

C^* -Algebras and Mackey's Axioms

ROGER J. PLYMEN

Mathematical Institute, University of Oxford

Received August 30, 1967

Abstract. A non-commutative version of probability theory is outlined, based on the concept of a Σ^* -algebra of operators (sequentially weakly closed C^* -algebra of operators). Using the theory of Σ^* -algebras, we relate the C^* -algebra approach to quantum mechanics as developed by KADISON with the probabilistic approach to quantum mechanics as axiomatized by MACKEY. The Σ^* -algebra approach to quantum mechanics includes the case of classical statistical mechanics; this important case is excluded by the W^* -algebra approach. By considering the Σ^* -algebra, rather than the von Neumann algebra, generated by the given C^* -algebra A in its reduced atomic representation, we show that a difficulty encountered by GUENIN concerning the domain of a state can be resolved.

1. Introduction

The C^* -algebra approach to quantum mechanics, inaugurated by SEGAL [17] and developed by KADISON [11], has received much attention since the publication of the paper by HAAG and KASTLER [8]. HAAG and KASTLER emphasized the importance of *abstract* C^* -algebras because of the physical equivalence of all faithful representations. On the other hand, the probabilistic formulation of quantum mechanics, begun by VON NEUMANN, has been clarified by MACKEY's axiomatization [13]. It is shown in this paper that an important class of abstract C^* -algebras, called Σ^* -algebras, lend themselves to a probabilistic formulation along such lines.

The key concept in KADISON's theory of quantum dynamics [11] is the *dynamical system* $(A, S_0, t \rightarrow V_t)$, where A is an *abstract* C^* -algebra, S_0 is a full family of states of A , and $t \rightarrow V_t$ is a weakly continuous, one-parameter group of automorphisms of S_0 (the *dynamical group*). In order to discuss bounded observables only, we strengthen slightly axioms 3,6 of MACKEY. We also drop axiom 7, which is ad hoc, and does not express the structural features of quantum mechanics in the same way as the other axioms. We call the modified set of eight axioms, obtained in this way, the *essential axioms of MACKEY* (see Section 3). A Σ^* -algebra of operators is a *sequentially* weakly closed C^* -algebra A of operators. That is, given any *sequence* $x_n \in A$ which converges to the bounded operator x

in the weak operator topology, we then have that $x \in A$. A state f of A such that $f(x_n) \rightarrow f(x)$ for all sequences such as x_n will be called a σ -state of A . Now Σ^* -algebras may be characterized abstractly (see Section 2). The theory of Σ^* -algebras and σ -states may be regarded as providing a basis for a non-commutative probability theory. The main result of this paper is

Theorem 1. *The dynamical system $(A, S_0, t \rightarrow V_t)$ satisfies the essential axioms of Mackey if*

- (i) *A is an abstract Σ^* -algebra,*
- (ii) *S_0 is the family of all σ -states of A .*

Let X be a topological space, and consider the Borel structure underlying the topology of X . The algebra $B\{X\}$ of all bounded Borel functions on X is a commutative Σ^* -algebra (see Section 2). The σ -states of $B\{X\}$ are precisely the probability Borel measures on X . In particular, let X be phase space of a classical statistical mechanical system. Then each real function in $B\{X\}$ represents a bounded observable, and each σ -state of $B\{X\}$ represents a physical state of the system. Thus Theorem 1 includes classical statistical mechanics in the case of bounded observables. The C^* -algebra $B\{X\}$ is in general not the dual of a Banach space, hence not a W^* -algebra [16]. Consequently the W^* -algebra approach to quantum mechanics *excludes* this important case.

Given an abstract C^* -algebra A , E. B. DAVIES [1] constructs a canonical Σ^* -algebra A^\sim containing A . Thus it is always possible to enlarge an abstract C^* -algebra A so that condition (i) of Theorem 1 is satisfied; and it is always possible to extend each state of A to a σ -state of A^\sim so that condition (ii) of Theorem 1 is satisfied. In the case where A is a separable commutative C^* -algebra with identity, A^\sim can be identified with the Σ^* -algebra $B\{X\}$ of all bounded Borel functions on the spectrum X of A [1].

In Section 2 we discuss Σ^* -algebras and sketch from first principles the spectral theory of self-adjoint elements in a Σ^* -algebra (see Lemma 2.5). In Section 3 we prove Theorem 1 and derive a corollary concerning W^* -algebras (Corollary 3.10). In Section 4 we discuss the τ -envelope A^\sim of a C^* -algebra A . By considering A^\sim rather than the weak closure of A in its *reduced atomic* representation, we show that a difficulty encountered by GUENIN [7] concerning the domain of a state of a C^* -algebra, can be completely resolved. The set of all projections in a Σ^* -algebra forms a σ -complete orthocomplemented lattice. For a thorough discussion of this fact and of Piron's axioms, the reader is referred to R. J. PLYMEN [15].

We should like to thank J. T. LEWIS, G. W. MACKEY and E. B. DAVIES for several very helpful discussions. We should particularly like to thank E. B. DAVIES for showing us the unpublished manuscript of [1].

2. On Σ^* -Algebras

For the general theory and notation concerning C^* -algebras, we shall make systematic use of DIXMIER's book [2]. Since it is no restriction to assume that the C^* -algebra A has an identity *we shall always assume our C^* -algebras have identities denoted by 1*. A state f of a C^* -algebra A is a linear functional on A such that $f(1) = 1$ and $f(x) \geq 0$ when $x \geq 0$. A state f of A has norm 1, and so lies in the continuous dual A^* of A . Let A be a C^* -algebra acting on the Hilbert space H , and let ξ be a unit vector in H . The state $x \rightarrow (x\xi, \xi)$ of A is called a vector state, and denoted ω_ξ . We denote by $\mathcal{B}(H)$ the C^* -algebra of all bounded operators on the Hilbert space H . We shall be concerned with the weak operator topology on $\mathcal{B}(H)$, the weakest topology on $\mathcal{B}(H)$ such that the mappings $x \rightarrow (x\xi, \xi)$ are continuous for each ξ in H . If $x_n \rightarrow x$ in the weak operator topology, we shall say $x_n \rightarrow x$ *weakly*.

Now let A be a C^* -algebra and denote by \mathcal{F} the set of all ordered pairs $\{x_n, x\}$ consisting of a sequence $x_n \in A$ and a point $x \in A$. If $\mathcal{G} \subseteq \mathcal{F}$ we denote by \mathcal{G}^σ the set of all states f on A such that for all $\{x_n, x\} \in \mathcal{G}$ we have $f(x_n) \rightarrow f(x)$.

Definition 2.1. A Σ^* -algebra A is a C^* -algebra together with a subset $\mathcal{G} \subseteq \mathcal{F}$, called the set of σ -convergent sequences in A and denoted $x_n \rightarrow x$, such that the following properties hold:

- (i) if $x_n \rightarrow x$ then there is a constant K such that for all n we have $\|x_n\| \leq K < \infty$;
 - (ii) if $x_n \rightarrow x$ and $y \in A$ then $x_n y \rightarrow xy$;
 - (iii) if $x_n \in A$ is a sequence such that $f(x_n)$ converges for all $f \in \mathcal{G}^\sigma$ then there is some $x \in A$ such that $x_n \rightarrow x$.
 - (iv) if $0 \neq x \in A$ then there is some $f \in \mathcal{G}^\sigma$ such that $f(x) \neq 0$.
- \mathcal{G}^σ is called the set of σ -states of the Σ^* -algebra A .

Example 2.2. A set A of bounded operators on the Hilbert space H shall be called σ -closed if given any sequence $x_n \in A$ which converges weakly to $x \in \mathcal{B}(H)$, we then have that $x \in A$. Given any set A there is a smallest σ -closed set containing it, which we call its σ -closure and denote by $\sigma(A)$. Let A be a sub- C^* -algebra of $\mathcal{B}(H)$ such that $A = \sigma(A)$. A becomes a Σ^* -algebra if we define the σ -convergent sequences to be the weakly convergent sequences. We call such algebras Σ^* -algebras of operators; clearly $\mathcal{B}(H)$ is itself a Σ^* -algebra of operators. By a σ -representation π of the Σ^* -algebra A on the Hilbert space H we shall mean a representation such that if $x_n \rightarrow x$ then $\pi(x_n) \rightarrow \pi(x)$. By a *faithful σ -representation* we shall mean a faithful representation such that $\pi(A)$ is σ -closed and $x_n \rightarrow x$ if and only if $\pi(x_n) \rightarrow \pi(x)$.

Lemma 2.3. Every Σ^* -algebra A has a faithful σ -representation as a Σ^* -algebra of operators on a Hilbert space.

The proof of this Lemma is in [1].

Example 2.4. Let X be a space with a given σ -ring of subsets. The space $B\{X\}$ of all bounded measurable functions on X is a commutative C^* -algebra in an obvious sense. We say that a sequence u_n in $B\{X\}$ is σ -convergent to u in $B\{X\}$ if and only if $\|u_n\| \leq K$ for some K and all n , and u_n also converges pointwise to u . Then $B\{X\}$ is a Σ^* -algebra; and the family of σ -states is exactly the set of probability measures on X . There is a discussion of this example in [1]. Note that $B\{X\}$ is in general not the dual of a Banach space, hence not a W^* -algebra [16].

Return to Example 2.2. Let A be a Σ^* -algebra of operators on the Hilbert space H .

The set S_0 of all σ -states of A contains the weakly continuous states of A . Now the weakly continuous states of the C^* -algebra A have the form $\lambda_1 \omega_{\xi_1} + \cdots + \lambda_n \omega_{\xi_n}$ where $0 \leq \lambda_j \leq 1$, ξ_j is a unit vector in H , and $\lambda_1 + \cdots + \lambda_n = 1$. Now S_0 is convex and $x \geq 0$ if $f(x) \geq 0$ for all f in S_0 ; hence S_0 is a full family of states of A [11]. The states weakly continuous on the unit ball of A have the form $\sum_{n=1}^{\infty} \lambda_n \omega_{\xi_n}$ where $\lambda_n \geq 0$, $\sum_{n=1}^{\infty} \lambda_n = 1$ and each ξ_n is a unit vector in H [3, p. 54]. Thus, each such state of A is a norm limit of weakly continuous states. Since S_0 is a norm closed subset of A^* (see Lemma 3.4), S_0 contains all the states weakly continuous on the unit ball of A .

We describe next the spectral theory of self-adjoint elements of a Σ^* -algebra A of operators on H . The concepts of Σ^* -algebra and σ -representation help to clarify this theory. Let x be a fixed self-adjoint element in A . Let Ω be the spectrum of x , $C(\Omega)$ the C^* -algebra of complex continuous functions on Ω . Recall that Ω is a compact separable metric space. Let $B\{\Omega\}$ be the Σ^* -algebra of complex bounded Borel functions on Ω (see Example 2.4). There exists a unique representation π of $C(\Omega)$ on H such that $\pi(1) = 1$, $\pi(\iota) = x$, where ι is the function $\lambda \rightarrow \lambda$ on Ω . This representation is faithful. Its image is the sub- C^* -algebra of A generated by 1 and x , hence is composed of normal elements [2, p. 10]. There exists a unique σ -representation π^\sim of $B\{\Omega\}$ on H such that π^\sim extends π . The image of π^\sim is a sub- C^* -algebra of A , and is composed of normal elements [1]. By writing $u(x) = \pi^\sim(u)$ we introduce the functional calculus of bounded Borel functions in A . Let χ_M be the characteristic function of the Borel subset M of Ω , and write $E^x(M) = \chi_M(x)$. Let (M_n) be a sequence of Borel subsets of Ω , disjoint in pairs, $M = \bigcup_{n=1}^{\infty} M_n$.

Then

$$\chi_{M_1} + \cdots + \chi_{M_n} \rightarrow \chi_M \quad \text{in the } \Sigma^*\text{-algebra } B\{\Omega\}.$$

Since π^\sim is a σ -representation,

$$\pi^\sim(\chi_{M_1}) + \cdots + \pi^\sim(\chi_{M_n}) \rightarrow \pi^\sim(\chi_M) \quad \text{weakly}.$$

Thus

$$E^x(M_1) + \cdots + E^x(M_n) \rightarrow E^x(M) \text{ weakly.}$$

Thus $M \rightarrow E^x(M)$ is a projection-valued measure in Ω , σ -additive in the weak operator topology. Let u lie in $B\{\Omega\}$. Given $\varepsilon > 0$, there exists a simple function $\lambda_1 \chi_{M_1} + \cdots + \lambda_n \chi_{M_n}$ in $B\{\Omega\}$ such that

$$\|u - \lambda_1 \chi_{M_1} - \cdots - \lambda_n \chi_{M_n}\| < \varepsilon.$$

Since $\|\pi^\sim(u)\| \leq \|u\|$ for all u in $B\{\Omega\}$,

$$\|u(x) - \lambda_1 E^x(M_1) - \cdots - \lambda_n E^x(M_n)\| < \varepsilon.$$

Thus $u(x) = \int u(\lambda) dE^x(\lambda)$ where the integral is defined in the norm topology. In particular $x = \int \lambda dE^x(\lambda)$ so that E^x is the *spectral measure* of X .

Let now E be a *compact real spectral measure* in A (i.e. E is a projection-valued measure, σ -additive in the weak operator topology, whose range lies in A , whose domain is the σ -ring of all Borel subsets of a compact subset of the real line). Then $\int \lambda dE(\lambda)$ exists. Let $x = \int \lambda dE(\lambda)$. Then x is the norm limit of linear combinations of projections $E(M)$. But A is norm closed, hence x lies in A . Moreover x is self-adjoint and $E = E^x$ [9; p. 65]. This establishes the following Lemma.

Lemma 2.5. *Let A be a Σ^* -algebra of operators. There is a canonical 1-1 mapping $E \rightarrow \int \lambda dE(\lambda)$ of the set of compact real spectral measures in A onto the set of self-adjoint elements in A .*

3. Mackey's Axioms

MACKEY develops Quantum Mechanics axiomatically as follows [13, chapter 2]. Let \mathcal{B} be the set of all Borel subsets of the real line R . We suppose we are given two abstract sets \mathcal{O} and \mathcal{S} and a function p which assigns a real number $p(x, f, M)$ in $0 \leq \lambda \leq 1$ to each triple x, f, M , where x is in \mathcal{O} , f is in \mathcal{S} , and M is in \mathcal{B} . We assume that p has certain properties which we list as axioms. Physically \mathcal{O} is to be thought of as the set of all observables of our system, and \mathcal{S} as the set of all states. $p(x, f, M)$ is the probability that a measurement in the state f of x will lead to a value in M .

Axiom 1. $p(x, f, \emptyset) = 0$, $p(x, f, R) = 1$

$$p(x, f, M_1 \cup M_2 \cup \dots) = \sum_{n=1}^{\infty} p(x, f, M_n)$$

whenever the M_n are Borel sets that are disjoint in pairs. Axiom 1 simply states that for each x in \mathcal{O} and each f in \mathcal{S} , $M \rightarrow p(x, f, M)$ is a probability measure.

Lemma 3.1. *Let A be a Σ^* -algebra of operators on H , f a σ -state of A , x a self-adjoint element of A . Then there exists a unique probability measure $M \rightarrow p(x, f, M)$ in the spectrum of x such that $f(x^n) = \int \lambda^n dp(x, f, \lambda)$ for all $n = 0, 1, 2, \dots$.*

Proof. Let Ω be the spectrum of x . Since x is self-adjoint, Ω is a compact subset of the real line. Let E^x be the compact real spectral measure of x , defined on the σ -ring of all Borel subsets of Ω . Let $p(x, f, M) = f(E^x(M))$. Let (M_n) be a sequence of Borel subsets of Ω , disjoint in pairs, $M = \bigcup_{n=1}^{\infty} M_n$. Then $E^x(M_1) + \dots + E^x(M_n) \rightarrow E^x(M)$ weakly. Since f is a σ -state, $f(E^x(M_1)) + \dots + f(E^x(M_n)) \rightarrow f(E^x(M))$. Thus $p(x, f, M) = \sum_{n=1}^{\infty} p(x, f, M_n)$. Thus $M \rightarrow p(x, f, M)$ is a probability Borel measure in Ω . Now $x^n = \int \lambda^n dE^x(\lambda)$ where the integral is defined in the norm topology on A . By norm continuity of f ,

$$f(x^n) = \int \lambda^n d(fE^x(\lambda)) = \int \lambda^n dp(x, f, \lambda).$$

To prove uniqueness, let p, p' be probability measures in the spectrum Ω of x such that $\int \lambda^n dp(\lambda) = \int \lambda^n dp'(\lambda)$ for all $n = 0, 1, 2, \dots$. By the Weierstrass approximation theorem for compact subsets of the real line, the set of real polynomials on Ω is norm dense in the set of real continuous functions on Ω . Hence, by linearity and continuity, $\int u(\lambda) dp(\lambda) = \int u(\lambda) dp'(\lambda)$ for all real continuous functions u on Ω . Therefore $p = p'$.

Axiom 2. *If $p(x, f, M) = p(x', f, M)$ for all f in \mathcal{S} and all M in \mathcal{B} then $x = x'$. Similarly if $p(x, f, M) = p(x, f', M)$ for all x in \mathcal{O} and all M in \mathcal{B} , then $f = f'$. Axiom 2 says that two states, to be different, must assign different probability distributions to at least one observable and that two observables, to be different, must have different probability distributions in at least one state.*

Lemma 3.2. *Let A be a Σ^* -algebra acting on H , S_0 the set of all σ -states of A .*

(i) *If $p(x, f, M) = p(x', f, M)$ for all f in S_0 and all M in \mathcal{B} then $x = x'$.*

(ii) *If $p(x, f, M) = p(x, f', M)$ for all self-adjoint x in A and all M in \mathcal{B} then $f = f'$.*

Proof. (i) By Lemma 3.1, $f(x) = \int \lambda dp(x, f, \lambda) = \int \lambda dp(x', f, \lambda) = f(x')$ hence $f(x - x') = 0$ for all f in S_0 . Now S_0 contains the vector states of A . Hence $x - x' = 0$.

(ii) $f(x) = f'(x)$ for all self-adjoint x in A . Hence $f(x) = f'(x)$ for all x in A . Thus $f = f'$.

Axiom 3 (modified). *Let x be any member of \mathcal{O} and let u be any real bounded Borel function on the line. Then there exists y in \mathcal{O} such that $p(y, f, M) = p(x, f, u^{-1}(M))$ for all f in \mathcal{S} and all M in \mathcal{B} .*

It follows from Axiom 2 that y is uniquely determined by x and we shall denote it by $u(x)$. Physically the observable $u(x)$ is constructed from x as follows. Whatever we do to measure x we measure $u(x)$ by applying the function u to the result of measuring x .

Lemma 3.3. *Let A be a Σ^* -algebra acting on H , x a self-adjoint element of A . Let u be a real bounded Borel function on the line. Then there exists a self-adjoint element y in A such that $p(y, f, M) = p(x, f, u^{-1}(M))$ for all σ -states f and all M in \mathcal{B} .*

Proof. Let $v = u^n$ with n a natural number. Then

$$f(v(x)) = f(\int v(\lambda) dE^x(\lambda)) = \int v(\lambda) d(fE^x(\lambda)) = \int v(\lambda) dp(x, f, \lambda)$$

by norm continuity of f . Thus

$$f(u^n(x)) = \int u(\lambda)^n dp(x, f, \lambda) = \int \lambda^n dp(x, f, u^{-1}(\lambda))$$

by change of variable. But $f(u^n(x)) = f(u(x)^n) = \int \lambda^n dp(u(x), f, \lambda)$. Therefore $\int \lambda^n dp(x, f, u^{-1}(\lambda)) = \int \lambda^n dp(u(x), f, \lambda)$ for all natural numbers n . By the uniqueness assertion in Lemma 3.1, $p(u(x), f, M) = p(x, f, u^{-1}(M))$ for each σ -state f and each M in \mathcal{B} . Since x is self-adjoint and u is real-valued, $u(x)$ is self-adjoint.

Axiom 4. *If f_1, f_2, \dots are members of \mathcal{S} and $\lambda_1 + \lambda_2 + \dots = 1$ where $0 \leq \lambda_n \leq 1$, then there exists f in \mathcal{S} such that $p(x, f, M) = \sum_{n=1}^{\infty} \lambda_n p(x, f_n, M)$ for all x in \mathcal{O} and all M in \mathcal{B} .*

It follows from Axiom 2 that f is uniquely determined by the f_n and λ_n . We denote it by $\sum_{n=1}^{\infty} \lambda_n f_n$. It corresponds physically to a state in which we know that we are in the state f_n with probability λ_n .

Lemma 3.4. *Let A be a Σ^* -algebra of operators on H , and let (f_n) be a sequence of σ -states of A . Let (λ_n) be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Then there exists a σ -state f such that $p(x, f, M) = \sum_{n=1}^{\infty} \lambda_n p(x, f_n, M)$ for all self-adjoint x in A and all M in \mathcal{B} .*

Proof. We prove that the set S_0 of all σ -states of A is a norm closed convex subset of A^* . Let (g_n) be a sequence in S_0 and let $g_n \rightarrow g$ in norm. Then $g_n(y) \rightarrow g(y)$ for each y in A whence g is a state of A . Let (x_j) be a weakly convergent sequence in A , $x_j \rightarrow x$, then x lies in A . We prove that (x_j) is uniformly bounded. Now $(x_j \xi, \eta) \rightarrow (x \xi, \eta)$ for each ξ, η in H . Fix ξ and consider the sequence of vectors $(x_j \xi)$. By the uniform boundedness theorem we have that the sequence of norms $(\|x_j \xi\|)$ is bounded. This is true for each ξ in H . Again by the uniform boundedness theorem we have that the sequence of norms $(\|x_j\|)$ is bounded. Now

$$\begin{aligned} |g(x_j) - g(x)| &\leq |g(x_j) - g_n(x_j)| + |g_n(x_j) - g_n(x)| \\ &\quad + |g_n(x) - g(x)| \\ &\leq \|g - g_n\| K + |g_n(x_j) - g_n(x)| \end{aligned}$$

since the sequence (x_j) is uniformly bounded. Choose n such that $\|g - g_n\| K < \varepsilon/2$. Since g_n is a σ -state, there exists j_0 such that $|g_n(x_j) - g_n(x)| < \varepsilon/2$ whenever $j > j_0$. Hence $|g(x_j) - g(x)| < \varepsilon$ whenever $j > j_0$. Thus g is a σ -state. Hence S_0 is norm closed.

Return to the Lemma. If $\lambda_n = 0$ for all but finitely many n then $\sum_{n=1}^{\infty} \lambda_n f_n$ lies in S_0 since S_0 is convex. Now consider $\lambda_n > 0$ for all n . Let $g_n = (\lambda_1 f_1 + \cdots + \lambda_n f_n) / (\lambda_1 + \cdots + \lambda_n)$. Then (g_n) is a Cauchy sequence in S_0 . S_0 is complete, hence $g_n \rightarrow g$ in norm, where g lies in S_0 . But $g = \sum_{n=1}^{\infty} \lambda_n f_n$.

Finally, let x be a self-adjoint element of A and put $y = E^x(M)$ for some M in \mathcal{B} . Then $\sum_{n=1}^{\infty} \lambda_n f_n(y) = g(y)$, hence $\sum_{n=1}^{\infty} \lambda_n f_n(E^x(M)) = g(E^x(M))$. Thus

$$\sum_{n=1}^{\infty} \lambda_n p(x, f_n, M) = p(x, g, M)$$

for all self-adjoint x in A and all M in \mathcal{B} .

Recall that MACKEY begins with an abstract set \mathcal{O} of observables and an abstract set \mathcal{S} of states. We realize \mathcal{O} as the set of self-adjoint elements of a Σ^* -algebra A , and \mathcal{S} as the family of all σ -states of A . With this realization, some concepts defined by MACKEY become well-known concepts in the theory of operator algebras.

(i) A question e is an observable in \mathcal{O} such that $p(e, f, \{0, 1\}) = 1$ for all f in \mathcal{S} . e is a question if and only if $e^2 = e$. Thus a question in \mathcal{O} is realized as a self-adjoint idempotent, i.e. a projection, in the Σ^* -algebra A . The set \mathcal{Q} of all questions is realized as the set $\mathcal{L}(A)$ of all projections in A .

(ii) Let e lie in \mathcal{Q} and let f lie in \mathcal{S} . Let us put $m_f(e) = p(e, f, \{1\})$. In our model, $m_f(e) = p(e, f, \{1\}) = \int \lambda \, d p(e, f, \lambda) = f(e)$ hence $m_f(e)$ is realized as the expectation value of e in the state f .

(iii) The set S_0 of all σ -states of the Σ^* -algebra A contains all the vector states of A . But the vector states define positivity. Thus, with e, e' in $\mathcal{L}(A)$, $e \leq e'$ if and only if $f(e) \leq f(e')$ for all f in S_0 . Hence the partial ordering defined by MACKEY on \mathcal{Q} is realized as the partial ordering by positivity on $\mathcal{L}(A)$.

(iv) Let e, e' be in \mathcal{Q} and $e \leq 1 - e'$. MACKEY calls e, e' disjoint questions. In our model, $e \leq 1 - e'$ implies $e(1 - e') = e$ hence $ee' = 0$. Thus disjoint questions in \mathcal{Q} are realized as orthogonal projections in $\mathcal{L}(A)$.

Remark 3.5. $\mathcal{L}(A)$ is actually a σ -complete orthocomplemented lattice (each sequence in $\mathcal{L}(A)$ has a least upper bound and a greatest lower bound in $\mathcal{L}(A)$). But the lattice $\mathcal{L}(A)$ of all projections in the Σ^* -algebra

A is in general neither *complete* or *atomic*. For a full discussion of these facts, and for a full consideration of PIRON's lattice-theoretical axioms for quantum mechanics, the reader is referred to [15]. We prove here that $\mathcal{L}(A)$ is a lattice. Pass to a faithful σ -representation of A as a Σ^* -algebra of operators on the Hilbert space H . Let e_1, e_2 lie in $\mathcal{L}(A)$. Since $\|e_1 e_2 e_1\| \leq 1$, the powers of $e_1 e_2 e_1$ form a decreasing sequence of positive operators. It follows that $(e_1 e_2 e_1)^n$ is weakly convergent to, say, e . Since A is sequentially weakly closed, e lies in A . It is easy to verify that e is a projection and that e is the greatest lower bound $e_1 \wedge e_2$ of e_1 and e_2 . The least upper bound $e_1 \vee e_2$ of e_1 and e_2 is $1 - ((1 - e_1) \wedge (1 - e_2))$. Thus $\mathcal{L}(A)$ is a lattice.

Let (e_n) be a sequence in \mathcal{Q} such that e_m, e_n are disjoint whenever $m \neq n$. If e in \mathcal{Q} satisfies $m_f(e) = \sum_{n=1}^{\infty} m_f(e_n)$ for all f in \mathcal{S} then MACKAY writes $e = \sum_{n=1}^{\infty} e_n$.

Axiom 5. Let (e_n) be any sequence of questions such that e_m, e_n are disjoint whenever $m \neq n$. Then the question $\sum_{n=1}^{\infty} e_n$ exists.

Lemma 3.6. Let (e_n) be any sequence of projections in the Σ^* -algebra A such that e_m, e_n are orthogonal whenever $m \neq n$. Then there exists a projection e in A such that $f(e) = \sum_{n=1}^{\infty} f(e_n)$ for all σ -states f .

Proof. Let the Σ^* -algebra A act on H , and consider the complete lattice of all projections in $\mathcal{B}(H)$. Let e be the least upper bound, taken in this lattice, of the sequence (e_n) of projections. Then $e_1 + \cdots + e_n \rightarrow e$ weakly [9, p. 49]. Since A is a Σ^* -algebra, e lies in A . If f is a σ -state of A then $f(e_1) + \cdots + f(e_n) \rightarrow f(e)$. This completes the proof.

Let E be a function from \mathcal{B} to \mathcal{Q} such that

- (i) $M \cap N = \emptyset$ implies $E(M), E(N)$ disjoint
- (ii) $M_m \cap M_n = \emptyset$ for $m \neq n$ implies $E\left(\bigcup_{n=1}^{\infty} M_n\right) = \sum_{n=1}^{\infty} E(M_n)$,
- (iii) $E(\emptyset) = 0, E(R) = 1$.

Then E is called a *question-valued measure*. If E has compact support then we call E a *compact question-valued measure*.

Axiom 6 (modified). If E is any compact, question-valued measure then there exists an observable x in \mathcal{O} such that $\chi_M(x) = E(M)$ for all M in \mathcal{B} , where χ_M is the characteristic function of M .

It is easy to verify that a compact question-valued measure E is realized, in the Σ^* -algebra model, as a compact projection-valued measure, σ -additive in the weak operator topology. Thus E is realized as a compact real spectral measure in A .

Lemma 3.7. *Let E be a compact real spectral measure in the Σ^* -algebra A . Then there exists a self-adjoint element x in A such that $\chi_M(x) = E(M)$ for all M in \mathcal{B} .*

Proof. Let $x = \int \lambda dE(\lambda)$. By Lemma 2.5, x is a self-adjoint element in A and $E = E^x$. Thus $\chi_M(x) = E^x(M) = E(M)$ for all M in \mathcal{B} . This completes the proof.

For reference, we include Axiom 7.

Axiom 7. *The partially ordered set \mathcal{Q} of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a separable, infinite-dimensional complex Hilbert space.*

Axiom 8. *If e is any question different from 0 then there exists a state f in \mathcal{S} such that $m_f(e) = 1$.*

Lemma 3.8. *Let e be a non-zero projection in the Σ^* -algebra A . Then there exists a σ -state f of A such that $f(e) = 1$.*

Proof. Let A act on H . The range eH of e is non-zero. Let ξ be a unit vector in eH . Let $f = \omega_\xi: x \rightarrow (x\xi, \xi)$. Then $f(e) = (e\xi, \xi) = (\xi, \xi) = 1$, and the vector state f is a σ -state.

In order to formulate quantum dynamics, MACKEY considers a one-parameter group $t \rightarrow V_t$ of transformations of \mathcal{S} onto \mathcal{S} .

Axiom 9. *For each sequence (f_n) of members of \mathcal{S} and each sequence (λ_n) of non-negative real numbers whose sum is 1, $V_t \left(\sum_{n=1}^{\infty} \lambda_n f_n \right) = \sum_{n=1}^{\infty} \lambda_n V_t(f_n)$ for all $t \geq 0$; and for all x in \mathcal{O} , f in \mathcal{S} , and M in \mathcal{B} , $t \rightarrow p(x, V_t(f), M)$ is continuous.*

If the state of a physical system is f at time t_1 , then it is $V_{t_2-t_1}(f)$ at time $t_2 \geq t_1$. Axiom 9 asserts that each V_t preserves convex combinations of states; and for each fixed triple x, f, M , the probability $p(x, f, M)$ changes only slightly in a short time interval.

Lemma 3.9. *Let $(A, S_0, t \rightarrow V_t)$ be a dynamical system, where A is a Σ^* -algebra of operators, and S_0 is the family of all σ -states of A . Let (f_n) be a sequence in S_0 . Let (λ_n) be a sequence of non-negative real numbers whose sum is 1. Then*

(i) *$t \rightarrow p(x, V_t(f), M)$ is continuous for each self-adjoint x in A , each f in S_0 , each M in \mathcal{B} .*

(ii) *$V_t \left(\sum_{n=1}^{\infty} \lambda_n f_n \right) = \sum_{n=1}^{\infty} \lambda_n V_t(f_n)$ for all real t .*

Proof. (i) Since the mapping $t \rightarrow V_t$ is weakly continuous, $t \rightarrow (V_t(f))y$ is continuous for each f in S_0 and each y in A . Let x be a self-adjoint element of A , and choose $y = \chi_M(x)$. Now $(V_t(f)) \chi_M(x) = p(x, V_t(f), M)$ hence $t \rightarrow p(x, V_t(f), M)$ is continuous for each self-adjoint x in A , each f in S_0 , each M in \mathcal{B} .

(ii) Consider a single element V of the dynamical group. V is an affine w^* isomorphism of the full family S_0 . V is a continuous mapping of S_0 onto S_0 , when S_0 has the topology induced by the w^* topology on the dual A^* of A . Firstly, let $\lambda_n = 0$ for all but finitely many n . Then (ii) follows since V is affine. Secondly, let $\lambda_n > 0$ for all n . Let $g_n = \lambda_1 f_1 + \cdots + \lambda_n f_n$, $h_n = \lambda_1 V(f_1) + \cdots + \lambda_n V(f_n)$. Then $g_n \rightarrow f$ in norm with f in S_0 . Hence $g_n/(\lambda_1 + \cdots + \lambda_n) \rightarrow f$ in norm and in the w^* topology. Hence $V(g_n/(\lambda_1 + \cdots + \lambda_n)) \rightarrow V(f)$ in the w^* topology, since V is w^* continuous. Hence $h_n/(\lambda_1 + \cdots + \lambda_n) \rightarrow V(f)$ in the w^* topology, since V is affine. Let $h_n \rightarrow h$ in norm with h in S_0 . Then $h_n/(\lambda_1 + \cdots + \lambda_n) \rightarrow h$ in norm and in the w^* topology. Therefore $V(f) = h$ since S_0 is w^* Hausdorff. Thus $h_n \rightarrow V(f)$ in norm. That is, $\sum_{n=1}^{\infty} \lambda_n V(f_n) = V\left(\sum_{n=1}^{\infty} \lambda_n f_n\right)$, where the convergence on both sides is in the norm topology.

Theorem 1 is a consequence of Lemma 2.3 together with the preceding eight Lemmas.

Each von Neumann algebra A is a Σ^* -algebra of operators. Now each norm bounded increasing net x_α in A has a least upper bound x in A . States f of A such that $f(x) = \text{lub} f(x_\alpha)$ for all such nets x_α are called *normal states* of A . The normal states of A are precisely those of the form $\sum_{n=1}^{\infty} \lambda_n \omega_{\xi_n}$, where λ_n is a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$, and ξ_n is a sequence of unit vectors in the Hilbert space on which A acts [3, p. 54]. Thus each normal state of A is a σ -state of A . Now von Neumann algebras may be characterized abstractly [10, 16]. SAKAI's characterization is as follows: An abstract C^* -algebra A has a faithful representation as a von Neumann algebra if and only if A is the dual of a Banach space. Such abstract C^* -algebras are called *W^* -algebras*. Now the set of all normal states of A is a norm closed convex subset of A^* [2, p. 337]. Recalling the proofs of Lemmas 3.1 to 3.9 (especially 3.4), we obtain the following Corollary to Theorem 1.

Corollary 3.10. *The dynamical system $(A, S_0, t \rightarrow V_t)$ satisfies the essential axioms of MACKAY if*

- (i) *A is an abstract W^* -algebra,*
- (ii) *S_0 is the family of all normal states of A .*

If f is a σ -state of a von Neumann algebra which is *countably decomposable* (in particular for each von Neumann algebra acting on a separable Hilbert space) then f is normal. This follows from DIXMIER [3, p. 65, Exercise 9].

In this special case, the normal states are precisely the σ -states. In general, the author expects that the family of all normal states of a W^* -algebra A is strictly smaller than the family of all σ -states of A .

Now the normal states of $\mathcal{B}(H)$ are precisely those of the form

$$\sum_{n=1}^{\infty} \lambda_n \omega_{\xi_n}$$

where λ_n is a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$ and ω_{ξ_n} is a sequence of vector states of $\mathcal{B}(H)$ [3; p. 54]. Thus, when H is separable, the normal states ($= \sigma$ -states) of $\mathcal{B}(H)$ are in 1-1 correspondence with the *von Neumann density matrices* [13, p. 80].

Remark 3.11. Recall Lemma 3.8. Let H be a fixed arbitrary Hilbert space. Let e be a nonzero projection in the von Neumann algebra $\mathcal{B}(H)$. If ξ is a unit vector in the range of e then the vector state ω_{ξ} is normal, pure, and such that $\omega_{\xi}(e) = 1$. Thus in the special case when $A = \mathcal{B}(H)$, there exists a *pure*, normal state f of A such that $f(e) = 1$. This is in general false. For let A be a type II_1 factor. Then A possesses no pure, normal states [4, p. 273]. Each vector state of $\mathcal{B}(H)$ is pure, though not every pure state of $\mathcal{B}(H)$ is a vector state [17; p. 944]. The extreme points of the convex set S_0 of all normal states of $\mathcal{B}(H)$ are precisely the vector states of $\mathcal{B}(H)$ [2, p. 84].

4. σ -Envelope of a C^* -Algebra

HAAG and KASTLER [8] argue that, in the C^* -algebra approach to quantum mechanics, the abstract C^* -algebra A is the relevant object, and not one of its faithful representations. STREATER [18] uses an abstract uniformly-hyperfinite C^* -algebra to discuss the infinite ferromagnetic lattice. KADISON [11] formalizes a concept of representation of a physical system (A, S_0) and of a dynamical system $(A, S_0, t \rightarrow V_t)$ and emphasizes considering an abstract physical system as independent of its specific representations. It is for such reasons that we have stated Theorem 1 and Corollary 3.10 in terms of abstract Σ^* -algebras and abstract W^* -algebras.

In the following argument, A denotes a fixed abstract C^* -algebra. We consider the following problem, raised by G. W. MACKEY in conversation: Find a canonical C^* -algebra B containing A such that when A is commutative and separable, B determines and is determined by the Borel structure underlying the topology on the spectrum \hat{A} of A . The point is that in classical statistical mechanics, the set \mathcal{O} of observables determines the Borel structure underlying the topology of phase space. For the Borel subsets of phase space may be identified with the *questions* in \mathcal{O} . We are able to give in this section a complete solution to this problem. But we first consider two candidates for B .

(i) Let Q denote the set of all positive linear functionals on A , and let $\phi = \bigoplus_{f \in Q} \pi_f$, where π_f is the canonical cyclic representation defined by f . Then ϕ is called the *universal representation* of A . Consider the von Neumann algebra $\phi(A)^-$ generated by $\phi(A)$. Then $\phi(A)^-$ and the second dual A^{**} of A are isomorphic as Banach spaces [2, p. 236]. We embed A in A^{**} and identify $\phi(A)^-$ with A^{**} . Then A^{**} is a W^* -algebra containing A , but when A is commutative and separable, A^{**} seems to the author to be too large. Note that each state of A has a unique extension to a normal state of A^{**} , and that each normal state of A^{**} arises in this way.

(ii) Following [2], we define the spectrum \hat{A} of A as the set of unitary equivalence classes of irreducible representations of A . The *reduced atomic representation* ψ [6, 12] of A is defined as the direct sum of the irreducible representations ψ_α of A , taking one from each unitary equivalence class. By abuse of notation, $\psi = \bigoplus_{\alpha \in \hat{A}} \psi_\alpha$. Let H_α be the representation space of ψ_α . Then we know that $\psi(A)^- = \bigoplus_{\alpha \in \hat{A}} \mathcal{B}(H_\alpha)$ [6, p. 549]. If we identify A with $\psi(A)$ then $\psi(A)^-$ is a type I W^* -algebra containing A . When A is commutative, $\mathcal{B}(H_\alpha) = C_\alpha$, the algebra of complex numbers. Hence $\psi(A)^- = \bigoplus_{\alpha \in \hat{A}} C_\alpha = W^*$ -algebra of all *bounded* complex functions on \hat{A} . Thus we lose *all* the structure of \hat{A} , a compact Hausdorff space, and we cannot recover the Borel structure underlying the topology of A . This event has the following interesting consequence.

When A is commutative, the projections in $\psi(A)^-$ are precisely the characteristic functions of all *subsets* of \hat{A} . Thus a σ -state on $\psi(A)^-$ induces a probability measure on \hat{A} such that every subset of \hat{A} is measurable. This excludes Lebesgue measure on the unit interval. Now GUENIN [7], working from the lattice-theoretic approach to quantum mechanics, considers a W^* -algebra of the form $\bigoplus \mathcal{B}(H_\alpha)$, where the H_α are separable Hilbert spaces. We can recover this W^* -algebra from our own considerations by requiring that A be separable. Then $\psi(A)^- = \bigoplus_{\alpha \in \hat{A}} \mathcal{B}(H_\alpha)$ and each H_α is separable. Yet GUENIN cannot define a state as a probability measure on the set of *all* projections in $\bigoplus \mathcal{B}(H_\alpha)$, for he would then encounter the difficulty described above, when the H_α are a family of 1-dimensional Hilbert spaces indexed by the unit interval [7, p. 275]. To circumvent this difficulty, GUENIN chooses to *redefine* a state as a probability measure on a *sublattice* of the lattice of all projections in $\bigoplus \mathcal{B}(H_\alpha)$. But the difficulty can be completely *avoided* if we consider, not $\psi(A)^-$, but the σ -envelope A^\sim described in (iii) of this Section.

By construction of ψ , each pure state of A has a unique extension to a normal state of $\psi(A)^-$. Note that ψ is a multiplicity-free representation, and that ψ is a subrepresentation of ϕ .

(iii) Let A^\sim be the σ -closure of A in its universal representation. A^\sim is called the σ -envelope of A . A^\sim is isomorphic, as a Σ^* -algebra, with the σ -closure of A in its reduced atomic representation. Note that each state of A has a unique extension to a σ -state of A^\sim , and each σ -state of A^\sim arises in this way [1].

Let A be commutative and separable. Then A^\sim may be identified with the Σ^* -algebra of all bounded Borel functions on the spectrum \hat{A} of A [1]. Now the projections in A^\sim are precisely the characteristic functions of Borel subsets of \hat{A} . Thus a σ -state on A^\sim induces a probability Borel measure on \hat{A} . Thus we completely avoid the difficulty which GUENIN encountered. A^\sim determines and is determined by the Borel structure underlying the topology of \hat{A} , for the set $\mathcal{L}(A^\sim)$ of projections in A^\sim may be identified with the set of Borel subsets of \hat{A} . The Borel structure underlying the topology of \hat{A} is a standard Borel space [2, p. 357]. Hence, if \hat{A} is uncountable, the Borel structure underlying the topology of \hat{A} is isomorphic, as a Borel space, with the Borel structure underlying the usual topology of the unit interval $[0, 1]$ [2, p. 357].

Consider finally the general case, A a fixed C^* -algebra. Then $\mathcal{L}(A^\sim)$ is in general not a complete lattice, hence fails to satisfy Piron's lattice-theoretic axioms [14]. But in [15] the author proposes to weaken slightly one of Piron's axioms, replacing completeness by σ -completeness. The resulting set of five axioms we call the *essential axioms of PIRON*. Indeed if A is a separable type I C^* -algebra, we establish in [15] that $\mathcal{L}(A^\sim)$ satisfies the essential axioms of PIRON. There is a full discussion of these matters in [15].

References

1. DAVIES, E. B.: On the Borel structure of C^* -algebras. Commun. Math. Phys. 8, 147—163 (1968).
2. DIXMIER, J.: Les C^* -algèbres et leurs représentations. Paris: Gauthier-Villars 1964.
3. — Les algèbres d'opérateurs dans l'espace hilbertien. Paris: Gauthier-Villars 1957.
4. DYE, H. A.: The Radon-Nikodym theorem. Trans. Am. Math. Soc. 72, 243—280 (1952).
5. GLEASON, A. M.: Measures on the closed subspaces of a Hilbert space. J. Math. Mech. 6, 885—893 (1957).
6. GLIMM, J., and R. V. KADISON: Unitary operators in C^* -algebras. Pacific J. Math. 10, 547—556 (1960).
7. GUENIN, M.: Axiomatic foundations of quantum theories. J. Math. Phys. 7, 271—282 (1966).
8. HAAG, R., and D. KASTLER: An algebraic approach to quantum field theory. J. Math. Phys. 5, 848—861 (1964).
9. HALMOS, P.: Introduction to Hilbert space. New York: Chelsea publishing company 1957.
10. KADISON, R. V.: Operator algebras with a faithful weakly-closed representation. Ann. Math. 64, 175—181 (1956).

11. KADISON, R. V.: Transformations of states in operator theory and dynamics. *Topology* **3**, suppl. 2, 177—198 (1965).
12. —, and J. R. RINGROSE: Derivations and automorphisms of operator algebras. *Commun. Math. Phys.* **4**, 32—63 (1967).
13. MACKEY, G. W.: *Mathematical foundations of quantum mechanics*. New York: Benjamin 1963.
14. PIRON, C.: Axiomatique quantique. *Helv. Phys. Acta* **37**, 439—468 (1964).
15. PLYMEN, R. J.: A modification of Piron's axioms. *Helv. Phys. Acta* **41**, 69—74 (1968).
16. SAKAI, S.: A characterization of W^* -algebras. *Pacific J. Math.* **6**, 763—773 (1956).
17. SEGAL, I. E.: Postulates for general quantum mechanics. *Ann. Math.* **48**, 930—948 (1947).
18. STREATER, R. F.: The Heisenberg ferromagnet as a quantum field theory. *Commun. Math. Phys.* **6**, 233—247 (1967).

DR. R. J. PLYMEN

Present address:

Department of Mathematics

The University

Manchester 13, Great Britain