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Quantum Field Model with Unrenormalizable Interaction

M. K. Volkov
Joint Institute for Nuclear Research, Dubna

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Abstract. The unitary relativistic model of quantum field theory with rapidly increasing spectral function (i.e. it grows faster than any finite power of momentum) is investigated. It is shown that there exist nontrivial Lagrangians, leading to this kind of spectral functions and allowing to construct the local theory without the ultraviolet divergences on their basis. In this theory the S-matrix is unitary and not e qual identically to unity.

1. Introduction

The problem arising through the attempts to construct finite unrenormalizable theory, as well as through ascertaining the connection among the nonlocal theories and the unrenormalizable theories have attracted the attention of many authors [1—13, 20]. Some years ago one supposed that the unrenormalizable theories were nonlocal. But recently one discovered that there exist some region where the field theories with rapidly increasing spectral functions were local [12—13]. Also at the same time construction of the finite unrenormalizable field theories was attempted at [1, 6, 7, 9]. At present there are yet many uncertainties in these questions.

Due to the big complication of these problems it is interesting to consider a simple model in order to explain some general properties of the unrenormalizable theories.

We investigate here a model of the quantum field theory with the Lagrangian [14—16]

$$L(x) = L_0(x) + L_{\text{int}}(x)\,, \tag{1.1}$$

where $L_0(x)$ is the Lagrangian of the free fields and

$$L_{\text{int}}(x) = -g :\bar{\psi}(x)\,\tau_1\,\gamma_\nu\,\psi(x)\,\partial_\nu\,\varphi(x): - \Delta m :\bar{\psi}(x)\,\tau_3\,\psi(x):\,. \tag{1.2}$$

Here τ_1 and τ_3 are the isotopic spin matrices, γ_ν are the Dirac matrices, $\psi(x)$ is the spinor field operator, and $\varphi(x)$ is the scalar field operator.

The Lagrangian (1.1) has the following remarkable property: when $\Delta m = 0$, it is reduced to the diagonal form

$$L_{\Delta m = 0}(x) = L_0(\psi'(x),\, \varphi(x)) \tag{1.3}$$

by means of the unitary transformation

$$\psi'(x) = \psi(x)\exp\{ig\tau_1\,\varphi(x)\}\,. \tag{1.4}$$

The part of the Lagrangian containing Δm takes on the form of the essentially non-linear interaction in the field $\varphi(x)$ after the transformation (1.4). Thus,

$$L_{\mathrm{int}}(x) = \Delta m : \bar{\psi}'(x)\, \tau_3 \exp\{-i\, 2g\tau_1\, \varphi(x)\}\, \psi'(x) : \tag{1.5}$$

The sign of the normal product is not ascribed to the operators $\varphi(x)$. So we get the theory with non-polynomial interaction in the field $\varphi(x)$. As a result, there appears a rapidly increasing spectral function.

As the Green functions and the scattering amplitudes in that theory have essential singularity, so the principial problem concentrates on the construction of the Fourier transforms of these functions and on the definition of the integrals of their products in higher orders of the perturbation theory.

It will be shown that the ultraviolet divergences are absent in the model and the unitarity, locality and causality conditions are fulfilled.

2. Scalar Particles Scattering Amplitude

(Second Order of the Perturbation Theory)

To avoid the appearance of infinite factors in calculation of the physical quantities, we shall suppose that the sign of the normal product is ascribed to all operators in (1.5). The scalar particles scattering amplitude is obtained by the functional integration method [14, 15]

$$\begin{aligned} &f(p', q' \mid p, q) \\ &= \Pi(s) + \Pi(t) + \Pi(u) + \Pi(0) - \Pi(p^2) - \Pi(q^2) - \Pi(p'^2) - \Pi(q'^2) \end{aligned} \tag{2.1}$$

where

$$\Pi(p^2) = i\, 8g^4\, (\Delta m)^2 \int d^4x\, Sp\{S^c(x)\, S^c(-x)\} \exp i\{px - (2g)^2 \Delta^c(x)\} \tag{2.2}$$

$$S^c(x-x') = i\langle T(\psi(x)\, \bar{\psi}(x'))\rangle_0; \quad \Delta^c(x-x') = i\langle T(\varphi(x)\, \varphi(x'))\rangle_0 \tag{2.3}$$

and $s = (p+q)^2$, $t = (p-p')^2$, $u = (p-q')^2$.

In the case of massless particles, we have

$$\Pi(p^2) = -8(4\varkappa \Delta m)^2 F(p^2) \tag{2.4}$$

where

$$F(p^2) = i \int d^3\mathrm{x} \int_R dx_0\, e^{ipx} \frac{\exp\left\{-\dfrac{4\varkappa}{x^2}\right\}}{(x^2)^3}. \tag{2.5}$$

$\varkappa = \left(\dfrac{g}{2\pi}\right)^2$ and the contour R is shown in Fig. 1. Notice that if we define (2.5) with the help of its power series expansion in $\varkappa$ we get the usual perturbation theory with simple pole singularities in each term of the series. That is why we will operate with (2.5) as a whole.

Let us introduce an intermediate regularization in (2.5). We will define $F(p^2)$ as a limit of the following expression

$$F(p^2) = \lim_{\delta\to 0}\{\alpha F_\delta^{(1)}(p^2) + \beta F_\delta^{(1)}(p^2)\} \tag{2.6}$$

$$\alpha + \beta = 1 \quad \text{and} \quad \operatorname{Re}(\alpha - \beta) = 0 \, . \tag{2.7}$$

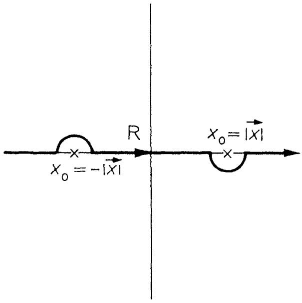

Fig. 1

The second equation in (2.7) is the unitarity condition. Taking into account (2.7) we have

$$F(p^2) = \frac{1}{2}\lim_{\delta\to 0}\{F_\delta^{(1)}(p^2) + F_\delta^{(2)}(p^2) + ia[F_\delta^{(1)}(p^2) - F_\delta^{(2)}(p^2)]\} \tag{2.8}$$

where a is some arbitrary real constant. $F_\delta^{\binom{1}{2}}(p^2)$ is as follows

$$F_\delta^{\binom{1}{2}}(p^2) = i\int d^3\mathbf{x}\int_R dx_0\, e^{ipx}\frac{\exp\left\{-\dfrac{4\varkappa}{x^2 \pm i\delta}\right\}}{(x^2 \pm i\delta)^3} \, . \tag{2.9}$$

Here δ is smaller than the radii of the semicircles in the contour R.

Let us calculate these integrals in an unphysical domain $p^2 < 0$. The result can be easily analytically continued on the whole domain p. In the unphysical domain we choose the coordinate system $p = \{0, \mathbf{p}\}$. In that coordinate system we turn the contour R through the angle $-\pi/2$ so that it completely coincides with the imaginary axis. In the obtained Euclidean space we can easily calculate the integrals over angles. Then we get

$$F_\delta^{\binom{1}{2}}(p^2) = -\frac{2\pi^2}{p}\int_0^\infty d\lambda\sqrt{\lambda}J_1(p\sqrt{\lambda})\frac{\exp\left\{\dfrac{4\varkappa}{\lambda \mp i\delta}\right\}}{(\lambda \mp i\delta)^3} \tag{2.10}$$

where $p = \sqrt{-p^2} = \sqrt{\mathbf{p}^2}$ and $J_1(p\sqrt{\lambda})$ is the Bessel function. Making use of the Mellin-Barnes integral representation for $J_1(p\sqrt{\lambda})$ [17]

$$J_1(p\sqrt{\lambda}) = i\frac{p\sqrt{\lambda}}{4}\int_{-\alpha-i\infty}^{-\alpha+i\infty} dz\frac{\left(\dfrac{\lambda\mathbf{p}^2}{4}\right)^z}{\sin\pi z\,\Gamma(1+z)\,\Gamma(2+z)} \cdot (0 < \alpha < 1) \tag{2.11}$$

(here $\Gamma(z)$ is the gamma-function) and taking into account the absolute convergence of the integrals, we can rewrite (2.10) in the following form

$$F_\delta^{\binom{1}{2}}(p^2) = i\frac{\pi^2}{8}\int\limits_{-\alpha+i\infty}^{-\alpha-i\infty} dz \frac{(\mathrm{p}^2)^z}{\sin\pi z\,\Gamma(1+z)\,\Gamma(2+z)}\int\limits_0^\infty dt\, t^{1+z}\frac{\exp\left\{\dfrac{\varkappa}{t\mp i\delta}\right\}}{(t\mp i\delta)^3}$$
$$= \int\limits_{-\alpha+i\infty}^{-\alpha-i\infty} dz\,\chi_p(z)\, f_\delta^{\binom{1}{2}}(z)\,. \quad \left(t=\frac{\lambda}{4}\right) \tag{2.12}$$

The function $f_\delta^{\binom{1}{2}}(z)$ corresponds to the integral over variable t, and is defined in the region $-2 < \operatorname{Re} z < 1$

$$f_\delta^{(2)}(z) = \int\limits_0^\infty dt\, t^{1+z}\frac{\exp\left\{\dfrac{\varkappa}{t+i\delta}\right\}}{(t+i\delta)^3}\,. \tag{2.13}$$

The integrand has a cut along the negative real axis, an essential singularity on the negative imaginary axis and tends to zero when $|t|\to\infty$. Therefore, we can perform rotation of the integration contour so that the minus sign appears in the exponential of the integrand (turned through the angle $+\pi$) and then we put δ to zero. The resulting integral is easily calculated:

$$\lim_{\delta\to 0} f_\delta^{(2)}(z) = -e^{i\pi z}\int\limits_0^\infty dt\, t^{z-2}e^{-\varkappa/t} = -e^{i\pi z}\varkappa^{z-1}\Gamma(1-z) = f^{(2)}(z)\,. \tag{2.14}$$

The function $f_\delta^{(2)}(z)$ can be analytically continued throughout the whole right half z-plane, with the exception of the positive real axis, where it possesses poles. In that region also we can apply the above method of taking the limit $\delta\to 0$ and obtain always the same function $f^{(2)}(z)$.

So we obtain the following prescription to find a limit of $F_\delta^{(2)}(p^2)$ at $\delta\to\infty$. Defining $f_\delta^{(2)}(z)$ as an analytic function on the whole right half z-plane, with the exception of the positive real axis, we deform the contour in (2.12) so that it passes around real positive axis (see Fig. 2). After that we can put δ to zero.

$$F_{\delta=0}^{(2)}(p^2) = \lim_{\delta\to 0}\int\limits_{-\alpha+i\infty}^{-\alpha-i\infty} dz\,\chi_p(z)\, f_\delta^{(2)}(z) = \lim_{\delta\to 0}\int\limits_L dz\,\chi_p(z)\, f_\delta^{(2)}(z)$$
$$= \int\limits_L dz\,\chi_p(z)\, f^{(2)}(z)\,. \tag{2.15}$$

Substituting (2.14) in that expression and making analytic continuation of $F_{\delta=0}^{(2)}(p^2)$ throughout the whole region p we get

$$F_{\delta=0}^{(2)}(p^2) = i\frac{\pi}{8\varkappa}\int\limits_L dz\frac{\Gamma(-z)\,\Gamma(1-z)}{\Gamma(2+z)}\left[\varkappa(p^2+i\varepsilon)\right]^z$$
$$= \frac{\pi^2}{4\varkappa}G_{03}^{20}(\varkappa(p^2+i\varepsilon)\mid 1, 0, -1)\,, \tag{2.16}$$

where $G_{03}^{20}(\varkappa(p^2 + i\varepsilon) \mid 1, 0, -1)$ is the Meijer's G-function [17]. Also the expression for $F_{\delta=0}^{(1)}(p^2)$ is

$$F_{\delta=0}^{(1)}(p^2) = \frac{\pi^2}{4\varkappa} G_{03}^{20}(\varkappa(p^2 + i\varepsilon)\, e^{-i2\pi} \mid 1, 0, -1)\,. \tag{2.17}$$

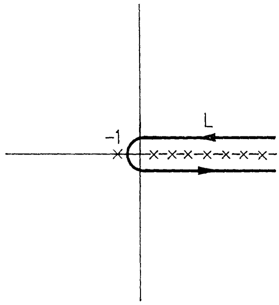

Fig. 2

Substituting (2.17), (2.16), (2.8) into (2.4) and using the representation of the Meijer's G-function as power series, we get

$$\Pi(p^2) = -2(4\pi\varkappa\Delta m)^2 (p^2 + i\varepsilon) \sum_0^\infty \frac{[\varkappa(p^2 + i\varepsilon)]^n}{n!(n+1)!(n+2)!} \tag{2.18}$$
$$\cdot \{\ln[c\varkappa(p^2 + i\varepsilon)\, e^{-i\pi}] - \psi(n+1) - \psi(n+2) - \psi(n+3)\} - 2\varkappa(4\pi\Delta m)^2.$$

where $\psi(n)$ is the Euler function and c is arbitrary dimensionless constant connected with a (see (2.8)).

This result is more simply obtained with the help of analytic continuation procedure for the quantity (2.5) over the value $\varkappa'$, where $\varkappa' = -\varkappa > 0$. The function $F_{\varkappa'}(p^2)$ exists and is perfectly well behaved

$$F_{\varkappa'}(p) = -\frac{\pi^2}{4\varkappa'} G_{03}^{20}(\varkappa'(p^2 + i\varepsilon)\, e^{-i\pi} \mid 1, 0, -1)\,. \tag{2.19}$$

To analytically continue to the region $\varkappa' < 0$ one notice that $G_{03}^{20}(\varkappa'(p^2 + i\varepsilon)\, e^{-i\pi} \mid 1, 0, -1)$ has a cut along the negative real axis in the $\varkappa'$-plane. So the analytically continued quantity is as follows

$$\begin{aligned} F_\varkappa(p^2) = \frac{\pi^2}{4\varkappa} \{&\alpha G_{03}^{20}(\varkappa(p^2 + i\varepsilon) \mid 1, 0, -1) \\ &+ \beta G_{03}^{20}(\varkappa(p^2 + i\varepsilon)\, e^{-i2\pi} \mid 1, 0, -1)\} \end{aligned} \tag{2.20}$$

where $\alpha + \beta = 1$ and where, from the unitary condition, $\mathrm{Re}(\alpha - \beta) = 0$. As a result we arrive again at (2.18).

The scalar particles scattering amplitude is written as

$$f(p', q' \mid p, q) = f(s) + f(t) + f(u) \tag{2.21}$$

where

$$f(s) = -2(4\pi\varkappa\Delta m)^2 s \sum_0^\infty \frac{(\varkappa s)^n}{n!(n+1)!(n+2)!} \cdot [\ln(c\varkappa s e^{-i\pi}) - \psi(n+1) - \psi(n+2) - \psi(n+3)] . \tag{2.22}$$

3. Unitary, Causality and Locality of the Theory

(Second Order of the Perturbation Theory)

The expression (2.22) satisfies the unitarity condition. Really from the unitarity condition $SS^+ = 1$ one easily obtains in the second order of the perturbation theory in Δm $\left(S = \sum\limits_0^\infty (\Delta m)^n S_n\right)$ the following:

$$\delta^{(4)}(p' + q' - p - q) \operatorname{Im} f(s)$$

$$= 2\pi^2 \sqrt{\omega_{p'}\omega_{q'}\omega_p\omega_q} \int \sum_0^\infty \langle a_p a_q \mid S_1 \mid n b_r^+ b_l^+ \rangle \langle b_l b_r n \mid S_1^+ \mid a_{p'}^+ a_{q'}^+ \rangle \tag{3.1}$$

where a_p^+ (a_p) is the production (annihilation) operator of a scalar particle; b_r^+ (b_r) is the production (annihilation) operator of a spinor particle; ω_p is the energy of a scalar particle.

The right hand side of (3.1) contains the sum of invariant phase volumes of particles $\Omega_{n+2}(k)$ (n-particles are scalar and two-particles are spinor)

$$\begin{aligned} &2\pi^2 \sqrt{\omega_{p'}\omega_{q'}\omega_p\omega_q} \int \sum_0^\infty \langle a_p a_q \mid S_1 \mid n b_r^+ b_l^+ \rangle \langle b_l b_r n \mid S_1^+ \mid a_{p'}^+ a_{q'}^+ \rangle \\ &\qquad = (4\pi\varkappa\,\Delta m)^2\, \delta^{(4)}(p' + q' - p - q) \sum_0^\infty \frac{(\varkappa/\pi)^n}{n!}\, \Omega_{n+2}(p+q) \end{aligned} \tag{3.2}$$

where

$$\Omega_{n+2}(k) = 2\frac{(\pi k^2)^{n+1}}{(n+1)!(n+2)!}\,\theta(k^2)\,\theta(k^0) . \tag{3.3}$$

Comparing (3.2) with the imaginary part of (2.22) we see that the Eq. (3.1) is valid. Thus, we have proved that in the second order of perturbation expansion in Δm the theory is unitary.

Let us investigate the asymptotic behaviour of the scattering amplitude, at $s \to \infty$, in order to show that $f(s)$ obeys also the locality and causality conditions of field theory.

It turns out that, at $s \to \infty$, it grows as follows [15]

$$f_{s\to\infty}(s) \approx B\varkappa(\Delta m)^2 \frac{\exp[3(\varkappa s)^{1/3}]}{(\varkappa s)^{1/3}} \left(1 + O\left(\frac{1}{(\varkappa s)^{1/3}}\right)\right) . \tag{3.4}$$

Here B is the dimenionless constant. This behaviour of the scattering amplitude satisfies the condition arising from the generalized causality and locality principle of the theory [12, 13].

4. Spectral Representation of the Functions $\Psi(p)$ and $\Phi(p)$

In order to investigate the higher orders of perturbation expansion, in addition to the function $\Pi(p^2)$ studied in 2, 3, two more functions are needed, which we denote by $\Psi(p)$ and $\Phi(p)$. In this section we shall obtain for them the necessary integral and spectral representations. The function $\Pi(p^2)$ has similar representations, but they are not written down here.

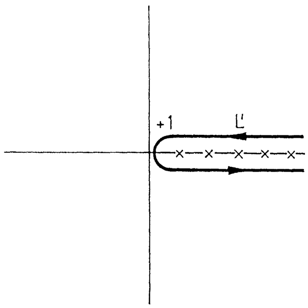

Fig. 3

Let us consider, first of all, the spinor Green function $\Psi(p)$

$$\Psi(p) = \int d^4x\, S^c(x) \exp i[px - (2g)^2 \Delta^c(x)]\,. \tag{4.1}$$

Using the method developed in the second section it is easy to obtain the following integral representation for the regularized function $\Psi_\beta(p)$

$$\Psi_\beta(p) = -i\hat{p}\frac{\varkappa}{2}\int\limits_{L'} dz \frac{e^{-i\pi z}(p^2 + i\varepsilon)^{z-2}}{\sin\pi z\,\Gamma(z)\,\Gamma(z+1)}\bar{f}_\beta(z) \tag{4.2}$$

where the contour L' is shown in Fig. 3, $\hat{p} = p^\nu \gamma_\nu$ and $\bar{f}_\beta(z)$ in the range $0 \leqq \mathrm{Re}\, z < 2$ has the representation

$$\bar{f}_\beta(z) = \frac{1}{2}\int\limits_0^\infty d\lambda\, \lambda^z \left[(1 + ia)\frac{\exp\left(\frac{\varkappa}{\lambda + i\beta}\right)}{(\lambda + i\beta)^3} + (1 - ia)\frac{\exp\left(\frac{\varkappa}{\lambda - i\beta}\right)}{(\lambda - i\beta)^3}\right] \tag{4.3}$$

and can be analytically continued throughout the whole right half z-plane, for the exception of the real positive axis. Then the function $\Psi(p)$ is a limit of $\Psi_\beta(p)$ at $\beta \to 0$. In this case, in (4.3) it is necessary to perform beforehand rotations of the integration contours so that the minus sign appears in the exponential of the integrand.

Instead of (4.3) we can introduce the function

$$f_\delta(z) = -\pi\varkappa^{z-2}\frac{\cos\pi z - a\sin\pi z}{\sin(1+\delta)\,\pi z\,\Gamma(z-1)}\left(\lim_{\beta\to 0}\bar{f}_\beta(z) = \lim_{\delta\to 0} f_\delta(z)\right). \tag{4.3'}$$

Put $\delta > 1,5$. Then in the integral (4.2) the integration contour may be straightened so that it will be parallel to the imaginary axis (Fig. 4). In investigating higher perturbation orders we use the Green function representations of the type of (4.2) with the straightened integration contour C. Then in order to go over to the limit $\delta \to 0$ we should firstly integrate over all the momenta and after that return again to the contour L'.

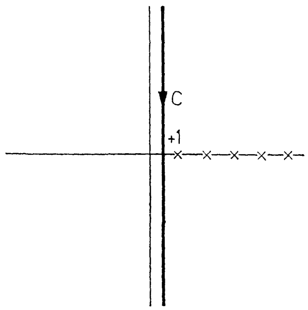

Fig. 4

To Eq. (4.1) there may correspond another parametrized function $\tilde{\Psi}_\delta(p)$. It is obtained from $\Psi_\beta(p)$ if we shift the integration contour to the right by unity, single out from (4.2) a term corresponding to the first order pole of the integrand at $z = 1$ and let β tend to zero. In the remaining integral we straighten the integration contour, introducing the parameter δ. Thus

$$\tilde{\Psi}_\delta(p) = -\hat{p}\left\{\frac{1}{p^2 + i\varepsilon} + i\frac{\varkappa}{2}\int\limits_C dz \frac{e^{-i\pi z}(p^2 + i\varepsilon)^{z-1}}{\sin\pi z\,\Gamma(z+1)\,\Gamma(z+2)}\tilde{f}_\delta(z+1)\right\}. \quad (4.4)$$

Noticing that in the region $0 < \operatorname{Re} z < 1$ we have the integral equation

$$-\pi(p^2 + i\varepsilon)^{z-1}\frac{e^{-i\pi z}}{\sin\pi z} = \int\limits_0^\infty dm^2 \frac{m^{2(z-1)}}{m^2 - p^2 - i\varepsilon} \quad (4.5)$$

we can rewrite (4.4) in the form

$$\tilde{\Psi}_\delta(p) = -\hat{p}\left\{\frac{1}{p^2 + i\varepsilon} + \frac{\varkappa}{2\pi i}\int\limits_C dz \frac{\tilde{f}_\delta(z+1)}{\Gamma(z+1)\,\Gamma(z+2)}\int\limits_0^\infty dm^2 \frac{m^{2(z-1)}}{m^2 - p^2 - i\varepsilon}\right\}. \quad (4.6)$$

Eq. (4.6) may be considered as a spectral representation of the function $\tilde{\Psi}_\delta(p)$.

Now we consider the scalar function $\Phi(p)$

$$\Phi(p) = i\int d^4x \exp i[px + (2g)^2 \Delta^c(x)]. \quad (4.7)$$

Similar procedures lead to the following spectral representation of the parametrized function corresponding to $\Phi(p)$

$$\tilde{\Phi}_\delta(p) = i(2\pi)^4\,\delta^{(4)}(p) + (4\pi)^2\varkappa\left\{\frac{1}{p^2+i\varepsilon} - \frac{\varkappa}{2\pi i}\int\limits_C dz\,\frac{\varphi_\delta(z+1)}{\Gamma(z+1)\,\Gamma(z+2)}\int\limits_0^\infty dm^2\,\frac{m^{2(z-1)}}{m^2-p^2-i\varepsilon}\right\} \tag{4.8}$$

where

$$\varphi_\delta(z) = -\frac{\pi\varkappa^{z-2}}{\sin(1+\delta)\,\pi z\,\Gamma(z-1)}\,. \qquad (\delta > 0{,}5) \tag{4.9}$$

We give also another type of the parametrized function $\Phi_\delta(p)$ similar to (4.2) for $\Psi(p)$

$$\Phi_\delta(p) = i(2\pi)^4\,\delta^{(4)}(p) - i\,8\pi^2\varkappa^2\int\limits_C dz\,\frac{e^{-i\pi z}\,(p^2+i\varepsilon)^{z-2}}{\sin\pi z\,\Gamma(z)\,\Gamma(z+1)}\,\varphi_\delta(z)\,. \tag{4.10}$$

The most essential difference of the function $\Phi(p)$ from $\Psi(p)$ and $\Pi(p^2)$ consists in that the scalar particle propagator in the exponential has opposite sign. Due to this fact, it is unnecessary in the integral (4.7) to introduce an intermediate regularization as we have to do in finding the functions $\Psi(p)$ and $\Pi(p^2)$. The parametrization is here necessary only for straightening the integration contour in the z-plane from L' to C.

Using the obtained representations for the two-point Green functions it is easy to prove the unitarity of the S-matrix and the absence of ultraviolet divergences in the model studied.

5. Unitarity and Absence of the Ultraviolet Divergences

(Higher Orders of the Perturbation Theory)

a) We consider the matrix element of interaction of spinor particle with vacuum in the third perturbation order in Δm (Fig. 5). The unitarity condition gives the equation for the matrix elements

$$\mathrm{Re}\langle b_p \mid S_3 \mid b_{p'}^+\rangle = -\mathrm{Re}\langle b_p \mid S_2 S_1^+ \mid b_{p'}^+\rangle\,. \tag{5.1}$$

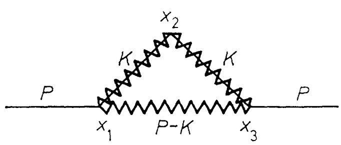

Fig. 5. ⁓⁓ — $\Phi(p-k)$; ⩔⩔ — $\Psi(k)$

$\xrightarrow{P}$ — $v_p^{\alpha\pm}(\mathrm{p})\,\dfrac{e^{\pm ipx}}{(2\pi)^{3/2}}$

The left-hand side of Eq. (5.1) is expressed in terms of the well known functions $\tilde{\Psi}_\delta(p)$ and $\tilde{\Phi}_\delta(p)$

$$\begin{aligned}\operatorname{Re}\langle b_p \mid S_3 \mid b^+_{p'}\rangle \\ = \tau_3 \frac{\delta^{(4)}(p-p')}{(2\pi)^3} \bar{v}^{\alpha+}_\nu(\mathbf{p})\, v^{\beta-}_\mu(\mathbf{p}) \lim_{\delta\to 0} \operatorname{Re} \int dk^4\, \tilde{\Psi}^2_\delta(k)\, \tilde{\Phi}_\delta(p-k)\end{aligned} \tag{5.2}$$

where $\bar{v}^{\alpha+}_\nu(\mathbf{p})$ and $v^{\beta-}_\mu(\mathbf{p})$ are the orthonormalized spinors. We write (5.2) as an integral of the product of the real and imaginary parts of the functions $\Psi(p)$ and $\Phi(p)$ and the corresponding functions $\theta(p^0)$ $\begin{pmatrix}\theta(p^0)=1,\, p^0 \geqq 0\\ \theta(p^0)=0,\, p^0<0\end{pmatrix}$. For this it is convenient, using the spectral representations of the two-point Green functions derived in the previous section, to divide them into parts corresponding to $G^{\text{ret}}(p)$ and $G^+(p)$, namely

$$\tilde{\Psi}_\delta(p) = \hat{p} \int_0^\infty dm^2\, \varrho_\delta(m^2) \left[\frac{1}{m^2-p^2-i2\varepsilon p^0} + i2\pi\, \delta(m^2-p^2)\, \theta(p^0)\right] \tag{5.3}$$

and we do the same for $\tilde{\Phi}_\delta(p)$. Using the property

$$\int d^4k (\tilde{\Psi}^{\text{ret}}_\delta(k))^2\, \tilde{\Phi}^{\text{ret}}_\delta(k-p) = 0 \tag{5.4}$$

it is easy to write (5.2) in the form ($p^0>0$)

$$\begin{aligned}\lim_{\delta\to\infty} \operatorname{Re} \int d^4k\, \tilde{\Psi}^2_\delta(k)\, \tilde{\Phi}_\delta(p-k) \\ = -2(2\pi)^4 \operatorname{Re}\Psi(p) \operatorname{Im}\Psi(p) \\ -4 \int d^4k \operatorname{Re}\Psi(k) \operatorname{Im}\Psi(k) \operatorname{Im}{}'\Phi(p-k)\, \theta(k^0)\, \theta(p^0-k^0)\,.\end{aligned} \tag{5.5}$$

where

$$'\Phi(p) = \Phi(p) - i(2\pi)^4\, \delta^{(4)}(p)\,. \tag{5.6}$$

Introducing the intermediate states between S_2 and S_1^+ we get for the right-hand side of (5.1)

$$\begin{aligned}\operatorname{Re} \sum_0^\infty \langle b_p \mid S_2 \mid n\rangle \langle n \mid S_1^+ \mid b^+_{p'}\rangle = \tau_3\, 2\pi^2\, \delta^{(4)}(p-p')\, \bar{v}^{\alpha+}_\nu(\mathbf{p})\, v^{\beta-}_\mu(\mathbf{p}) \\ \cdot \int d^4k \operatorname{Re}\Psi(k) \sum_0^\infty \frac{(\varkappa/\pi)^n}{n!} \hat{\Omega}_{n+1}(k) \sum_0^\infty \frac{(-\varkappa/\pi)^m}{m!} \Omega_m(p-k)\,.\end{aligned} \tag{5.7}$$

where $\hat{\Omega}_{n+1}(k)$ and $\Omega_m(p-k)$ are the phase volumes of particles (in $\hat{\Omega}_{n+1}(k)$ n-particles are scalar and one is spinor; in $\Omega_m(p-k)$ all particles are scalar). Inserting the expressions for the imaginary parts of the functions $\Psi(p)$ and $\Phi(p-k)$ into (5.5) and for the phase volumes into (5.7) [16] it is easy to see that Eq. (5.1) is valid. Thus, we have proved that in the third perturbation order in Δm the theory is unitary. The author expects that the obtained spectral representations for the two-point Green functions would provide the validity of the unitarity in higher order in Δm as well.

b) Consider the integral contained in the right-hand side of (5.2) and calculate it using the integral representations of the functions $\Psi_\delta(p)$ and $\Phi_\delta(p-k)$ (see (4.2) and (4.10))

$$\int d^4k\, \Psi_\delta^2(k)\, '\Phi_\delta(p-k) = i2\pi^2\varkappa^4 \int_{C_1}\int_{C_2}\int_{C_3} dz_1\, dz_2\, dz_3$$

$$\cdot \frac{\exp[-i\pi(z_1+z_2+z_3)]\, f_\delta(z_1)\, f_\delta(z_2)\, \varphi_\delta(z_3)}{\sin\pi z_1 \sin\pi z_2 \sin\pi z_3\, \Gamma(z_1)\, \Gamma(z_2)\, \Gamma(z_3)} \frac{1}{\Gamma(z_1+1)\,\Gamma(z_2+1)\,\Gamma(z_3+1)} \tag{5.8}$$

$$\cdot \int d^4k\,(k^2+i\varepsilon)^{z_1+z_2-3}\,[(p-k)^2+i\varepsilon]^{z_3-2}\,.$$

The requirement that the integral over k must be free of ultraviolet divergences imposes the following restrictions on the variables z_i

$$\mathrm{Re}\,(z_1+z_2+z_3) < 3\,. \tag{5.9}$$

Since the contours c_i lie in the range $0 \leq \mathrm{Re}\, z_i < 1$, this requirement is fulfilled. The integral over k is:

$$\int d^4k\,(k^2+i\varepsilon)^{z_1+z_2-3}\,[(p-k)^2+i\varepsilon]^{z_3-2} \tag{5.10}$$

$$= i\pi^2(p^2+i\varepsilon)^{z_1+z_2+z_3-3}\,\frac{\Gamma(z_3-1)\,\Gamma(z_1+z_2-2)\,\Gamma(3-z_1-z_2-z_3)}{\Gamma(1-z_3)\,\Gamma(2-z_1-z_2)\,\Gamma(z_1+z_2+z_3-1)}\,.$$

Inserting (5.10) into (5.8) and rotating the contours C_i so that they pass around the real positive axes in the appropriate z_i-planes and using the residues at the poles we can explicitly calculate all the integrals.

Let us prove that ultraviolet divergences are absent in the n-th perturbation order in Δm too. To this end we consider a diagram with n vertices, two external spinor lines and an arbitrary number of external scalar lines (Fig. 6). All the vertices are connected in pair by lines each

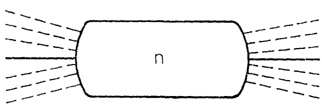

Fig. 6. — — — external scalar line, ———— external spinor line, n-number of vertices

of which corresponds to one of the Green functions considered by us and having the integral representation like (4.2) or (4.10). We consider the case when all the vertices are connected by a continuous spinor line corresponding to the $n-1$ functions $\Psi(k_i)$. Then, in addition to these functions, $(n-1)(n-2)/2$ scalar functions like $\Phi(k_j)$ will correspond to the diagram. The product of all these functions will have the sign of $2(n-1)(n-2)$ fold integral over k_l. Then the requirement of the absence of ultraviolet divergences is written as

$$2(n-1)(n-2)+n-1-2n(n-1)+2\sum_{1}^{n(n-1)/2} z_i < 0\,, \tag{5.11}$$

hence, it follows that

$$\operatorname{Re} \sum_{1}^{n(n-1)/2} z_i < \frac{3}{2}(n-1)\,. \tag{5.12}$$

Assuming all z_i to be equal, ($z_i = z$) we get

$$\operatorname{Re} z < \frac{3}{n}\,. \tag{5.13}$$

Since the contour C in the integral representations of our Green functions may exactly coincide with the imaginary axis, the condition (5.13) is well satisfied.

We prove in a similar manner the absence of ultraviolet divergences in higher orders in Δm in terms somewhat different. To this end, following the work by N. N. Bogolubov and D. V. Shirkov [18], we introduce the notion of maximum vertex index and calculate it in the framework of our model:

$$\omega_i^{\max} = \frac{1}{2} \sum_{l_{\text{int}}} (r_l + 2\bar{z}_l) - 4 = -3\,. \tag{5.14}$$

Here the summation is made over internal lines, r_l is unity for spinor lines and zero for scalar lines and $\bar{z}_l = \operatorname{Re} z_l$ are assumed to be zero. From the inequality $\omega_i^{\max} < 0$ it follows that if ultraviolet divergences are absent in lower perturbation orders then they can not appear in higher orders too.

Thus, it is proved that in the considered model ultraviolet divergences are absent in any perturbation order in Δm.

6. Generalization of the Model to Case of Massive Spinor Particles

In the previous sections the finite two-point Green functions were found, in terms of which all physical quantities of field theory are expressed. The spectral representation for these functions were constructed and the integrals of their products were determined. It was shown that our model of field theory was unitary and free of ultraviolet divergences.

A particular case was investigated: the rest masses of all particles were assumed to be zero. This case is the most convenient for investigation since the propagators have a very simple form. However as far as the interaction Lagrangian includes the mass difference of nucleons in two different states it is interesting to generalize the model to the case of spinor non-zero masses. This section is just devoted to this problem. Let us discuss the spinor Green function $\Psi_m(p)$

$$\Psi_m(p) = \int d^4x\, S_m^c(x) \exp i\,[px - (2g)^2 \Delta^c(x)] \tag{6.1}$$

where $S_m^c(x)$ and $\Delta^c(x)$ are the propagators of the spinor and scalar fields, respectively. The rest mass of a scalar particle is zero, while that of a spinor one differs from zero.

In order to illustrate our model by simpler example we consider, instead of Eq. (6.1), the function $D(p)$

$$D(p) = \int d^4x \, \Delta_m^c(c) \exp i \left[px - (2g)^2 \Delta^c(x)\right] \tag{6.2}$$

where $\Delta_m^c(x)$ is a scalar propagator with non-zero rest mass. The calculation for Eq. (6.2) is easily extended to the integral (6.1).

We consider Eq. (6.2) in a physical domain $p^2 > 0$. This integral reduces to an integral in the momentum space of the product of two functions $\Delta_m(p-k)$ and $\bar{\Phi}_\delta(k)$

$$D(p) = \frac{i}{(2\pi)^4} \int d^4k \, \frac{\bar{\Phi}_\delta(k)}{(p-k)^2 - m^2 + i\varepsilon} \tag{6.3}$$

where

$$\bar{\Phi}_\delta(k) = i(2\pi)^4 \delta^{(4)}(k) + i8(\pi\varkappa)^2 \int\limits_{\alpha - i\infty}^{\alpha + i\infty} dz \, \frac{e^{-i\pi z}(k^2 + i\varepsilon)^{z-2}}{\sin\pi z \, \Gamma(z) \, \Gamma(z+1)} f_\delta(z) \tag{6.4}$$

$\varkappa = (g/2\pi)^2, 0 \leqq \alpha < 1, \Gamma(z)$ is the gamma function, $f_\delta(z)$ is the parametrized function (see (4.3)′). Inserting Eq. (6.4) into Eq. (6.3) we obtain

$$D(p) = \frac{1}{m^2 - p^2 - i\varepsilon} + \frac{\varkappa^2}{2\pi^2} \int\limits_{\alpha + i\infty}^{\alpha - i\infty} dz \, \frac{e^{-i\pi z} f_\delta(z)}{\sin\pi z \, \Gamma(z) \, \Gamma(z+1)} d(z, p) \tag{6.5}$$

where

$$d(z, p) = \int d^4k (k^2 + i\varepsilon)^{z-2} \left[(p-k)^2 - m^2 + i\varepsilon\right]^{-1} . \tag{6.6}$$

In the region $0 \leqq p^2 < m^2$ $d(z, p)$ is as follows:

$$d(z, p) = -i\pi^3 m^{2(z-1)} \frac{e^{i\pi z}}{\sin\pi z} \sum_0^\infty \left(\frac{p^2}{m^2}\right)^n \frac{\Gamma(z) \, \Gamma(z-1)}{n!(n+1)! \, \Gamma(z-n) \, \Gamma(z-n-1)} \tag{6.7}$$

and in the region $p^2 > m^2$

$$d(z, p) = -i\pi^2 p^{2(z-1)} \sum_0^\infty \left\{ \pi \frac{e^{i\pi z}}{\sin\pi z} \left(\frac{m^2}{p^2}\right)^{n+1} \frac{\Gamma(z) \, \Gamma(z-1)}{n!(n+1)! \, \Gamma(z-n) \, \Gamma(z-n-1)} \right.$$
$$\left. + \left(1 - \frac{m^2}{p^2}\right)^{2z+n-1} \frac{\Gamma(n+z) \, \Gamma(n+z-1)}{n! \, \Gamma(n+2z)} \right\} . \tag{6.8}$$

Substitute it into Eq. (6.5) and rotate the integration contour so that it passes around the real positive axis. Then we may go to the limit $\delta \to 0$ and calculate Eq. (6.5) as the sum of the residues at the poles. The result has the form of a well convergent double series.

In the region $0 < p^2 < m^2$, $D(p)$ may be written as

$$D(p) = \frac{(\varkappa m)^2}{2} \sum_0^\infty \frac{(\varkappa m^2)^k}{(\varkappa + 2)!} \sum_0^k \frac{\left(\frac{p^2}{m^2}\right)^n}{n!(n+1)!\,(k-n)!\,(k-n+1)!}$$
$$\cdot \{\pi^2 + \psi'(k+3) + \psi'(k-n+2) + \psi'(k-n+1)$$
$$- [\ln(\varkappa m^2) - \psi(k+3) - \psi(k-n+2) - \psi(k-n+1)]^2\} \quad (6.9)$$
$$+ \frac{1}{m^2} \sum_0^\infty \frac{(\varkappa m^2)^k}{k!} \sum_k^\infty \left(\frac{p^2}{m^2}\right)^n \frac{(n-k)!\,(n-k+1)!}{n!(n+1)!}$$
$$- \varkappa \sum_0^\infty \frac{(\varkappa p^2)^n}{n![(n+1)!]^2} [\ln(\varkappa m^2) - \psi(n+1) - 2\psi(1)]\,.$$

Here we put $a = 0$ (in (4.3)) for simplicity.

For $p^2 > m^2$ we give only $\operatorname{Im} D(p)$. (Here $D(p)$ is complex).

$$\operatorname{Im} D(p) = \pi \left\{ \delta(p^2 - m^2) + \varkappa h \right.$$
$$\left. + \varkappa^2 p^2 h^3 \sum_0^\infty \frac{(\varkappa p^2)^n}{n!(n+1)!\,(n+2)!} \sum_{2n}^\infty h^k \frac{(k-n)!\,(k-n+1)!}{(k+3)!\,(k-2n)!} \right\} \quad (6.10)$$
$$\left(h = 1 - \frac{m^2}{p^2}\right).$$

From the unitarity condition it follows that the imaginary part must be

$$\operatorname{Im} D(p) = \frac{\pi}{2} \sum_0^\infty \frac{(\varkappa/\pi)^n}{n!} \Omega_{n+1}(p) \quad (6.11)$$

where $\Omega_{n+1}(p)$ is the phase volume of $n + 1$ particles. One particle has non-zero rest mass and n-particles have zero rest masses. Such phase volumes are calculated in ref. [19]. Substituting their values into (6.11) we make sure of the validity of Eq. (6.10). Thus, the function obtained by us obeys the unitarity condition.

The functions $D(p)$ obeys also the locality and causality conditions of the field theory [12, 13].

In conclusion we note that the integral (6.2) may be calculated not going to the momentum representation but simply using the following integral representation for the propagator $\Delta_m^c(x)$

$$\Delta_m^c(x) = \frac{m^2}{2^5 \pi} \int_{L'} dz \frac{\left[\frac{m^2}{4}(x^2 - i\varepsilon)\, e^{i\pi}\right]^{z-2}}{\sin^2 \pi z\, \Gamma(z)\, \Gamma(z-1)}\,. \quad (6.12)$$

7. Conclusion

Thus, on the basis of the model with unrenormalizable interaction, we have constructed the local unitary quantum field theory free of the ultraviolet divergences. The scattering amplitudes and the Green functions in this theory are nonanalytical in the coupling constant g. This fact forbids utilization of the ordinary perturbation theory with expansion in this constant. The investigation of the asymptotic behaviour of the scattering amplitude at higher energy shows that one has an essential singularity at infinity. But the model belongs to the class of the local and causal theories defined in the axiomatic method [12, 13].

From the spectral Green function representation it follows the unitarity and the causality of our model. The Green function representations in the Mellin-Barnes integral forms permit to generalize the notion of the maximum vertex index so that our model can be described by a method close to the renormalizable theory, and the ultraviolet divergence absence in any perturbation order in Δm can be proved.

The dimensionless parameter c in (2.18) and (2.22) is a consequence of the fact that the scattering amplitude is not defined at the origin of the light cone $x = 0$.

Similar situations often take place, when we work with the time-ordered operator functions (see e.g. [7]). But our method gives only one arbitrary parameter, whereas the usual methods in the unrenormalizable theories give infinite number of them.

The method demonstrated here on the basis of field theoretic model can be applied to some real unrenormalizable interactions. For instance, the Bethe-Salpeter amplitude $R(x - y) = \langle 0 \mid T\psi_A(x)\,\bar{\psi}_A(y) \mid A\bar{A}\rangle$ for the scattering $A + \bar{A} \to A + \bar{A}$ (A is the spinor particle) obeys, in the ladder approximation, the equation [7]

$$(L_{\text{int}}(x) = g j_A(x)\, j_B(x),\; j_A(x) = \,:\bar{\psi}_A(x)\,\gamma_5\psi_A(x):,\; m_A = m_B = 0,$$

$$p = p_1 + p_2 = 0) \tag{7.1}$$

$$\Box R(x) = \frac{4g}{(x^2 - i\varepsilon)^3}\, R(x)\ .$$

This equation has the following solution

$$R(x) = a \exp\left\{\frac{\sqrt{g}}{x^2 - i\varepsilon}\right\} + b \exp\left\{-\frac{\sqrt{g}}{x^2 - i\varepsilon}\right\} \tag{7.2}$$

This function is easily described by our method.

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Dr. M. K. Volkov
Joint Institute for Nuclear Research
Laboratory of Theoretical Physics
Head Post Office, P.o. Box 79
Moskwa, UdSSR