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On Subalgebras which Survive Contraction*

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Abstract. We prove the following theorem: if a subalgebra $\mathbf{B}$ of an algebra $\mathbf{G}$ is spanned by root vectors, then if X is a regular element of $\mathbf{G}$, the limit $\lim\limits_{t\to\infty} \mathrm{Ad}\exp tX(\mathbf{B})$ exists and is isomorphic to $\mathbf{B}$, i.e. $\mathbf{B}$ "survives" contraction with X. The algebra $\mathbf{SL}(2C)$ is considered as an example. In particular it is shown that $\mathbf{SL}(2C)$ itself survives and applications to relativistic scattering theory are indicated.

1. Introduction

The investigation of the contraction of Lie groups [1, 2] is of considerable interest in theoretical physics. Apart from the classical examples, (e.g. contraction of the Poincaré group to the Galilean group in the limit of infinite light velocity) it was recently realized [3, 4] that group contraction is an important concept in a relativistic scattering theory.

It is well known that the little group (the isotropy group of the four momentum P) of the Poincaré group is isomorphic to $O(3)$, or $SU(2)$ if we wish to describe half-integral spins, provided the total four momentum, P, is time-like. As the total invariant mass of a system, $\sqrt{P^2}$, tends to zero, the group $SU(2)$ contracts to $E(2)$, the group of rigid motions in a plane. The sudden change in the structure of the little group at $P^2 = 0$ gives rise to certain singularities in the partial wave amplitudes, which contradict the conventionally assumed analytic properties of the scattering amplitude. It may be shown [4] that these "unwanted" singularities can be eliminated if, at the contraction point, the partial wave amplitudes, or rather the Regge-pole terms, can be grouped together to form families which transform according to representations of a group which does not change its structure at the contraction point.

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These considerations raise the important question: "What are the algebraic structures which survive group contraction?" The purpose of this note is to give a partial answer to this question.

According to the generally accepted usage in physics, we formulate and answer the question for Lie algebras and ignore the global structure of the groups in question.

II. A No-Contraction Theorem

Let us recall a few definitions. Let $\mathbf{G}$ be a Lie algebra with Cartan decomposition $\mathbf{G} = \mathbf{K} \oplus \mathbf{P}$ and X be a regular element of $\mathbf{G}$.

Definition. *If* $\mathbf{B} \subset \mathbf{G}$ *is a subalgebra of* $\mathbf{G}$ *we say that the limit*

$$\mathbf{B}(\infty) = \lim_{t \to \infty} \operatorname{Ad} \exp X t (\mathbf{B}) , \tag{2.1}$$

if it exists[1], *is a contraction of* $\mathbf{G}$ *with* X. *This definition is a special case of the one given in* HERMANN*'s book* [3]. *In this paper we shall not consider more general contractions.*

We now introduce the

Definition. *We say that the subalgebra* $\mathbf{B}$ *survives the contraction if* $\mathbf{B}(\infty)$ *is isomorphic to* $\mathbf{B}$.

Evidently if $\mathbf{B}$ is to survive it is first of all necessary that every element $b \in \mathbf{B}$ have a limit in the sense of (2.1). Further, if $b_i, b_j, b_k \in \mathbf{B}$ then the algebra $\mathbf{B}$ survives if the left and right-hand sides of a Lie-bracket relation,

$$[b_i, b_j] = \sum_k B_{ij}^k b_k ,$$

have the same limit, where B_{ij}^k is a structure constant of the subalgebra $\mathbf{B}$.

Let W_i and W_j be two root vectors with corresponding roots λ_i and λ_j,

$$[X, W_i] = \lambda_i W_i ; \quad [X, W_j] = \lambda_j W_j .$$

Consider the contraction of the Lie bracket $[W_i, W_j]$. Evidently

$$\operatorname{Ad} \exp X t ([W_i, W_j]) = \exp(\lambda_i + \lambda_j) [W_i, W_j] .$$

On the other hand we have

$$[W_i, W_j] = N_{i+j} W_{i+j} \tag{2.2}$$

where W_{i+j} is a root vector with eigenvalue $\lambda_i + \lambda_j$,

$$[X, W_{i+j}] = (\lambda_i + \lambda_j) W_{i+j} .$$

Thus

$$\operatorname{Ad} \exp X t (W_{i+j}) = \exp(\lambda_i + \lambda_j) t W_{i+j}$$

[1] In particular it is known that $\mathbf{K}(\infty)$ exists and is equal to $\mathbf{K}(X) \oplus \mathbf{N}^+(X)$ where $\mathbf{K}(X)$ is the centraliser of X in $\mathbf{K}$ and $\mathbf{N}^+(X)$ is the nilpotent subalgebra generated by the root vectors of X with positive roots.

and the Lie-bracket (2.2) of root vectors survives contraction since both left and right-hand sides are simply multiplied by $\exp(\lambda_i + \lambda_j)t$. Therefore a subalgebra **W**, spanned by root vectors W_i, which form a basis, survives.

Obviously if some of the eigenvalues are degenerate i.e. the linearly independent root vectors $W_{i_1} \ldots W_{i_n}$ $(n \geqq 1)$ belong to the same eigenvalue, then a change of basis "mixing" only root vectors belonging to the same eigenvalue is permissible. Thus we have proved the following

Theorem. *A subalgebra* $\mathbf{W} \subset \mathbf{G}$ *survives contraction if it is spanned by root vectors.*

In the case where the root vectors also span **G**, as a vector space, it is straightforward to prove that the above contraction is also necessary. However we are unable to prove the necessity of this condition in the general case.

III. Example: The Algebra SL(2C)

We employ the notation familiar in physics i.e. denote by M_i the — Hermitean — elements of $\mathbf{K} = \mathrm{SU}(2)$ and by N_i the elements of **P**. (The elements M_i are generators of rotations and N_i generate pure Lorentz transformations along the i^{th} coordinate axis.)

The Lie-bracket relations are of the form

$$\begin{aligned}
[M_i, M_j] &= i\varepsilon_{ijk} M_k \\
[M_i, N_j] &= i\varepsilon_{ijk} N_k \quad (i, j, k = 1, 2, 3) \\
[N_i, N_j] &= -i\varepsilon_{ijk} M_k \,,
\end{aligned}$$

where we adopt the usual summation convention and the symbol ε_{ijk} denotes the totally antisymmetric tensor in three dimensions.

We may choose the regular element N_3 to play the role of X in the previous section. The root vectors can be constructed very easily and we list them with the corresponding roots,

$$\begin{aligned}
W_1 &= M_1 + M_2 + N_2 - N_1 & \lambda_1 &= i \\
W_2 &= M_1 - M_2 + N_2 + N_1 & \lambda_2 &= i \\
W_3 &= M_1 + M_2 + N_1 - N_2 & \lambda_3 &= -i \\
W_4 &= M_2 - M_1 + N_1 + N_2 & \lambda_4 &= -i \\
W_5 &= M_3 & \lambda_5 &= 0 \\
W_6 &= N_3 & \lambda_6 &= 0 \,.
\end{aligned}$$

(Evidently X itself and the elements of the centralizer $\mathbf{K}(X)$ can be considered root vectors with zero root.)

The root vectors span several subalgebras, for example the Abelian algebra spanned by M_3, N_3, the Lie algebra of the group of rigid motions in the Euclidean plane $\mathbf{E}(2)$, with elements conventionally written as $\mathbf{E}(2) = \{M_1 + N_2, M_2 - N_1, M_3\}$. (The first two elements are obtained by adding and subtracting W_1 and W_2. This is permissible since W_1 and W_2 belong to the same root.) We may obtain "another copy" of $\mathbf{E}(2)$, in the same way, from W_3, W_4 and W_5. One can verify by explicit calculation that these subalgebras — and only these — survive. (The algebra $\mathbf{K} = \mathbf{SU}(2)$, as is well known, *does not* survive but contracts to $\mathbf{E}(2)$.) The most important point is that the algebra spanned by the root vectors $W_1 \ldots W_6$ is isomorphic to $\mathbf{SL}(2C)$ and therefore the whole algebra $\mathbf{SL}(2C)$ survives. (This remark is non-trivial as one can easily construct counterexamples.)

We finally add a remark about the physical applications. In the physical problem mentioned in the introduction, one has to require, on physical grounds, that the surviving algebra contain $\mathbf{SU}(2)$ as a subalgebra. In the case of $\mathbf{SL}(2C)$ this is the full algebra itself. As a consequence, the zero mass Regge pole bound states of a relativistic scattering amplitude have to form a basis for a representation of the Lie algebra of the homogeneous Lorentz group [4]. This phenomenon has recently become known as "conspiracy" in the physical literature. Our considerations show that the solution to the conspiracy problem is unique.

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