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A Classification of the Unitary Irreducible Representations of $SO_0(N, 1)$

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Abstract. All inequivalent continuous unitary irreducible representations of the group $SO_0(N, 1)$, $N \geqq 3$, and its universal covering group are classified.

Introduction

Besides the identity representation all unitary irreducible representations (UIR) of $SO_0(N, 1)$ are infinite dimensional. In this paper we will classify these infinite dimensional representations and calculate the matrix elements of corresponding infintesimal operators in a certain Hilbert space. Previously the unitary representations have been derived [1] for groups $SO_0(N, 1)$ up to $N = 5$. The method of calculation which is used here demands that we make reservations for the case of $SO_0(2, 1)$ repeatedly and deal with it separately. As the representations of $SO_0(2, 1)$ are already classified [1] we omit this case and consider only $N \geqq 3$. It is not that our method fails for $N = 2$ but the discussion would be more difficult to survey if also that case is included.

The UIR of the universal covering group of $SO_0(N, 1)$ are derived via the UIR of the Lie algebra so $(N, 1)$. According to theorems by HARRISH-CHANDRA and NELSON [2] there exists a one-to-one correspondence between these representations.

DIXMIER has shown [3] that when an UIR of $SO_0(N, 1)$ is restricted to the subgroup $SO(N)$, each UIR of this subgroup occurs at most once. As a Hilbert space for the representation of $so(N, 1)$ we may therefore choose the direct sum of the Hilbert spaces of an apropriate set of inequivalent representations of $so(N)$.

The representations of $SO(N)$ and its twofold covering group have been classified by GELFAND and ZETLIN [4]. We will in the following use the notation of GELFAND and ZETLIN for the representations of $SO(N)$ and $so(N)$.

The present paper contains a derivation of the matrix elements of a representation of the generators of $so(N, 1)$. In these calculations we first exploit the full content of the commutation relations and then the requirement for unitarity. After that conditions for irreducibility and inequivalence are imposed. We end the paper with a discussion of the UIR of $SO_0(5,1)$ to exhibit the results in an explicit example.

Gelfand-Zetlin Patterns for *so* (*N*)

The infinitesimal generators $I_{i,k}$ of $SO(N)$, corresponding to an infinitesmal rotation in the (i, k)-plane, satisfy the following commutation relation

$$[I_{i,j}, I_{k,l}] = \delta_{j,k} I_{i,l} + \delta_{i,l} I_{j,k} - \delta_{i,k} I_{j,l} - \delta_{j,l} I_{i,k}$$

We shall exhibit the irreducible representations of these generators in a certain finite-dimensional Hilbert space $\mathscr{H}$.

The vectors $\xi(\alpha)$ of anorthormal basis of $\mathscr{H}$ are labeled by a Gelfand-Zetlin pattern α. When $N = 2p + 1$ is odd α has the form

$$\begin{bmatrix} l_{2p,1} & l_{2p,2} & \cdots & l_{2p,p} \\ l_{2p-1,1} & l_{2p-1,2} & \cdots & l_{2p-1,p} \\ & l_{2p-2,1} & \cdots\; l_{2p-2,p-1} & \\ & l_{2p-3,1} & \cdots\; l_{2p-3,p-1} & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ & l_{4,1} \quad l_{4,2} & & \\ & l_{3,1} \quad l_{3,2} & & \\ & l_{2,1} & & \\ & l_{2,1} & & \end{bmatrix}$$

and when $N = 2p$ is even α has the form

$$\begin{bmatrix} l_{2p-1,1} & l_{2p-1,2} & \cdots & l_{2p-1,p} \\ & l_{2p-2,1} & \cdots\; l_{2p-2,p-1} & \\ & l_{2p-3,1} & \cdots\; l_{2p-3\,p-1} & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ & l_{4,1} \quad l_{4,2} & & \\ & l_{3,1} \quad l_{3,2} & & \\ & l_{2,1} & & \\ & l_{1,1} & & \end{bmatrix}$$

where all the numbers l_{ij} are either integral or half-integral and they further fullfill the inequalities

$$l_{2k+1,i} \geqq l_{2k,i} > l_{2k+1,i+1} \qquad i = 1, \ldots, k-1$$

$$l_{2k+1,k} \geqq l_{2k,k} > |l_{2k+1,k+1}|$$

$$l_{2k,i} > l_{2k-1,i} \geqq l_{2k,i+1} \qquad i = 1, \ldots, k-1$$

$$l_{2k,k} > l_{2k-1,k} > - l_{2k,k} \, .$$

The uppermost row of the array α determines the representation. The corresponding representation space is then spanned by all vectors with allowed patterns. The operators $I_{2k+1,2k}$ and $I_{2k+2,2k+1}$ act on the basis vectors $\xi(\alpha)$ in the following way

$$I_{2k+1,2k}\,\xi(\alpha) = \sum_{j=1}^{k} A_{2k-1,j}(\alpha)\,\xi(\alpha_{2k-1}^{+j}) - \sum_{j=1}^{k} A_{2k-1,j}(\alpha_{2k-1}^{-j})\,\xi(\alpha_{2k-1}^{-j})$$

$$I_{2k+2,2k+1}\,\xi(\alpha) = \sum_{j=1}^{k} B_{2k,j}(\alpha)\,\xi(\alpha_{2k}^{+j}) - \sum_{j=1}^{k} B_{2k,j}(\alpha_{2k}^{-j})\,\xi(\alpha_{2k}^{-j}) + iC_{2k}(\alpha)\xi(\alpha)$$

where

$$A_{2k-1,j}(\alpha) = \frac{1}{2}\left|\prod_{r=1}^{k-1}(l_{2k-2,r} - l_{2k-1,j} - 1)\,(l_{2k-2,r} + l_{2k-1,j})\right|^{1/2} \cdot \left|\prod_{r=1}^{k}(l_{2k,r} - l_{2k-1,j} - 1)\,(l_{2k,r} + l_{2k-1,j})\right|^{1/2} \cdot \left|\prod_{r\neq j}(l_{2k-1,r}^2 - l_{2k-1,j}^2)\,(l_{2k-1,r}^2 - (l_{2k-1,j}+1)^2)\right|^{-1/2}$$

$$B_{2k,j}(\alpha) = \left|\prod_{r=1}^{k}(l_{2k-1,r}^2 - l_{2k,j}^2)\prod_{r=1}^{k+1}(l_{2k+1,r}^2 - l_{2k,j}^2)\right|^{1/2} \cdot \left|l_{2k,j}^2(4l_{2k,j}^2 - 1)\prod_{r\neq j}(l_{2k,r}^2 - l_{2k,j}^2)\,((l_{2k,r}-1)^2 - l_{2k,j}^2)\right|^{-1/2}$$

$$C_{2k}(\alpha) = \prod_{r=1}^{k} l_{2k-1,r}\prod_{r=1}^{k+1} l_{2k+1,r}\left[\prod_{r=1}^{k} l_{2k,r}\,(l_{2k,r}-1)\right]^{-1}$$

α_i^{+j} and α_i^{-j} are the arrays obtained from the array α by changing $l_{i,j}$ to $l_{i,j}+1$ and $l_{i,j}-1$ respectively. Since any generator $I_{i,j}$ can be written as a commutator between operators of the kind $I_{r+1,r}$, the action of all the other generators can be derived from the equations above. The denominators in the matrix elements A and B seem to vanish for certain values of the l's. However in these elements one pattern is not allowed so such matrix elements are zero. When $l_{2k,k}$ is one then $C_{2k}(\alpha)$ is zero.

The infinitesimal generators $I_{k,N+1}$ of $SO_0(N,1)$, have commutators of the following from

$$[I_{i,j}, I_{k,N+1}] = \delta_{j,k}\,I_{i,N+1} - \delta_{i,k}\,I_{j,N+1}$$

and

$$[I_{i,N+1}, I_{k,N+1}] = I_{i,k} \quad \text{where } i, j \text{ and } k \leqq N.$$

They satisfy the antihermiticity condition

$$(I_{k,\,N+1})^* = -\,I_{k,\,N+1}\,.$$

As a Hilbert space H for the representations of the Lie algebra $so(N, 1)$ we choose

$$H = \sum_{(l_{N-1,1},\, l_{N-1,2},\, \ldots) \in \Gamma} \oplus\, H(l_{N-1,1}, l_{N-1,2}, \ldots) = \sum_{\alpha} \oplus\, \xi(\alpha)$$

where $H(l_{N-1,1}, l_{N-1,2}, \ldots)$ is the Hilbert space of the representation of $so(N)$ which is labeled by $(l_{N-1,1}, l_{N-1,2}, \ldots)$. Γ is the set representations of $so(N)$, which appear in the representation of $so(N, 1)$ and $\xi(\alpha)$ is a vector in the Hilbert space of such a representation.

We next determine the action of the generator $I_{N+1,\,N}$ on the basis vectors of H by exploiting the commutation relation. That is, we shall calculate the matrix elements $\varphi(\alpha', \alpha)$ of the generator $I_{N+1,\,N}$ where

$$I_{N+1,\,N}\, \xi(\alpha) = \sum_{\alpha'} \varphi(\alpha', \alpha)\, \xi(\alpha')\,.$$

We have to study the two cases $so(2p+1)$ and $so(2p, 1)$ separately.

Conditions from the Commutation Relations for the Case $so\,(2p+1, 1)$

By considering the matrix elements of the relation

$$[I_{N+1,\,N}, I_{i,\,i-1}] = 0; \quad i < N$$

we find by Schur's lemma that $I_{N+1,\,N}$ shifts the indices in the uppermost row only. Therefore, the two matrix elements of $I_{i,\,i-1}$ which appear in the commutator are equal, and it follows (whether these elements are zero or not) that the matrix elements of $I_{N+1,\,N}$ can depend only on the two uppermost lines.

We now exploit the matrix elements of the relation

$$[[I_{2p+2,\,2p+1}, I_{2p+1,\,2p}], l_{2p+1,\,2p}] = -\,I_{2p+2,\,2p+1}\,. \tag{1}$$

With the notation

$$I_{2p+1,\,2p}\, \xi(\alpha) = \sum_{\alpha'} \chi(\alpha', \alpha)\, \xi(\alpha')$$

the relation reads

$$\begin{aligned} \sum_{\alpha'} \sum_{\alpha''} \chi(\alpha', \alpha)\, \chi(\alpha'', \alpha')\, \varphi(\alpha''', \alpha'') + \sum_{\beta'} \sum_{\beta''} \varphi(\beta', \alpha)\, \chi(\beta'', \beta')\, \chi(\alpha''', \beta'') \\ - 2 \sum_{\gamma'} \sum_{\gamma''} \chi(\gamma', \alpha)\, \varphi(\gamma'', \gamma')\, \chi(\alpha''', \gamma'') = -\,\varphi(\alpha''', \alpha)\,. \end{aligned} \tag{2}$$

Suppose that $\varphi(\alpha, \alpha')$ can be different from zero when the uppermost rows of α and α' are $(l_{2p,1}, l_{2p,2}, \ldots, l_{2p,r}, \ldots)$ and $(l_{2p,1} + s_1, l_{2p,2} + s_2, \ldots, l_{2p,r} + s_r, \ldots)$ respectively. We will then derive the conditions that the s_i have to satisfy.

First suppose that some $s_{j+1} \geqq 2$. Let us then choose the numbers in the two uppermost rows of α and α''' in the following way.

$$\begin{pmatrix} l_{2p,1}, l_{2p,2}, \ldots, l_{2p,j+1}, \ldots \\ l_{2p-1,1}, l_{2p-1,2}, \ldots, l_{2p-1\,j}, \ldots \end{pmatrix}$$

for α and

$$\begin{pmatrix} l_{2p,1} + s_1, l_{2p,2} + s_2, \ldots, l_{2p,j+1} + s_{j+1}, \ldots \\ l_{2p-1,1}, l_{2p-1,2}, \ldots, l_{2p-1,j} + 2, \ldots \end{pmatrix}$$

for α'''. Then the right hand side of the commutation relation (2) is zero and at most one term contributes in each sum. Put

$$l_{2p-1,j} = l_{2p,j} - 1$$

then $\chi(\alpha', \alpha)$ and $\chi(\gamma', \alpha)$ are zero. If $l_{2p,j} - 1 \geqq l_{2p,j+1} + s_{j+1}$ then $\chi(\beta'', \beta')\,\chi(\alpha''', \beta'')$ is different from zero. But if $l_{2p,j} - 1 < l_{2p,j+1} + s_{j+1}$ then both α and β' can not be alowed arrays. So we have

$$\varphi(\beta', \alpha) = 0 \quad \text{if} \quad l_{2p-1,j} = l_{2p,j} - 1$$

and where the two upper rows of β' are

$$\begin{pmatrix} l_{2p,1} + s_1, l_{2p,2} + s_2, \ldots \\ l_{2p-1,1}, l_{2p-1,2}, \ldots, l_{2p-1,j}, \ldots \end{pmatrix}.$$

We next proceed to relax the special choice of the numbers $l_{2p-1,j}$. Put

$$l_{2p-1,j} = l_{2p,j} - 2$$

then $\chi(\alpha', \alpha)\,\chi(\alpha'', \alpha')$ and $\chi(\gamma', \alpha)\,\varphi(\gamma'', \gamma')\,\chi(\alpha''', \gamma)$ are zero, and we have again

$$\varphi(\beta', \alpha) = 0 \quad \text{if} \quad l_{2p-1,j} = l_{2p,j} - 2\,.$$

We may continue this procedure until we reach the minimal value $l_{2p,j+1} + s_{j+1}$ by relations where two of the φ's have previously been proved to vanish. Therefore, we have

$$\varphi(\beta', \alpha) = 0 \quad \text{if} \quad s_j > 1\,.$$

In a similar way we can show that

$$\varphi(\beta', \alpha) = 0 \quad \text{if} \quad s_j < -1\,.$$

Suppose $s_i = s_j = +1$. Then we choose the second row of α''' to be

$$(l_{2p-1,1}, l_{2p-1,2}, \ldots, l_{2p-1,i} + 1, \ldots, l_{2p-1,j} + 1, \ldots)\,.$$

Then the right hand side of (2) is zero and at most two terms in each sum give contributions. First put

$$l_{2p-1,j} = l_{2p,j} - 1$$
$$l_{2p-1,i} = l_{2p,i} - 1$$

then $\chi(\alpha', \alpha)$ and $\chi(\gamma', \alpha)$ are zero and we must have

$$\varphi(\beta', \alpha) = 0\,.$$

Put then

$$l_{2p-1,j} = l_{2p,j} - 2$$
$$l_{2p-1,i} = l_{2p,i} - 1$$

then $\chi(\alpha', \alpha)\,\chi(\alpha'', \alpha')$ and $\chi(\gamma', \alpha)\,\varphi(\gamma'', \gamma')\,\chi(\alpha''', \gamma'')$ vanish and we have once again

$$\varphi(\beta', \alpha) = 0\,.$$

Repeating these arguments we can cover the allowed domain in its entirety for $(l_{2p-1,i}, l_{2p-1,j})$, and we find that the equality

$$\varphi(\beta', \alpha) = 0 \quad \text{if} \quad s_i = s_j = +1$$

holds for the general case. In a similar way we can exclude the other choices of sign for s_i and s_j. So that we have $\varphi(\beta', \alpha) = 0$ if more than one s_r is different from zero.

In summary we have found that

$$I_{2p+2,2p+1}\,\xi(\alpha) = \sum_r \varrho_r(\alpha)\,\xi(\alpha_{2p}^{+r}) + \sigma(\alpha)\,\xi(\alpha) + \sum_r \tau_r(\alpha)\,\xi(\alpha_{2p}^{-r})$$

where ϱ, σ and τ depend only on the uppermost rows of α. Next consider the matrix element of the commutator (1) between the states

$$\begin{pmatrix} \ldots, l_{2p,r}, \ldots \\ \ldots, l_{2p-1,i}, \ldots \end{pmatrix} \text{ and } \begin{pmatrix} \ldots, l_{2p,r}+1, \ldots \\ \ldots, l_{2p-1,i}-2, \ldots \end{pmatrix}.$$

(We will sometimes exhibit only those l's in the α's which are of special interest for the discussion.) It gives

$$\begin{aligned} A_{2p-1,i}&\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-2 \end{pmatrix} A_{2p-1,i}\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-1 \end{pmatrix} \varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i} \end{pmatrix} \\ &- 2A_{2p-1,i}\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-2 \end{pmatrix} A_{2p-1,i}\begin{pmatrix} l_{2p,r}+1 \\ l_{2p-1,i}-1 \end{pmatrix} \varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-1 \end{pmatrix} \\ &+ A_{2p-1,i}\begin{pmatrix} l_{2p,r}+1 \\ l_{2p-1,i}-2 \end{pmatrix} A_{2p-1,i}\begin{pmatrix} l_{2p,r}+1 \\ l_{2p-1,i}-1 \end{pmatrix} \varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-2 \end{pmatrix} = 0 \end{aligned}$$

and hence

$$\begin{aligned} \varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-2 \end{pmatrix} &= \frac{A_{2p-1,i}\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-2 \end{pmatrix}}{A_{2p-1,i}\begin{pmatrix} l_{2p,r}+1 \\ l_{2p-1,i}-2 \end{pmatrix}} \\ &\cdot \left[2\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-1 \end{pmatrix} - \frac{A_{2p-1,i}\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-1 \end{pmatrix}}{A_{2p-1,i}\begin{pmatrix} l_{2p,r}+1 \\ l_{2p-1,i}-1 \end{pmatrix}} \varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i} \end{pmatrix} \right]. \end{aligned}$$

By iteration we have

$$\begin{aligned} \varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-n \end{pmatrix} &= \left[\frac{(l_{2p,r}-l_{2p-1,i}+1)\,(l_{2p,r}+l_{2p-1,i}-n)}{(l_{2p,r}-l_{2p-1,i}+n)\,(l_{2p,r}+l_{2p-1,i}-1)}\right]^{1/2} \qquad (3) \\ &\cdot \left[n\,\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}-1 \end{pmatrix} - (n-1)\left[\frac{(l_{2p,r}-l_{2p-1,i})\,(l_{2p,r}-l_{2p-1,i}-1)}{(l_{2p,r}-l_{2p-1,i}+1)\,(l_{2p,r}-l_{2p-1,i})}\right]^{1/2} \varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i} \end{pmatrix} \right]. \end{aligned}$$

Particularly, when $i = r$ and

$$l_{2p-1,r} = l_{2p,r} = 1$$

we get

$$\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,r} - 2\end{pmatrix} = 2\sqrt{\frac{l-1}{2l-1}}\,\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,r} - 1\end{pmatrix}$$

and

$$\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,r} - n\end{pmatrix} = \sqrt{\frac{n(2l-n)}{2l-1}}\,\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,r} - 1\end{pmatrix}.$$

We now return to the commutation relation (1) and consider the matrix element between the states

$$\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i},\ l_{2p-1,j}\end{pmatrix} \text{ and } \begin{pmatrix} l_{2p,r}+1 \\ l_{2p-1,i}+1,\ l_{2p-1,j}+1\end{pmatrix}.$$

This yields the relation

$$\begin{aligned}
&(l_{2p-1,i} - l_{2q-1,j}) \\
&\cdot\left[\frac{(l_{2p,r} - l_{2p-1,j})\,(l_{2p,r} + l_{2p-1,j} + 1)\,(l_{2p,r} - l_{2p-1,i})\,(l_{2p,r} + l_{2p-1,i} + 1)}{(l_{2p,r} - l_{2p-1\,j} - 1)\,(l_{2p,r} + l_{2p-1,j})\,(l_{2p,r} - l_{2p-1,i} - 1)\,(l_{2p,r} + l_{2p-1,i})}\right]^{1/2} \\
&\cdot\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i},\ l_{2p-1,j}\end{pmatrix} - (l_{2p-1,i} - l_{2p-1,j} + 1)\left[\frac{(l_{2p,r} - l_{2p-1\,i})\,(l_{2p,r} + l_{2p-1,i} + 1)}{(l_{2p,r} - l_{2p-1,i} - 1)\,(l_{2p,r} + l_{2p-1,i})}\right]^{1/2} \\
&\cdot\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i},\ l_{2p-1,j}+1\end{pmatrix} - (l_{2p-1,i} - l_{2p-1,j} - 1)\left[\frac{(l_{2p,r} - l_{2p-1,j})\,(l_{2p,r} + l_{2p-1,j} + 1)}{(l_{2p,r} - l_{2p-1,j} - 1)(_{2p,r} + l_{2p-1,j})}\right]^{1/2} \\
&\cdot\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}+1,\ l_{2p-1,j}\end{pmatrix} + (l_{2p-1,i} - l_{2p-1,j})\,\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}+1,\ l_{2p-1,j}+1\end{pmatrix} = 0. \qquad (4)
\end{aligned}$$

Here we can get a relation between two $\varrho_r(\alpha)$'s with consecutive values on $l_{2p-1,j}$ by choosing i equal to r

$$\varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,j}\end{pmatrix}\sqrt{|l^2_{2p,r} - (l_{2p-1,j}+1)^2|} = \varrho_r\begin{pmatrix} l_{2p,r} \\ l_{2p-1,j}+1\end{pmatrix}\sqrt{|l^2_{2p,r} - l^2_{2p-1,j}|}.$$

Together with the previous relation (3) this gives us the dependence on the second row of ϱ_r. We will use the notation $\varrho_r(l_{2p,1}, l_{2p,2}, \ldots)$ for the part of ϱ_r that does not depend on the second row. Then we have

$$\varrho_r\begin{pmatrix} l_{2p,1}, l_{2p,2}, \ldots \\ l_{2p-1,1}, l_{2p-1,2}, \ldots\end{pmatrix} = \sqrt{\prod_{j=1}^{p} |l^2_{2p,r} - l^2_{2p-1,j}|}\quad \varrho_r(l_{2p,1}, l_{2p,2}, \ldots).$$

In a similar way we find

$$\tau_r\begin{pmatrix} l_{2p,1}, l_{2p,2}, \ldots \\ l_{2p-1,1}, l_{2p-1,2}, \ldots\end{pmatrix} = \sqrt{\prod_{j=1}^{p} |l^2_{2p,r} - (l_{2p-1,j}-1)^2|}\quad \tau_r(l_{2p,1}, l_{2p,2}, \ldots).$$

We now turn to σ. In analogy with the derivation of relation (3) we can derive

$$\sigma\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}+n\end{pmatrix} = n\sigma\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}+1\end{pmatrix} - (n-1)\,\sigma\begin{pmatrix} l_{2p,r} \\ l_{2p-1,i}\end{pmatrix}$$

and find that σ is a linear function of n. So either σ is independent of some l in the second row or we have that

$$\sigma(\alpha) = \prod_{r=1}^{p} (l_{2p-1,r} + \lambda_r)\, \sigma(l_{2p,1}, \ldots)$$

where the λ_r's may depend on the $l_{2p,j}$'s. With the analogue of the equation (4) we find if $\sigma(\alpha)$ is independent of one $l_{2p-1,j}$ it is independent of all $l_{2p-1,j}$, and further that all λ_r's are equal. We denote them by λ.

The diagonal element of the commutation relation (1) yields that if $\sigma(\alpha)$ is independent of all $l_{2p-1,j}$ it is zero, and further that

$$2\sigma(l_{2p,1}, \ldots) \sum_i [(A_{2p-1,i}(\alpha))^2 - (A_{2p-1,i}(\alpha_{2p-1}^{-i}))^2] \prod_{r \neq j} (l_{2p-1,r} + \lambda) = -\sigma(l_{2p,1}, \ldots) \prod_{r=1}^{p} (l_{2p-1,r} + \lambda)$$

which hold identically in $l_{2p-2,r}, l_{2p-1,r}$ and $l_{2p,r}$. By choosing $l_{2p-2,r} = l_{2p-1,r} = l_{2p,r} - 1$ for $r = 1, \ldots, p-1$ and $l_{2p-1,p} = l_{2p,p} - 1$ we find that λ is zero.

We now turn to the commutation relation

$$[I_{2p+2,2p+1}, [I_{2p+2,2p+1}, I_{2p+1,2p}]] = I_{2p+1,2p}\,. \tag{5}$$

The matrix element of this relation between vectors with the arrays

$$\begin{pmatrix} l_{2p,i}, l_{2p,j} \\ l_{2p-1,k} \end{pmatrix} \text{ and } \begin{pmatrix} l_{2p,i} \pm 1, l_{2p,j} \pm 1 \\ l_{2p} - 1, k + 1 \end{pmatrix}$$

gives

$$(l_{2p,i} - l_{2p,j} + 1)\, \varrho_i(l_{2p,i}, l_{2p,j})\, \varrho_j(l_{2p,i} + 1, l_{2p,j}) = (l_{2p,i} - l_{2p,j} - 1)\, \varrho_i(l_{2p,i}, l_{2p,j} + 1)\, \varrho_j(l_{2p,i}, l_{2p,j})$$

$$(l_{2p,i} + l_{2p,j} - 2)\, \tau_i(l_{2p,i}, l_{2p,j})\, \varrho_j(l_{2p,i} - 1, l_{2p,j}) = (l_{2p,i} + l_{2p,j})\, \tau_i(l_{2p,i}, l_{2p,j} + 1)\, \varrho_j(l_{2p,i}, l_{2p,j})$$

$$(l_{2p,i} + l_{2p,j} - 2)\, \varrho_i(l_{2p,i}, l_{2p,j} - 1)\, \tau_j(l_{2p,i}, l_{2p,j}) = (l_{2p,i} + l_{2p,j})\, \varrho_i(l_{2p,i}, l_{2p,j})\, \tau_j(l_{2p,i} + 1, l_{2p,j})$$

$$(l_{2p,i} - l_{2p,j} - 1)\, \tau_i(l_{2p,i}, l_{2p,j})\, \tau_j(l_{2p,i} - 1, l_{2p,j}) = (l_{2p,i} - l_{2p,j} + 1)\, \tau_i(l_{2p,i}, l_{2p,j} - 1)\, \tau_j(l_{2p,i}, l_{2p,j})$$

which implies that

$$(l_{2p,i} - l_{2p,j} + 1)(l_{2p,i} + l_{2p,j} + 1)\, \varrho_j(l_{2p,i} + 1, l_{2p,j})\, \tau_j(l_{2p,i} + 1, l_{2p,j} + 1) = (l_{2p,i} - l_{2p,j} - 1)(l_{2p,i} + l_{2p,j})\, \varrho_j(l_{2p,i}, l_{2p,j})\, \tau_j(l_{2p,i}, l_{2p,j} + 1)$$

if $\varrho_i(l_{2p,i}, l_{2p,j})$, $\varrho_j(l_{2p,i}, l_{2p,j} + 1)$, $\tau_i(l_{2p,i} + 1, l_{2p,j} + 1)$ or $\tau_j(l_{2p,i} + 1, l_{2p,j})$ is different from zero. Therefore the expression

$$(l_{2p,i}^2 - l_{2p,j}^2)((l_{2p,i} - 1)^2 - l_{2p,j}^2)\, \varrho_j(l_{2p,i}, l_{2p,j})\, \tau_j(l_{2p,i}, l_{2p,j} + 1)$$

is independent of $l_{2p,i}$, and the product

$$\varrho_j(l_{2p,j})\,\tau_j(l_{2p,j}+1)\prod_{r\neq j}(l_{2p,r}^2-l_{2p,j}^2)\,((l_{2p,r}-1)^2-l_{2p,r}^2)$$

depends only on $l_{2p,j}$.

The matrix element of the commutation relation (5) between vectors with the array:

$$\begin{pmatrix} l_{2p,j} \\ l_{2p-1,k} \end{pmatrix} \text{ and } \begin{pmatrix} l_{2p,j}+1 \\ l_{2p-1,k}+1 \end{pmatrix}$$

implies that

$$(l_{2p,j}-1)\,\sigma(l_{2p,j}) = (l_{2p,j}+1)\,\sigma(l_{2p,j}+1)$$

so that

$$l_{2p,j}\,(l_{2p,j}-1)\,\sigma(l_{2p,j})$$

is independent of $l_{2p,j}$. Therefore,

$$\sigma(l_{2p,1}, l_{2p,2}, \ldots)\prod_{r=1}^{p} l_{2p,r}(l_{2p,r}-1) = \sigma$$

where σ is a constant. If $\sigma \neq 0$ we have

$$\sigma(l_{2p,1}, l_{2p,2}, \ldots) = \frac{\sigma}{\prod\limits_{r=1}^{p} l_{2p,r}(l_{2p,r}-1)}\,.$$

If $\sigma = 0$ then either $\sigma(l_{2p,1}, l_{2p,2}, \ldots) = 0$ or $l_{2p,p} = 1$, so that $\sigma(\alpha)$ vanishes.

Finally we take the matrix elements of the two members of the commutation relation (5) between vectors with the arrays

$$\begin{pmatrix} \cdots \\ l_{2p-1,k} \end{pmatrix} \text{ and } \begin{pmatrix} \cdots \\ l_{2p-1,k}+1 \end{pmatrix}$$

which yields

$$\sum_i \Bigg[-\frac{2l_{2p,i}+1}{l_{2p,i}^2-l_{2p-1,k}^2}\,\varrho_i(l_{2p,i})\,\tau_i(l_{2p,i}+1)\left|\prod_{r=1}^{p}(l_{2p-1,r}^2-l_{2p,i}^2)\right|$$

$$+\frac{2l_{2p,i}-3}{(l_{2p,i}-1)^2-l_{2p-1,k}^2}\,\varrho_i(l_{2p,i}-1)\,\tau_i(l_{2p,i})\left|\prod_{r=1}^{p}(l_{2p-1,r}^2-(l_{2p,i}-1)^2)\right|\Bigg]$$

$$= 1-\frac{1}{l_{2p-1,k}^2}\,\frac{\prod\limits_{r=1}^{p} l_{2p-1,r}^2}{\prod\limits_{r=1}^{p} l_{2p,r}^2(l_{2p,r}-1)^2}\,\sigma^2, \quad \text{if } l_{2p,p} > 1$$

and a similar relation without the second term in the right member when $l_{2p,p} = 1$. We first assume that no $l_{2p-1,r}$'s are constant in value for the representation, and consider the degenerate case later. Then identify terms with the same dependence on $l_{2p-1,r}$, for all r different from k. Using the notation

$$\beta_j(l_{2p,j}) = \varrho_j(l_{2p,j})\,\tau_j(l_{2p,j}+1)$$

$$\cdot\, l_{2p,j}^2(4l_{2p,j}^2-1)\prod_{r\neq j}(l_{2p,r}^2-l_{2p,j}^2)\,((l_{2p,r}-1)^2-l_{2p,j}^2)$$

where β_j depends on $l_{2p,j}$ only, we have

$$\sum_{i=1}^{p} \frac{(-1)^i}{2l_{2p,i}-1}\Bigg[-\frac{\beta_i(l_{2p,i})}{l_{2p,i}^2 \prod_{r\neq i}(l_{2p,r}^2-l_{2p,i}^2)((l_{2p,r}-1)^2-l_{2p,i}^2)} + \frac{\beta_i(l_{2p,i}-1)}{(l_{2p,i}-1)^2\prod_{r\neq i}(l_{2p,r}^2-(l_{2p,i}-1)^2((l_{2p,r}-1)^2-(l_{2p,i}-1)^2)}\Bigg] = (-1)^{p+1}\frac{\sigma^2}{\prod_{r=1} l_{2p,r}^2(l_{2p,r}-1)^2} \tag{6}$$

$$\sum_{i=1}^{p} \frac{(-1)^i}{2l_{2p,i}-1}\Bigg[-\frac{l_{2p,i}^{2\nu}\beta_i(l_{2p,i})}{\prod_{r\neq i}(l_{2p,r}^2-l_{2p,i}^2)((l_{2p,r}-1)^2-l_{2p,i}^2)} + \frac{(l_{2p,i}-1)^{2\nu}\beta_i(l_{2p,i}-1)}{\prod_{r\neq i}(l_{2p,r}^2-(l_{2p,i}-1)^2)((l_{2p,r}-1)^2-(l_{2p,i}-1)^2)}\Bigg] = 0 \tag{7}$$

for $\nu = 0, 1, \ldots, p-3$

$$\sum_{r\neq i}^{p} \frac{(-1)^i}{2l_{2p,i}-1}\Bigg[-\frac{l_{2p,i}^{2p-4}\beta_i(l_{2p,i})}{\prod_{r\neq i}(l_{2p,r}^2-l_{2p,i}^2)((l_{2p,r}-1)^2-l_{2p,i}^2)} + \frac{(l_{2p,i}-1)^{2p-4}\beta_i(l_{2p,i}-1)}{\prod_{r\neq i}(l_{2p,r}^2-(l_{2p,i}-1)^2)((l_{2p,r}-1)^2-(l_{2p,i}-1)^2)}\Bigg] = -1\,. \tag{8}$$

(Observe that terms with zero in the denominator do not occure in these equations. In the corresponding ϱ's or τ's there are unallowed patterns in that case).

We now let all indices but one $l_{2p,i}$ take their minimal values $l_{2p,r,\min}$ in Γ. By starting with the minimal value on $l_{2p,i}$ and then increasing it by one unit at a time we get a series of equations from which we can calculate $\beta_i(l_{2p,i})$. We find

$$\frac{(-1)^{i+1}\beta_i(l_{2p,i})}{\prod_{r\neq i}(l_{2p,r,\min}^2-l_{2p,i}^2)((l_{2,p,r,\min}-1)^2-l_{2p,i}^2)} = \sum_{l=l_{2p,i,\min}}^{l_{2p,i}}(2l-1) \cdot \sum_{j\neq i}\frac{(-1)^j\beta_j(l_{2p,j,\min})}{(2l_{2p,j,\min}-1)(l^2-l_{2p,j,\min}^2)((l-1)^2-l_{2p,j,\min}^2)\prod_{\substack{r\neq j\\ r\neq i}}(l_{2p,r,\min}^2-l_{2p,j,\min}^2)\cdot((l_{2p,r,\min}-1)^2-l_{2p,j,\min}^2)}$$

$$= \sum_{j\neq i}\frac{(-1)^j\beta_j(l_{2p,j,\min})(l_{2p,i,\min}^2-l_{2p,i}^2)((l_{2p,i,\min}-1)^2-l_{2p,i}^2)}{(2l_{2p,j,\min}-1)(l_{2p,j,\min}^2-l_{2p,i}^2)\prod_{r\neq j}(l_{2p,r,\min}^2-l_{2p,j,\min}^2)((l_{2p,r,\min}-1)^2-l_{2p,j,\min}^2)} + \frac{(-1)^i\beta_i(l_{2p,i,\min})((l_{2p,i,\min}-1)^2-l_{2p,i}^2)}{(2l_{2p,i,\min}-1)\prod_{r\neq i}(l_{2p,r,\min}^2-l_{2p,i,\min}^2)((l_{2p,r,\min}-1)^2-l_{2p,i,\min}^2)}$$

so that

$$(-1)^{i+1}\beta_i(l_{2p,i}) = \prod_{r=1}^{p}((l_{2p,r,\min}-1)^2-l_{2p,i}^2) \cdot \sum_{j=1}^{p}\frac{(-1)^i\beta_j(l_{2p,j,\min})\prod_{r\neq j}(l_{2p,r,\min}^2-l_{2p,i}^2)}{(2l_{2p,j,\min}-1)\prod_{r\neq j}(l_{2p,r,\min}^2-l_{2p,j,\min}^2)((l_{2p,r,\min}-1)^2-l_{2p,j,\min}^2)}\,.$$

Irrespective of the sign, we see that all the β's have the same functional dependence on the l's. They depend on l^2, and they are polynomials. We find that their degree is $2p+2$ by putting a very large $l_{2p,1}$ into the relations (7) and (8). The only possibility is then that

$$(-1)^{i+1}\,\beta_i(l_{2p,i}) = b\prod_{r=1}^{p+1}(l^2_{2p+1,r} - l^2_{2p,i})$$

where $l_{2p+1,r} = l_{2p,r,\min} - 1$ for $r = 1, 2, \ldots p$, and where b is a constant which can be determined by the relation (8) to $(-1)^p$.

We will calculate σ^2 from the relation (6). We first observe that as (6) is fullfilled in an infinitive set of points there is only one rational function that can interpolate and extrapolate the right member on the whole real line. Comparing the limits of the two members as $l_{2p,1}$ goes to one, we find

$$\sigma^2 = \prod_{r=1}^{p+1} l^2_{2p+1,r}\,.$$

We next consider the cases when some of the numbers $l_{2p-1,r}$ are constant in a representation. The result of the calculations in these cases is the same as in the previous case. If n of the numbers $l_{2p-1,r}$ are constant then the calculations are analogous to those of the non-degenerate case for the algebra $(2p-2n+1, 1)$. Some extra factors enter and modify the result. The number of equations of the type (7) decreases, and there is none left in the degenerate case when

$$\begin{aligned} l_{2p,p-r} &= l + r \quad \text{for} \quad r = 0, 1, \ldots, p-2 \\ l_{2p-1,p-r} &= l + r - 1 \quad \text{for} \quad r = 1, \ldots, p-2 \quad \text{and} \quad l > 1\,. \end{aligned}$$

Let us outline the calculations in this case. Instead of the equations (6), (7) and (8) we now have

$$\begin{aligned} &-\beta_1(l_{2p,1})\left[(l^2_{2p,2} - l^2_{2p,1})\prod_{r=2}^{p}((l_{2p,r}-1)^2 - l^2_{2p,1})\right]^{-1} \\ &+\beta_1(l_{2p,1}-1)\left[(l^2_{2p,2} - (l_{2p,1}-1)^2)\prod_{r=2}^{p}((l_{2p,r}-1)^2 - (l_{2p,1}-1)^2\right]^{-1} \\ &= (-1)^p(2l_{2p,1}-1) \end{aligned} \tag{9}$$

and

$$\begin{aligned} &-\beta_1(l_{2p,1})\left[l^2_{2p,1}(l^2_{2p,2} - l^2_{2p,1})\prod_{r=2}^{p}((l_{2p,r}-1)^2 - l^2_{2p,1})\right]^{-1} \\ &+\beta_1(l_{2p,1}-1) \\ &\cdot\left[(l_{2p,1}-1)^2(l^2_{2p,2} - (l_{2p,1}-1)^2)\prod_{r=2}^{p}((l_{2p,r}-1)^2 - (l_{2p,1}-1)^2)\right]^{-1} \\ &= (-1)^p\sigma^2\left[l^2_{2p,1}\,l^2_{2p,2}\prod_{r=1}^{p}(l_{2p,r}-1)^2\right]^{-1} \end{aligned} \tag{10}$$

when $l_{2p,1}$ does not take its minimal value $l_{2p,1\,\min}$. If $l_{2p,1\,\min} > l + p - 1$ (9) and (10) also hold for $l_{2p,1\,\min}$. For $l_{2p,1\,\min} = l + p - 1$ we have instead

$$- \beta_1(l_{2p,1,\min}) \left[\prod_{r=2}^{p} ((l_{2p,r} - 1)^2 - l_{2p,1\,\min}^2) \right]^{-1} = (-1)^p (2 l_{2p,1\,\min} - 1) \left(\sigma^2 \left[\prod_{r=1}^{p} (l_{2p,r} - 1)^2 \right]^{-1} - 1 \right) \tag{11}$$

From the equations (9), (10) and (11) we again deduce that

$$\beta_1(l_{2p,1}) = (-1)^p \prod_{r=1}^{p+1} (l_{2p+1,p}^2 - l_{2p,1}^2)$$

and

$$\sigma^2 = \prod_{r=1}^{p+1} l_{2p+1,r}^2 \,.$$

In the most degenerate case with $l = 1$ one finds $l_{2p-1,p} = 0$. When $l_{2p,1} > l_{2p,1,\min}$ or $l_{2p,1,\min} > p$ the equation (9) remains valid. If $l_{2p,1,\min} = p$ we get no equation to determine $\beta_1(l_{2p,1,\min})$. Different values on this quantity therefore give rise to different representation But the freedom of choosing $\beta_1(l_{2p,1,\min})$ is a consequence of the freedom of choosing the factor $(l_{2p+1,p+1}^2 - l_{2p,1,\min}^2)$ or rather the constant $l_{2p+1,p+1}$. Therefore it is possible to classify also these representations with the numbers $l_{2p+1,r}$.

Conditions for Unitarity, Irreducibility and Inequivalence

The unitarity of the representations requires that σ is imaginary and that

$$\varrho_j(\alpha) = - \overline{\tau(\alpha_{2p}^{+j})} \,. \tag{12}$$

We can change the phases of the vectors $\xi(\alpha)$ by multiplying them by a factor $\prod_{r=2}^{p} \omega_r(l_{2p,r})$ of modulus one so that ϱ_j become positive or zero on Γ. We have, therefore,

$$I_{2p+2,2p+1}\xi(\alpha) = \sum_{j=1}^{p} B_{2p,j}(\alpha)\, \xi(\alpha_{2p}^{+j}) - \sum_{j=1}^{p} B_{2p,j}(\alpha_{2p}^{-j})\, \xi(\alpha_{2p}^{-j}) + C_{2p}\xi(\alpha)$$

where

$$B_{2p,j}(\alpha) = \left| \frac{\prod_{r=1}^{p} (l_{2p-1,r}^2 - l_{2p,j}^2) \prod_{r=1}^{p+1} (l_{2p+1,r}^2 - l_{2p,j}^2)}{l_{2p,j}^2 (4 l_{2p,j}^2 - 1) \prod_{r \neq j} (l_{2p,r}^2 - l_{2p,j}^2)((l_{2p,r} - 1)^2 - l_{2p,j}^2)} \right|^{1/2}$$

$$C_{2p} = \frac{\prod_{r=1}^{p} l_{2p-1,r} \prod_{r=1}^{p+1} l_{2p+1,r}}{\prod_{r=1}^{p} l_{2p,r}(l_{2p,r} - 1)} \,.$$

The numbers $l_{2p+1,r}$, $r = 1, \ldots, p$ are all integer or all half integer and ordered so that

$$l_{2p+1,1} > l_{2p+1,2} > \cdots > l_{2p+1,p} .$$

As C_{2p} is imaginary either $l_{2p+1,p+1}$ is imaginary or $l_{2p+1,p}$ is zero. Relation (12) implies that β is negative or zero in Γ. Thus, when $l_{2p+1,p}$ is zero, $l_{2p+1,p+1}$ can be real or imaginary. The sign of $l_{2p+1,p+1}$ has no consequence when $l_{2p+1,p}$ is zero. Sign conditions imply that when $l_{2p+1,p+1}$ is real and positive, β has to be zero for all positive integers smaller than $l_{2p+1,p+1}$.

That the representations obtained in this way are irreducible follows immediately from the fact that the representation of $so(2p+1)$ occur at the most once, and that B is identically zero only on the boundary of Γ.

To determine equivalence conditions we will first establish the condition which must be satisfied by a unitary transformation that transforms one Gelfand-Zetlin base into another Gelfand-Zetlin base. Clearly, it must not mix the irreducible spaces of the $so(2p+1)$ – subalgebra, since they correspond to inequivalent representations. This then implies that the unitary transformation reduces to a direct sum of unitary transformations in the irreducible spaces of $so(2p+1)$. We can now proceede to $so(2p)$, $so(2p-1)$, ... and repeat the argument, and we find that the unit operator is the only unitary transformation that transfers one Gelfand-Zetlin base into another Gelfand-Zetlin base. Therefore, two representations are inequivalent if and only if all the matrix elements are the same in the two representations. This leaves us with the following inequivalent representations.

The Principal Series

$D(p; l_{2p+1,1}, l_{2p+1,2}, \ldots, l_{2p+1,p+1})$
where $l_{2p+1,p+1} = i\tau$, τ real. If $l_{2p+1,p} = 0$ then $\tau \geqq 0$.
$l_{2p,1} \geqq l_{2p+1,1} + 1$, $l_{2p+1,r-1} \geqq l_{2p,r} \geqq l_{2p+1,r} + 1$, for $r = 2, \ldots, p$.

The Supplementary Series

$D(s; l_{2p+1,1}, l_{2p+1,2}, \ldots, l_{2p+1,p+1})$
where $l_{2p+1,p-r+1} = r - 1$ for $r = 1, \ldots, s$ and $l_{2p+1,p+1} = s$,
$l_{2p,1} \geqq l_{2p+1,1} + 1$, $l_{2p+1,r-1} \geqq l_{2p,r} \geqq l_{2p+1,r} + 1$ for $r = 2, \ldots, p - s$,
$l_{2p,p-r+1} = r$ for $r = 1, \ldots, s$, $s \leqq p - 1$.

The Exceptional Series

$D(e; l_{2p+1,1}, l_{2p+1,2}, \ldots, l_{2p+1,p+1})$
where $l_{2p+1,p-r+1} = r - 1$, for $r = 1, \ldots, t$ and $0 < l_{2p+1,p+1} < t$, for some positive integer $t \leqq p - 1$, $l_{2p,1} \geqq l_{2p+1,1} + 1$,
$l_{2p+1,r-1} \geqq l_{2p,r} \geqq l_{2p+1,r} + 1$, for $r = 2, \ldots, p$.

Construction of the UIR of $so(2p, 1)$

The derivation can be made step by step in a similar way as in the previous case. The only difference is that we here can prove that $\sigma(\alpha)$ is zero. For this we make use of the diagonal element of the commutation relation

$$[[I_{2p+1,2p}, I_{2p,2p-1}], I_{2p,2p-1}] = -I_{2p+1,2p}$$

which yields

$$2\sigma(l_{2p-1,1}, \ldots) \sum_i [(B_{2p-1,i}(\alpha))^2 - (B_{2p-2,i}(\alpha_{2p-2}^{-i}))^2] \prod_{r \neq i} (l_{2p-2,r} + \lambda)$$
$$= -\sigma(l_{2p-1,1}, \ldots) \prod_{r=1}^{p-1} (l_{2p-2,r} + \lambda)$$

where the entities λ and $\sigma(l_{2p-2,1}, \ldots)$ are defined as in the case $so(2p+1, 1)$.

First choose $l_{2p-3,r} + 1 = l_{2p-2,r} = l_{2p-1,r}$ for $r = 1, \ldots, p-1$. And we find that $\sigma(\alpha)$ is identically zero if not $\lambda = -l_{2p-1,s}$ for some s for which $l_{2p-2,s}$ is not constant in the representation.

Next choose the l's so that all but one of the equalities above hold and we have

$$l_{2p-2,q} = l_{2p-1,q} - 1 \quad \text{for} \quad q \neq s .$$

Then we find that $\sigma(\alpha)$ is zero if not all but one $l_{2p-2,r}$ are constant in the representation and thus $s = 1$.

Finally choose the l's so that all but one of the equalities in the first choice are satisfied and we have

$$l_{2p-3,1} + 2 = l_{2p-2,1} .$$

Then we find

$$(B_{2p-2,1}(\alpha_{2p-2}^{-1}))^2 \sigma(l_{2p-1,1}, \ldots) \prod_{r=2}^{p} (l_{2p-2,r} - l_{2p-1,1}) = 0$$

so that $\sigma(\alpha)$ is zero even in this case, and $\sigma(\alpha)$ is identically zero.

We omit the rest of the derivation as it is completely analogous to the derivation in the case $so(2p+1, 1)$. The final result is

$$I_{2p+1,2p}\, \xi(\alpha) = \sum_{j=1}^{p} A_{2p-1,j}(\alpha)\, \xi(\alpha_{2p-1}^{+j}) - \sum_{j=1}^{p} A_{2p-1,j}(\alpha_{2p-1}^{-j})\, \xi(\alpha_{2p-1}^{-j})$$

where

$$A_{2p-1,j}(\alpha)$$
$$= \frac{1}{2} \left| \frac{\prod_{r=1}^{p-1} (l_{2p-2,r} - l_{2p-1,j} - 1)(l_{2p-2,r} + l_{2p-1,j}) \prod_{r=1}^{p} (l_{2p,r} - l_{2p-1,j} - 1)(l_{2p,r} + l_{2p-1,j})}{\prod_{r \neq j} (l_{2p-1,r}^2 - l_{2p-1,j}^2)(l_{2p-1,r}^2 - (l_{2p-1,j} + 1)^2)} \right|^{1/2} .$$

The numbers $l_{2p,r}$, $r = 1, \ldots, p-1$ are all integers or all half integers and ordered so that

$$l_{2p,1} > l_{2p,2} > \cdots > l_{2p,p-1}\,.$$

Reality and sign conditions imply that $l_{2p,p}(l_{2p,p} - 1)$ is real and not too large. When $l_{2p,p}$ is real and positive, β has to be zero for all positive arguments smaller than $l_{2p,p} - 1$.

This leaves us with the following inequivalent representations:

$D(+; l_{2p,1}, \ldots, l_{2p,p})$
where $l_{2p,p}$ is real and integer or half integer at the same time as $l_{2p,1}$. $l_{2p,p-1} > l_{2p,p} \geqq \frac{1}{2}$, $l_{2p-1,1} \geqq l_{2p,1}$, $l_{2p,r-1} - 1 \geqq l_{2p-1,r} \geqq l_{2p,r}$ for $r = 2, \ldots, p$.

$D(-; l_{2p,1}, \ldots, l_{2p,p})$
where $l_{2p,p}$ is real and integer or half integer at the same time as $l_{2p,p}$ and $-l_{2p,p-1} + 2 \leqq l_{2p,p} \leqq \frac{1}{2}$, $l_{2p-1,1} \geqq l_{2p,1}$ $l_{2p,r-1} - 1 \geqq l_{2p-1,r} \geqq l_{2p,r}$ for $r = 2, \ldots, p-1$ and $l_{2p,p} - 1 \geqq l_{2p-1p} \geqq l_{2p,p-1} + 1$.

$D(s; l_{2p,1}, \ldots, l_{2p,p})$
where s is a positive integer and $s \leqq p-1$, $l_{2p,p-r+1} = r$ for $r = 1, \ldots, s$, $l_{2p,r}$ are all integers, $l_{2p-1,1} \geqq l_{2p,1}$, $l_{2p,r-1} - 1 \geqq l_{2p-1,r} \geqq l_{2p,r}$ for $r = 2, \ldots, p-s-1$, $l_{2p-1,p-r+1} = r-1$ for $r = 1, \ldots, s$.

$D(c; l_{2p,1}, \ldots, l_{2p,p})$
where $l_{2p,p} = \frac{1}{2} + i\tau$, $\tau > 0$; $l_{2p-1,1} \geqq l_{2p,1}$, $l_{2p,r-1} - 1 \geqq l_{2p-1,r} \geqq l_{2p,r}$ for $r = 2, \ldots, p-1$, and $l_{2p,p-1} - 1 \geqq l_{2p-1,p} \geqq -l_{2p,p} + 1$.

$D(e; l_{2p,1}, \ldots, l_{2p,p})$
where $l_{2p,p}$ is real, $\frac{1}{2} \leqq l_{2p,p} < t+1$ for some non-negative integer $t \leqq p-1$; and when t is positive $l_{2p,p-r} = r$ for $r = 1, \ldots, t$; $l_{2p,r}$ are all integers; $l_{2p-1,1} \geqq l_{2p,1}$, $l_{2p,r-1} - 1 \geqq l_{2p-1,r} \geqq l_{2p,r}$ for $r = 2, \ldots, p-1$ and $l_{2p,p-1} - 1 \geqq l_{2p-1,p} \geqq -l_{2p,p-1} + 1$.

The Representations of the Groups

In the previous sections we have classified all UIR of the Lie algebras $so(N, 1)$. According to theorems by HARISH-CHANDRA and NELSON [2] we then also have classified the continuous UIR of the universal covering group of $SO_0(N, 1)$. We do not dixcuss this point further, a review of results in this field may be found in [5].

Conclusion. All the infinite dimensional continuous unitary irreducible representations of $SO_0(N, 1)$, $N \geqq 3$, and its universal covering group have been derived with the following result.

$$N = 2p + 1 \textit{ odd}$$

$D(p; l_{2p+1,1}, \ldots, l_{2p+1,p+1})$; $l_{2p+1,p+1} = i\tau$, τ real; if $l_{2p+1,p} = 0$ then $\tau \geqq 0$;
$D(s; l_{2p+1,1}, \ldots, l_{2p+1,p+1})$; s positive integer, $s \leqq p-1$; $l_{2p+1,p-r+1} = r-1$, for $r = 1, \ldots, s$; $l_{2p+1,p+1} = s$;
$D(e; l_{2p+1,1}, \ldots, l_{2p+1,p+1})$; for some positive integer t $l_{2p+1,p-r+1} = r-1$ for $r = 1, \ldots t$; $0 < l_{2p+1,p+1} < t$;
where $l_{2p+1,1} > l_{2p+1,2} > \cdots > l_{2p+1,p}$ are all integers or all half integers, they are non-negative and $l_{2p,1} \leqq l_{2p+1,1} + 1$, $l_{2p+1,r-1} \geqq l_{2p,r} \geqq l_{2p+1,r} + 1$ for $r = 2, \ldots, p-s$, and also for $r = p-s+1, \ldots, p$ in the p and e cases while $l_{2p,p-r+1} = r$ for $r = 1, \ldots, s$ in the e case.

$N = 2p$ *even*

$D(+; l_{2p,1}, \ldots, l_{2p,p})$; $l_{2p,p-1} > l_{2p,p} \geqq \frac{1}{2}$; $l_{2p,p-1} - l_{2p,p}$ integral; $l_{2p,p-1} \geqq l_{2p-1,p} \geqq l_{2p,p}$;
$D(-; l_{2p,1}, \ldots, l_{2p,p})$; $-l_{2p,p-1} + 2 \leqq l_{2p,p} \leqq \frac{1}{2}$; $l_{2p,p-1} - l_{2p,p}$ integral; $l_{2p,p} - 1 \geqq l_{2p-1,p} \geqq -l_{2p,p-1} + 1$;
$D(s; l_{2p,1}, \ldots, l_{2p,p})$; s positive integer, $s \leqq p-1$; $l_{2p,p} = s$; $l_{2p,p-r} = r-1$ for $r = 1, \ldots, s$; $l_{2p-1,p} = 0$;
$D(c; l_{2p,1}, \ldots, l_{2p,p})$; $l_{2p,p} = \frac{1}{2} + i\tau$, $\tau > 0$; $l_{2p,p-1} - 1 \geqq l_{2p-1,p} \geqq -l_{2p,p-1} + 1$;
$D(e; l_{2p,1}, \ldots, l_{2p,p})$; for some non-negative integer $t \leqq p-1$, $\frac{1}{2} \leqq l_{2p,p} < t+1$ and when t is positive $l_{2p,p-r} = r-1$ for $r = 1, \ldots, t$; $l_{2p,p-1} - 1 \geqq l_{2p-1,p} \geqq -l_{2p,p-1} + 1$.
where $l_{2p,1} > l_{2p,2} > \cdots > l_{2p,p-1}$ are all integers or all half integers, they are non-negative and, $l_{2p-1,1} \geqq l_{2p,1}$ and $l_{2p,r-1} - 1 \geqq l_{2p-1,r} \geqq l_{2p,r}$ for $r = 2, \ldots, p-s$ and also for $r = p-s+1, \ldots, p-1$ in the $+$, $-$, c and e cases while $l_{2p-1,p-r+1} = r-1$ for $r = 1, \ldots, s$ in the s case.

Example: UIR of $SO_0(5,1)$. The representations of the algebra $so(5,1)$ is of physical interest because the algebra is isomorphic to the Dirac algebra over the real numbers in the momentum space [6]. It has the following inequivalent UIR besides the identy representation
$D(p; l_{5,1}, l_{5,2}, i\tau)$;
τ real; if $l_{5,2} = 0$ then $\tau \geqq 0$; $l_{5,1} > l_{5,2} \geqq 0$ are both integral or both half integral and $l_{4,1} \geqq l_{5,1} + 1$; $l_{5,1} \geqq l_{4,2} \geqq l_{5,2} + 1$;
$D(1; l_{5,1}, 0, 1)$;
$l_{5,1} > 0$ is integral and $l_{4,1} \geqq l_{5,1} + 1$; $l_{4,2} = 1$;
$D(e; l_{5,1}, 0, l_{5,3})$
$0 < l_{5,3} < 1$; $l_{5,1} > 0$ is integral; $l_{4,1} \geqq l_{5,1} + 1$;
$l_{5,1} \geqq l_{4,2} \geqq l_{5,2} + 1$.

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