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Remarks on the Quantum Field Theory in Lattice Space. I

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Abstract. We calculate the Gelfand functionals $E(f, g; a)$ for quantized fields ϕ in lattice space, a being the lattice constant. In the limit $a \to 0$ the functionals take on two different forms depending upon the "potential" $F[\phi]$ of the lattice Hamiltonian (coupling between different lattice sites not included). If $F[\phi]$ is of a short-range type (see text for definition) the limit functional is Gaussian. The corresponding representation of CCR is reducible and its realization apparently non-unique unless $F[\phi]$ is quadratic. The most natural realization is to represent the field as a linear combination of Fock fields whose masses are given by the excitation energies of the lattice Hamiltonian. If $F[\phi]$ is of a long-range type, the limit functional takes the more general form once studied by ARAKI.

I. Introduction and Summary

As an object of quantum mechanics the field is distinguished from any (finite) particle systems by the infinity of its degrees of freedom. It is sometimes asserted that the quantized local field should be dealt with as a limit of some *approximate* field [1] (finite box, finite cut-off or averaged bilocal interaction [2], [3]).

In the present series of papers we choose to consider the limit of a quantized field in a lattice space, by which we mean a set of canonical variables $\{\pi(\mathbf{s}), \phi(\mathbf{s})\}$ defined for each site $\mathbf{s}$ of a discrete lattice (simple cubic, lattice constant a, total volume $V < \infty$). We assume the commutation relations, (see [3]):

$$[\pi(\mathbf{r}), \phi(\mathbf{s})] = -ia^{-3}\delta_{r,s}, \quad \text{etc.} \tag{1.1}$$

Deferring the discussion of a coupling between different lattice sites to the subsequent paper, the first paper deals with the case of no coupling, the Hamiltonian of the system being of the form,

$$\mathscr{H}_0 = a^3 \sum_s \left\{\frac{1}{2}\pi(\mathbf{s})^2 + F[\phi(\mathbf{s})]\right\}. \tag{1.2}$$

We study the limit representation at $a \to 0$ of the canonical commutation relations in the following way: (1) We calculate the ground-state expectation functional $E(f, g; a)$ for the lattice field. (2) We let a approach zero to see what representation results for the limit functional[1]. Basic to this approach is of course the Gelfand reconstruction theorem (see for instance [4]).

If the potential $F[\phi]$ is $2^{-1}(\mu_0^2\phi^2 + \lambda\phi^4)$, or more generally if it has no long-range tail as defined in the text, then it turns out that the limit functional has the same Gaussian form as Klauder [5] has obtained in his treatment of the continuum theory under the restrictive assumption of "rotational invariance" in test function space. This is discussed in section II. The explicit construction of the representation of the canonical commutation relations corresponding to the limit functional is treated in section III. One finds the phenonemon emphasized by Klauder that the representation becomes reducible unless $F(\phi)$ is quadratic. The field can be expressed then in terms of several Fock fields but this realization is not unique. The most natural construction is to represent the field as a linear combination of Fock fields whose masses are given by the energy levels of a single lattice site.

If we allow $F[\phi]$ to have the long-range tail, then the limit functional can deviate from the Gaussian taking the more general form once studied by Araki [6] in the continuum theory (Sec. IV).

II. Short-Range Potential

In view of the commutation relations (1.1) it is convenient to use new variables $\{p_s, q_s\}$ defined by

$$\left.\begin{aligned}\pi(\mathbf{s}) &= \eta^{+1/2} a^{-3/2} p_s\,,\\ \phi(\mathbf{s}) &= \eta^{-1/2} a^{-3/2} q_s\end{aligned}\right\} \tag{2.1}$$

with a c-number η fixed later. They obey

$$[p_r, q_s] = -i\,\delta_{r,s}\,, \qquad \text{etc.} \tag{2.2}$$

Taking real valued, bounded, square-integrable functions, $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, we define smeared fields by

$$\left.\begin{aligned}\phi_a(f) &= \eta^{+1/2} a^3 \sum_s f(\mathbf{s})\,\phi(\mathbf{s}) = a^{3/2} \sum_s f(\mathbf{s})\, q_s\,,\\ \pi_a(g) &= \eta^{-1/2} a^3 \sum_s g(\mathbf{s})\,\pi(\mathbf{s}) = a^{3/2} \sum_s g(\mathbf{s})\, p_s\,.\end{aligned}\right\} \tag{2.3}$$

Note that they differ from the usual ones by the "renormalization" $\eta^{\pm 1/2}$. They satisfy

$$[\phi_a(f), \pi_a(g)] = -\,i a^3 \sum_s f(\mathbf{s})\, g(\mathbf{s})\,, \qquad \text{etc.} \tag{2.4}$$

which go over to the well-known canonical commutation relations for the fields in the continuum limit $a \to 0$.

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[1] Subsequently one may take the limit $V \to \infty$. This process is rather immaterial in this paper and will not be stated explicitly.

In terms of the new variables the Hamiltonian (1.2) becomes

$$\mathscr{H}_0 = \eta \sum_s h_s , \tag{2.5}$$

where h_s looks like a single-particle Hamiltonian

$$h_s = \frac{1}{2} p_s^2 + \mathscr{V}(q_s; a) \tag{2.6}$$

with a potential

$$\mathscr{V}(q_s; a) = \frac{a^3}{\eta} F[\eta^{-1/2} a^{-3/2} q_s] . \tag{2.7}$$

In the following we shall omit the subscript **s** wherever there is no confusion. We may assume that the lowest eigenvalue of h is zero[2], $\omega_0 = 0$.

Now, the "short-range" potential is defined to be such a $F[\phi]$ that gives an a-independent $\mathscr{V}(q; a)$ for a suitable choice of $\eta = \eta(a)$. An example:

$$F[\phi] = \frac{1}{2}(\mu_0^2 \phi^2 + \lambda \phi^4), \qquad (\lambda \geqq 0) , \tag{2.8}$$

for which

$$\eta = 1/a \tag{2.9}$$

serves the purpose,

$$h = \frac{1}{2}(p^2 + \mu_0^2 q^2 + \lambda q^4) . \tag{2.10}$$

Another example is: $F[\phi] = 2^{-1} \lambda \phi^{2n} (\lambda > 0, n = 3, 4, \ldots)$ for which $\eta = a^{-3(n-1)/(n+1)}$. If, however, one wants to put a mass term in, $F[\phi] = 2^{-1}(\mu_0^2 \phi^2 + \lambda \phi^{2n})$, then the potential cannot be of short range unless λ vanishes with a.

Suppose $F[\phi]$ is of short range so that the Hamiltonian (2.6) is a-independent: $h = 2^{-1} p^2 + \mathscr{V}(q)$. Let its normalized ground-state wave function be $u_0(q)$, then the ground state of $\mathscr{H}_0$ is given by

$$\Omega_a = \prod_s u_0(q_s) . \tag{2.11}$$

Note that the a-independence of h implies the same for u_0. Then, for the renormalized fields (2.3) we study the limit $a \to 0$ of the expectation functional,

$$E(f, g; a) = \langle \Omega_a, \exp[i\phi_a(f)] \exp[i\pi_a(g)] \Omega_a \rangle . \tag{2.12}$$

We get

$$E(f, g; a) = \exp\Big[\sum_s \log E_s(f, g; a)\Big] , \tag{2.13}$$

with

$$\begin{aligned} E_s(f, g; a) &= \langle 0| \exp[i a^{3/2} f(\mathbf{s})\, q_s] \exp[i a^{3/2} g(\mathbf{s})\, p_s] |0\rangle \\ &= 1 - \{(1/4)\,[\alpha f(\mathbf{s})^2 + \beta g(\mathbf{s})^2] - \langle 0| qp |0\rangle f(\mathbf{s}) g(\mathbf{s})\} a^3 + O(a^6) , \end{aligned}$$

[2] An additive constant in the potential will be suppressed throughout the paper.

where $|0\rangle = u_0(q)$ and

$$\alpha = 2\,\langle 0|\, q^2\, |0\rangle, \qquad \beta = 2\,\langle 0|\, p^2\, |0\rangle\ ; \tag{2.14}$$

the error estimate $O(a^6)$ is obtained by using the boundedness as well as the square-integrability of the test functions. Now, the time-reversal invariance of h tells us that

$$\langle 0|\, qp\, |0\rangle = -\,\langle 0|\, pq\, |0\rangle = \frac{1}{2}\,\langle 0|\, [q, p]\, |0\rangle = \frac{1}{2}\, i\,.$$

In the continuum limit, therefore, the functional (2.12) tends to

$$E(f, g) = \exp\left[-\,(1/4)\,\alpha\,\|f\|^2 - (1/4)\,\beta\,\|g\|^2 - \frac{1}{2}\, i\,(f, g)\right], \tag{2.15}$$

where

$$(f, g) = \lim_{a \to 0} a^3 \sum_s f(\boldsymbol{s})\, g(\boldsymbol{s}) = \int f(\boldsymbol{x})\, g(\boldsymbol{x})\, d\boldsymbol{x}$$

and

$$\|f\|^2 = (f, f)\,.$$

This limit functional has the same Gaussian form as the one KLAUDER obtained [5].

III. Construction of the Representation

We now wish to construct the representation corresponding to the limit functional (2.15). From the uncertainty relation as applied to (2.14) we know

$$\alpha\beta \geqq 1\,, \tag{3.1}$$

the equality holding if and only if $F[\phi]$ is positive quadratic. When $\alpha\beta = 1$, (2.15) is the functional for the Fock representation, which we denote by $E_F(f, g; \beta)$. In terms of the creation, annihilation operators $a^+(\boldsymbol{k})$ and $a(\boldsymbol{k})$, the field operators are given by

$$\phi_F(\boldsymbol{x}; \beta) = \frac{1}{(2\pi)^{3/2}}\, \frac{1}{\beta^{1/2}} \int [a(\boldsymbol{k}) + a^+(-\boldsymbol{k})]\, e^{i\boldsymbol{k}\boldsymbol{x}}\, d\boldsymbol{k} \tag{3.2}$$

and the corresponding expression for $\pi_F(\boldsymbol{x}; \beta)$. We denote the no-particle state by $|\Omega_F; \beta\rangle$, which is the ground state of

$$\mathscr{H}_\beta = \frac{1}{2} \int [\pi_F(\boldsymbol{x}; \beta)^2 + \beta^2 \phi_F(\boldsymbol{x}; \beta)^2]\, d\boldsymbol{x}\ ; \tag{3.3}$$

β may be called the mass.

When $\alpha\beta > 1$, the functional (2.15) is known to give a reducible representation [5, 6]. In fact, the representation is realized by the operators,

$$\begin{aligned} \phi(\boldsymbol{x}) &= \sum_n G_n^{1/2}\, 1 \otimes \cdots \otimes 1 \otimes \phi_F(\boldsymbol{x}; \omega_n) \otimes 1 \otimes \cdots \otimes 1\,, \\ \pi(\boldsymbol{x}) &= \sum_n G_n^{1/2}\, 1 \otimes \cdots \otimes 1 \otimes \pi_F(\boldsymbol{x}; \omega_n) \otimes 1 \otimes \cdots \otimes 1 \end{aligned} \tag{3.4}$$

in the space $\mathfrak{H}$ cyclically generated from the state,

$$|\Omega\rangle = |\Omega_F; \omega_1\rangle \otimes \cdots \otimes |\Omega_F; \omega_n\rangle \otimes \cdots, \tag{3.5}$$

where $\{\omega_1, G_1; \omega_2, G_2; \ldots\}$ is a set of paired positive numbers satisfying

$$\sum_n \omega_n^{-1} G_n = \alpha, \qquad \sum_n \omega_n G_n = \beta \tag{3.6}$$

and

$$\sum_n G_n = 1\,. \tag{3.7}$$

The solution of (3.6)—(3.7) is not unique. The above construction gives us therefore many different realizations of the representation. They are, however, all unitarily equivalent due to Gelfand's theorem. For instance, one may take the set to be of size two:

$$\left.\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right\} = \beta \pm \sqrt{\frac{\beta}{\alpha}(\alpha\beta - 1)} \tag{3.8}$$

and

$$G_n = \frac{1}{2} \quad (n = 1, 2)\,. \tag{3.9}$$

For this choice,

$$\begin{aligned}\phi'(\boldsymbol{x}) &= \sqrt{1/2}\,[\phi_F(\boldsymbol{x}; \omega_1) \otimes 1 - 1 \otimes \phi_F(\boldsymbol{x}; \omega_2)]\,,\\ \pi'(\boldsymbol{x}) &= \sqrt{1/2}\,[\pi_F(\boldsymbol{x}; \omega_1) \otimes 1 - 1 \otimes \pi_F(\boldsymbol{x}; \omega_2)]\end{aligned} \tag{3.10}$$

commute with the operators (3.4) and yet are not multiples of identity in $\mathfrak{H}$. The representation is thus reducible.

Remark. If we put $\alpha\beta = 1$ in (3.8) then we get $\omega_1 = \omega_2 = \beta$. In this case one will see that the operators (3.10) have no range in the space cyclically generated by the operators (3.4) from the state (3.5). The representation is thus irreducible in accordance with the previous realization (3.2).

The most natural set $\{\omega_n, G_n\}$ will be the one that is obtained by considering time-dependent expectation functional,

$$E_t(f, g) = \lim_{a\to 0} \langle \Omega, \exp[i\phi_a(f)] \exp\left[it \sum_s h_s\right] \exp[i\pi_a(g)]\, \Omega\rangle\,. \tag{3.11}$$

Note that the time translation is effected by a "renormalized" Hamiltonian $\eta^{-1}\mathscr{H}_0$ and not by $\mathscr{H}_0$ itself; otherwise the limit won't exist[3]. By a calculation similar to the one in the above, we get

$$E_t(f, g) = \exp[-(1/4)\,\alpha\,\|f\|^2 - (1/4)\,\beta\,\|g\|^2 - \langle 0|\, qp(t)\,|0\rangle\,(f, g)] \tag{3.12}$$

where

$$p(t) = e^{iht}\, p e^{-iht}\,.$$

[3] Another way for the renormalization will be discussed in the subsequent paper.

Now suppose we know the complete set of (discrete) eigenvalues and eigenfunctions, ω_n and u_n, of h, then

$$\langle 0|\, q q(t)\, |0\rangle = 2^{-1} \sum_{n=1}^{\infty} \omega_n^{-1} G_n e^{i\,\omega_n t}\,, \tag{3.13}$$

with

$$G_n = 2\,\omega_n |\langle n|\, q\, |0\rangle|^2\,. \tag{3.14}$$

One should recall that ω_0 is taken to be zero. By symmetry of the eigenfunctions we have $G_0/\omega_0 = 0$ too. Thus we are left with $\{\omega_1, G_1; \omega_2, G_2; \ldots\}$. The normalization (3.7) is guaranteed by the THOMAS-REICHE-KUHN sum rule. Further, the relation

$$\frac{d^2}{dt^2} \langle 0|\, q q(t)\, |0\rangle = \frac{d}{dt} \langle 0|\, q(-t)\, p\, |0\rangle = -\langle 0|\, p(-t)\, p\, |0\rangle\,,$$

together with (3.13) proves the conditions (3.6). Thus the set $\{\omega_n, G_n\}$ can be used to carry out the construction (3.4); the field operator is a sum of Fock operators whose masses are given by the energy eigenvalues of h. The reducibility of the representation is obvious.

IV. Long-Range Potential

We have seen in the previous sections that the short range $F[\phi]$ yields only a trivial sort of representation for canonical commutation relations. If, however, $F[\phi]$ is not short range then the ground state u_0 of the Hamiltonian (2.6) becomes dependent upon a so that we may get a more general expectation functional. Suppose u_0 is given by the positive square-root of

$$[u_0(q;a)]^2 = \varrho_1(q) + \frac{1}{\sigma^3}\,\varrho_2\left(\frac{q}{\sigma}\right), \quad (\sigma = a^{-3/2})\,, \tag{4.1}$$

where ϱ_1 and ϱ_2 are both real-valued even functions. They must satisfy the normalization condition,

$$\int \varrho_1(q)\, dq + \frac{1}{\sigma^2} \int \varrho_2(x)\, dx = 1\,. \tag{4.2}$$

We assume further that $\varrho_1(q) > 0$, $\varrho_2(x) \geqq 0$ and that both belong to the class $\mathscr{S}$ of functions; it follows immediately that $u_0(q;a)$ itself should also belong to $\mathscr{S}$ (recall that $\varrho_1 > 0$!).

By the use of the Schrödinger equation one can easily construct a potential $\mathscr{V}(q;a)$ of (2.6) such that the $u_0(q;a)$ is an eigenstate of h. The $u_0(q;a)$ must be the ground state since, by assumption, it has no nodes. Due to the second term in (4.1), the potential $\mathscr{V}(q;a)$ has a long-range part; it shows up at a large distance $q \sim \sigma$ that is ever growing with $a^{-1} \to \infty$, because $\varrho_1(q)$ is assumed to decrease faster than any power of $1/q$. We call such a $F[\phi]$ a long-range potential.

Let us proceed to calculate the expectation functional (2.12) by using

$$\Omega_a = \prod_s u_0(q_s; a) \tag{4.3}$$

in place of (2.11). By the reason explained below, the test function $f(\boldsymbol{x})$ has to be restricted to those having a compact support. We have to know the function $E_s(f, g; a)$ as calculated for $|0_a\rangle = u_0(q; a)$, to order a^3. Now,

$$E_s(f, g; a) = I + II + \cdots + IV ,$$

where

$$\begin{aligned}
I &= \langle 0_a| \exp[i a^{3/2} f(\mathbf{s})\, q]\, |0_a\rangle ,\\
II &= i a^{3/2} g(\mathbf{s})\, \langle 0_a| \exp[i a^{3/2} f(\mathbf{s})\, q]\, p\, |0_a\rangle ,\\
III &= -\frac{1}{2} a^3 g(\mathbf{s})^2 \langle 0_a| \exp[i a^{3/2} f(\mathbf{s})\, q]\, p^2\, |0_a\rangle ,\\
IV &= \langle 0_a| \exp[i a^{3/2} f(\mathbf{s})\, q] \Big\{ \exp[i a^{3/2} g(\mathbf{s})\, p] - 1 \\
&\qquad - i a^{3/2} g(\mathbf{s})\, p + \frac{1}{2} a^3 g(\mathbf{s})^2 p^2 \Big\} |0_a\rangle .
\end{aligned}$$

The contribution from IV is negligible:

$$|IV| = \mathrm{O}(a^{9/2}) , \tag{4.4}$$

because by TAYLOR's expansion theorem we have

$$IV = \int u_0 \exp[i a^{3/2} f(\mathbf{s})\, q] \frac{1}{3!} [a^{3/2} g(\mathbf{s})]^3 \left(\frac{\partial^3 u_0}{\partial \xi^3}\right)_{\xi = q + \theta a^{3/2} g(s)} dq$$

$(0 < \theta < 1)$ so that by the Schwartz inequality and the normalization of u_0

$$|IV|^2 \leqq a^9 \left[\frac{g(\mathbf{s})^3}{3!}\right]^2 \int \left[\left(\frac{\partial^3 u_0}{\partial \xi^3}\right)_{\xi = q + \theta a^{3/2} g(s)}\right]^2 dq$$

the convergence of the integral being guaranteed by $u_0 \in \mathscr{S}$. The expression for I is the Fourier transform, which we denote by a tilde:

$$\begin{aligned}
I &= \tilde{\varrho}_1(a^{3/2} f(\mathbf{s})) + a^3 \tilde{\varrho}_2(f(\mathbf{s})) \\
&= 1 + \left\{-\frac{1}{2} f(\mathbf{s})^2 \langle q^2\rangle_1 + \tilde{\varrho}_2(f(\mathbf{s})) - \tilde{\varrho}_2(0)\right\} a^3 + \mathrm{O}(a^6) ;
\end{aligned} \tag{4.5}$$

the second line is obtained by using (4.2), and the expectation value $\langle\ \rangle_1$ is taken with the wave function $\varrho_1(q)^{1/2}$. Further, II and III can be calculated in the same way as in the previous section. Then the formula (2.13) gives for the limit functional,

$$\begin{aligned}
E(f, g) = \exp\Big[&-(1/4)\, \alpha \|f\|^2 - (1/4)\, \beta \|g\|^2 - \frac{1}{2} i\, (f, g) \\
&+ \int \{\tilde{\varrho}_2(f(\boldsymbol{x})) - \tilde{\varrho}_2(0)\}\, d\boldsymbol{x}\Big] ;
\end{aligned} \tag{4.6}$$

the assumption that supp$f(\boldsymbol{x})$ should be compact is used here to assure the existence of the integral. The coefficients α and β are given by

$$\begin{aligned}\alpha &= \langle q^2\rangle_1 = \int q^2 \varrho_1(q)\, dq\,,\\ \beta &= \langle p^2\rangle_1 = -\int \varrho_1(q)^{1/2} \frac{d^2}{dq^2}\, \varrho_1(q)^{1/2}\, dq\,.\end{aligned} \tag{4.7}$$

We have thus seen that the long-range potential can in fact give an expectation functional that has a non-quadratic dependence on f. A functional of the type (4.6) was once studied by ARAKI [6].

Two remarks are in order. First, in the two-point function,

$$\begin{aligned}\left(\frac{\delta^2 E(f,g)}{\delta f(\boldsymbol{x})\, \delta f(\boldsymbol{y})}\right)_{f=g=0} &= \left\{-\langle q^2\rangle_1 + \left(\frac{d^2 \tilde{\varrho}_2(f)}{df^2}\right)_{f=0}\right\} \delta(\boldsymbol{x}-\boldsymbol{y})\\ &= -\langle q^2\rangle\, \delta(\boldsymbol{x}-\boldsymbol{y})\,,\end{aligned} \tag{4.8}$$

there appears the expectation value of q^2 with respect to the whole wave function $u_0(q; a)$:

$$\langle q^2\rangle = \lim_{a\to 0} \int q^2\, u_0(q;a)^2\, dq = \int q^2 \{\varrho_1(q) + \varrho_2(q)\}\, dq\,. \tag{4.9}$$

Second, if one agrees to take the "renormalized" Hamiltonian $\sum_s h_s$, then one can show that

$$\begin{aligned}&\lim_{a\to 0} \langle \Omega_a, \exp[i\phi_a(f)] \exp\left[it \sum_s h_s\right] \exp[i\phi_a(g)]\, \Omega_a\rangle\\ &\quad= \exp\Big[-(1/4)\,\alpha\, \|f+g\|^2 - \frac{1}{2} i\, \langle q[q(t)-q(0)]\rangle\, (f,g) \\ &\qquad + \int \{\varrho_2(f(\boldsymbol{x}) + g(\boldsymbol{x})) - \varrho_2(0)\}\, d\boldsymbol{x}\Big]\,,\end{aligned} \tag{4.10}$$

where the expectation value $\langle\ \ \rangle$ is in the sense of (4.9).

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