{Im} Q_i}{D_i}$$

$$i = 1, 2, \ldots, n-3\,. \tag{50}$$

Now the question arises whether the $3n - 6$ parameters D_i, Q_j can be solved for in terms of the scalars and pseudoscalars given in Eq. (49) and (50). From the set of $\frac{n(n-1)}{2}$ $p_i \cdot p_j$ scalar invariants (49) consider the $2n - 3$ scalars $p_i \cdot p_n$, $i = 1, 2, \ldots, n - 1$, and $p_j \cdot p_{n-1}$, $j = 1, 2, \ldots, n - 2$.

$$2p_{n-1} \cdot p_n = M_{n-1} M_n \left[D_{n-1}^2 + \frac{1}{D_{n-1}^2}\right], \tag{51}$$

$$2p_i \cdot p_n = M_i M_n \left[D_i^2 + \frac{1}{D_i^2} + |Q_i|^2\right], \tag{52}$$

$$2p_i \cdot p_{n-1} = M_i M_{n-1} \left[\frac{D_i^2}{D_{n-1}^2} + \frac{D_{n-1}^2}{D_i^2} + \frac{|Q_i|^2}{D_{n-1}^2}\right]. \tag{53}$$

Eq. (51) to (53) can be used to calculate $D_1, D_2, \ldots, D_{n-1}$, $|Q_1|^2, \ldots, |Q_{n-3}|^2$, $|Q_{n-2}|$, in terms of the $2n - 3$ scalars $p_i \cdot p_n$ and $p_j \cdot p_{n-1}$. The imaginary part of Q_i, $i = 1, 2, \ldots, n - 3$ can be found using the $n - 3$ pseudoscalars (50), and the real part of Q_i, $i = 1, 2, \ldots, n - 3$ can be found using the $n - 3$ scalars $p_i \cdot p_{n-2}$, $i = 1, 2, \ldots, n - 3$ where

$$2p_i \cdot p_{n-2} = M_i M_{n-2} \left[\frac{D_i^2}{D_{n-2}^2} + \frac{D_{n-2}^2}{D_i^2} + \frac{|Q_i|^2}{D_{n-2}^2} + \frac{|Q_{n-2}|^2}{D_i^2} - \frac{2|Q_{n-2}| \operatorname{Re} Q_i}{D_i D_{n-2}}\right]. \tag{54}$$

Thus, we have uniquely determined the $3n - 6$ parameters $D_1, D_2, \ldots, D_{n-1}$, $Q_i, Q_2, \ldots, Q_{n-3}$, and $|Q_{n-2}|$ in terms of $4n - 9$ invariants. Obviously, there exist other sets of invariants which could just as well be used to specify the $3n - 6$ parameters. (The extra $n - 3$ invariants, Eq. (50) serve to fix the sign of $\operatorname{Im} Q_i$.)

Then performing the integrations in Eq. (46) gives

$$|[M_1 s_1]\, p_1 \sigma_1\rangle \ldots |[M_n s_n]\, p_n \sigma_n\rangle = N \sum_{\substack{\alpha_1 \ldots \alpha_n \\ K, s, \sigma}} \sqrt{\frac{2s+1}{4\pi}}$$

$$D_{K\sigma}^{[s]*}(\Lambda_c^{-1}(p_D)\, SU(2)_D\, \Lambda_c\, \Lambda_c(p)) \prod_{i=1}^{n} D_{\alpha_i \sigma_i}^{[s_i]}(p_i, \Lambda_{D_i}\, SU(2)_D\, \Lambda_c) \tag{55}$$

$$\cdot\, |[M s]\, p\sigma;\, \alpha_i K \mu\rangle$$

with μ denoting the set of continuous scalar invariants.

It is not difficult to show that the rotation $\Lambda_c^{-1}(p_D)\, SU(2)_D\, \Lambda_c\, \Lambda_c(p)$ in the center of mass of the n-particle system rotates all of the momenta through the same angles, so that if one visualizes the momenta as being fixed in a rigid body (i.e. fixed with respect to each other), then the rotation rotates the rigid body.

In order to prove this statement we define a momentum vector p_i' to be

$$p_i' \equiv \Lambda_c^{-1}(p_D)\, p_{D_i} \tag{56}$$

with p_{D_i} defined as $p_{D_i} \equiv \Lambda_{D_i}^{-1}\, p_i$. Recall that $p_D = \sum_{i=1}^{n} \Lambda_{D_i}^{-1}\, p_i$ so that

p_{D_i} is just one of the n momenta which combine to give p_D. $\Lambda_c^{-1}(p_D)$ is the unique transformation which carries p_D to $p = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix}$ so that $\Lambda_c^{-1}(p_D)$ carries p_{D_i} to a well defined momentum vector in the center of mass system. Now the delta function $\delta^3(p_i - \Lambda_{D_i} SU(2)_D \Lambda_c p_i)$ implies $p_i = \Lambda_{D_i} SU(2)_D \Lambda_c p_i$, and the delta function $\delta^3(p_D - \Lambda_c p)$ implies $p_D = \Lambda_c p$ or $\Lambda_c = \Lambda_c(p_D) \Lambda_c^{-1}(p)$, since $p = \Lambda_c^{-1}(p_D) p_D = \Lambda_c^{-1}(p) p$; therefore

$$\begin{aligned} p_i' &\equiv \Lambda_c^{-1}(p_D) p_{D_i} \\ &= \Lambda_c^{-1}(p_D) \Lambda_{D_i}^{-1} p_i \\ &= \Lambda_c^{-1}(p_D) SU(2)_D \Lambda_c p_i \\ &= \Lambda_c^{-1}(p_D) SU(2)_D \Lambda_c \Lambda_c(p) p_{i_{cm}} \end{aligned} \tag{57}$$

where $p_{i_{cm}} = \Lambda_c^{-1}(p) p_i$ is a vector in the center of mass system. Thus, $\Lambda_c^{-1}(p_D) SU(2)_D \Lambda_c \Lambda_c(p)$ is the rotation which carries the vector $p_{i_{cm}}$ into the vector p_i', as was first discussed by WERLE [4].

Finally the remaining n rotations $(p_i, \Lambda_{D_i} SU(2)_D \Lambda_c)$ are defined by Eq. (28) and are uniquely specified by the n momenta p_i.

Conclusion

It has been shown that, for induced representations, Clebsch-Gordan coefficients can be obtained if it is possible to decompose an n-fold tensor product space and if it is possible to calculate the D-functions $D_{x_0 x}^{[\chi]}(g_c) = \langle [\chi]\, x_o |\, U(g_c)\, | [\chi]\, x \rangle$.

The decomposition of the n-fold tensor product representation consists of two steps, the decomposition into invariant subspaces labelled by double cosets and then the further decomposition of these subspaces into irreducible subspaces. This last decomposition depends on the subgroup inducing representations on the subspaces labelled by double cosets. This subgroup must be at least the identity subgroup which would then generate the regular representation. Thus, we have shown quite generally that if the decomposition of the regular representation of the non-compact group is known, then it is possible to decompose an n-fold tensor product representation into irreducible representations.

The method applied to the Poincaré group leads naturally to a set of equations relating Poincaré scalars to invariants parameters labelling the irreducible subspaces of the n-fold tensor product space. However, the number of equations relating the Poincaré scalars to the invariant parameters is much larger than the number of invariant parameters. Thus it is possible to solve for different sets of Poincaré scalars which uniquely determine the invariant parameters. If two of the particles' momenta are restricted to a z-axis, then the number of invariants necessary to completely specify the reduction is $3n - 10$ in agreement with ROHRLICH [13].

In a subsequent publication we will treat the problem of a generalized partial wave analysis for "2 in, n out" reactions.

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