

On Certain Non-Relativistic Quantized Fields*

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Abstract. We consider a Euclidean invariant interaction Hamiltonian which is a polynomial in smeared Fermion field operators (the smearing function providing an ultraviolet cut-off). By considering Guenin's perturbation series for the time-development of the theory, we show that time-displacements define a one-parameter group of automorphisms of the field algebra at $t = 0$, which acts continuously in the time-parameter. Results are obtained for any dimension of space and for both relativistic and nonrelativistic forms for the free Hamiltonian. In special cases the total Hamiltonian is a positive self-adjoint operator in Fock space, thus defining a concrete non-relativistic quantized field with non-trivial particle production.

I. Introduction

The recent axiomatic approach to quantum statistical mechanics [1]–[3] has increased interest in non-relativistic models of quantized fields. A pertinent question is whether or not a given Hamiltonian defines a time-development which is a continuous automorphism group of the local C^* -algebra of the theory. In a previous paper [4] we showed that if the Heisenberg theory of the ferromagnet is formulated in terms of local C^* -algebras of observables, then for a wide choice of spin couplings the Hamiltonian is a positive self-adjoint operator which does indeed generate a continuous automorphism.

In the present paper less trivial models, involving a fermion field, are constructed. The quasi-local algebra \mathfrak{A} is the C^* -algebra generated by the creation operators at time $t = 0$, smeared with test-functions in $\mathcal{D}(R^3)$. The interaction Hamiltonian is a polynomial in the smeared creation and annihilation operators, commuting with the Euclidean group but not necessarily conserving particle number. The general case therefore violates Galilean invariance. For certain choices, which annihilate the Fock vacuum, the total Hamiltonian is a non-negative

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self-adjoint operator in the Fock representation of the anticommutation relations. As in the Lee model, these cases exhibit no vacuum polarization, but the one-particle states do suffer a non-trivial energy shift.

The general polynomial, which does not annihilate the Fock vacuum, has no meaning as an operator in Fock space. We show that nevertheless it generates a continuous automorphism group of \mathfrak{A} . Our method of proof uses a form of the interaction picture introduced by M. GUENIN for this very purpose [5]. He was able to show that the perturbation series for the time-displaced operators converges for small t in the operator norm topology, at least in a two-dimensional space-time. The class of interactions considered in [5] is somewhat wider than that of this paper. Indeed, for our special models, we find that the proof goes through for any number of space dimensions, both for the relativistic and nonrelativistic forms for the free Hamiltonian.

II. Definition of the Models

We consider in R^3 the test-function space $\mathcal{D}(R^3)$ and the Fock representation of the anti-commutation relations in a Hilbert space \mathcal{H} . The Fock vacuum will be denoted by $\Omega \in \mathcal{H}$ and the creation and annihilation operators by $\psi^*(f)$, $\psi(f)$ respectively. Thus if $\mathbf{1}$ denotes the identity in \mathcal{H} ,

$$\{\psi^*(f), \psi(g)\}_+ = \int f(x) g(x) d^3x \mathbf{1} = (\bar{f}, g) \mathbf{1} . \quad (1)$$

$$\psi(f) \Omega = 0 \quad \text{if } f \in \mathcal{D}(R^3) . \quad (2)$$

Let D_n be the subset of vectors which are finite linear combinations of vectors of the form

$$\Psi = \psi^*(f_1) \dots \psi^*(f_n) \Omega . \quad (3)$$

Let $\mathcal{H}_n = \bar{D}_n$, the closure of D_n . Then

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n . \quad (4)$$

We take the free Hamiltonian to have either the non-relativistic form

$$H_0 = \frac{1}{2m} \int d^3x \nabla \psi^*(x) \cdot \nabla \psi(x) + mN \quad (5)$$

where $m > 0$ and N is the number operator, or the relativistic form

$$H_0 = \int d^3k \omega_k \tilde{\psi}^*(k) \tilde{\psi}(k) \quad (6)$$

where $\omega_k = \sqrt{k^2 + m^2}$ and $\tilde{\psi}(k)$ is the Fourier transform of $\psi(x)$. If D is the domain of finite linear combinations of vectors in $\bigcup_n D_n$, it is known that the operator H_0 , defined in a heuristic way by (5) or (6), is essentially self-adjoint on D ; and

$$(\Psi, H_0 \Psi) \geq mn \quad \text{if } \Psi \in D_n . \quad (7)$$

We now define an interaction term; divide space into a grid of unit cubes \mathcal{A}_i , the corners of which lie at points i of the subgroup Z^3 of R^3 . We pick a test-function $f \in \mathcal{D}(\mathcal{A}_0)$, invariant under rotations about the centre of \mathcal{A}_0 , and such that $\|f\|_2 = 1$. Define the translated functions by

$$f_i(x) = f(x - i). \tag{8}$$

We note that the functions f_i form part of a basis of $\mathcal{L}^2(R^3)$, so that

$$0 \leq \sum_i \psi^*(\bar{f}_i) \psi(f_i) \leq N. \tag{9}$$

Let P be any hermitian polynomial in the smeared creation and annihilation operators such that

$$\|P\| \leq 1, \quad [P, \psi^*(\bar{f}) \psi(f)] = 0. \tag{10}$$

We also assume that the test-functions used in P are invariant under rotations about a common point. For example, if g and h are normalized test-functions $\in \mathcal{D}(\mathcal{A}_0)$, rotation invariant and orthogonal to f , then we may take $P = \psi^*(\bar{g}) \psi^*(\bar{h}) + \psi(h) \psi(g)$. The interaction Hamiltonian is taken to be (λ is real)

$$V = \lambda \int d^3a \psi_a^*(\bar{f}) \psi_a(f) P_a \tag{11}$$

where $\psi_a(x) = \psi(x + a)$ and $P_a = U(a) P U^{-1}(a)$ is the translated operator. By simple computation, V and $H = H_0 + V$ have a meaning as symmetric operators when applied to D .

III. Positivity of the Hamiltonian

If A and B are bounded non-negative operators, and $[A, B] = 0$, then $AB \geq 0$. Thus if $\|B\| \leq 1$ then $1 \pm B \geq 0$, so that $A(1 \pm B) \geq 0$, giving

$$-A \leq AB \leq A. \tag{12}$$

Theorem 1. If $|\lambda| \leq m$, then H has a unique self-adjoint extension, and this extension is non-negative.

Proof. We apply (12) to $A_i^g = \psi_a^*(\bar{f}_i) \psi_a(f_i)$ and $B = P_{a+i}$. Thus

$$- \psi_a^*(\bar{f}_i) \psi_a(f_i) \leq \psi_a^*(\bar{f}_i) \psi_a(f_i) P_{a+i} \leq \psi_a^*(\bar{f}_i) \psi_a(f_i).$$

This holds for all i , and so using (9)

$$-N \leq - \sum_i \psi_a^*(\bar{f}_i) \psi_a(f_i) \leq \sum_i A_i^g P_{a+i} \leq \sum_i \psi_a^*(\bar{f}_i) \psi_a(f_i) \leq N.$$

Since $V = \lambda \int_{a \in \mathcal{A}} d^3a \sum_i A_i^g P_{a+i}$ we see that if $\Phi \in D$

$$-|\lambda|(\Phi, N\Phi) \leq (\Phi, V\Phi) \leq |\lambda|(\Phi, N\Phi).$$

Therefore

$$(\Phi, H\Phi) \geq (\Phi, H_0\Phi) - |(\Phi, V\Phi)| \geq (\Phi, H_0\Phi) - |\lambda|(\Phi, N\Phi) \geq 0$$

if $|\lambda| \leq m$, using (7).

Now a non-negative symmetric operator possesses a non-negative self-adjoint extension [6]. But we can prove more, namely that H is essentially self-adjoint on D , using the following slight modification¹ of Friedrich's theorem: Suppose D is a dense domain in \mathcal{H} , and $\langle \cdot, \cdot \rangle$ a non-negative sesquilinear form on $D \times D$, and X an essentially self-adjoint operator on D , such that

$$0 \leq \langle \Phi, \Phi \rangle \leq (\Phi, X\Phi), \quad \Phi \in D.$$

Then there exists a unique self-adjoint operator Y such that $\langle \Phi, \Psi \rangle = (\Phi, Y\Psi)$. We apply the theorem with $\langle \Phi, \Psi \rangle = (\Phi, H\Psi)$, noting that $0 \leq (\Phi, H\Phi) \leq (1 + |\lambda|/m) (\Phi, H_0\Phi)$ and that H_0 is essentially self-adjoint on D . This completes the proof.

In these models the interaction term is dominated by the free term; they therefore fall within the class discussed by KATO [7].

IV. The Time Evolution

Let \mathfrak{A} be the C^* -algebra generated by the fields $\psi^*(f)$, $f \in \mathcal{D}(R^3)$ at time $t = 0$. It is not immediately obvious that this is the same algebra as that generated by the field at a later time. Thus we have to show that if $A \in \mathfrak{A}$ then $A(t) = U(t) A U^{-1}(t) \in \mathfrak{A}$, where $U(t) = e^{iHt}$.

GUENIN introduced the following description of the time-development of the operators in a quantum system [5]. Let $A \in \mathfrak{A}$; then the family of bounded operators

$$A_G(t) = e^{-iH_0t} e^{iHt} A e^{-iHt} e^{iH_0t} \tag{13}$$

lie in \mathfrak{A} if and only if $A(t) \in \mathfrak{A}$, since the free motion is known to give an automorphism. Define $V(t) = e^{-iH_0t} V e^{iH_0t}$. Then $A_G(t)$ satisfies the integral equation

$$A_G(t) = A_G + i \int_0^t [V(t'), A_G(t')] dt' \tag{14}$$

leading to a perturbation series whose $(n + 1)$ th term is

$$i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n [V(t_1), [V(t_2), \dots, [V(t_n), A] \dots]]. \tag{15}$$

Theorem 2. Let H_0 be as in (5) or (6), and let P be any even polynomial in ψ and ψ^* smeared with test-functions in S . Let

$$V = \int da U(a) P U^{-1}(a), \quad \alpha \in L^2(R^3)$$

and $A = \psi(\alpha)$ or $\psi^*(\alpha)$. Then Guenin's series (15) converges for small enough t , the radius of convergence being independent of α .

Proof. If v is a monomial in ψ and ψ^* , of degree $2k$, then $[v, \psi]$ consists of at most $2k$ terms, each of degree $2k - 1$; thus if v_1, \dots, v_n

¹ H. ARAKI (private communication).

are monomials of maximum degree $2k$, then $[v_1[v_2[\dots[v_n, \psi] \dots]]]$ consists of at most $2k \cdot (2k - 1) 2k \dots (2(n - 1)k - 2n + 3) < n!(2k)^{2n}$ terms. Since V is a finite sum of monomials, there exists an integer K such that number of monomials in (15) is less than $K^n n!$

Let $h_j \in S$ denote a typical test-function smearing the fields, and write

$$G_{ij}(a, t) = \{\hat{\psi}_a^*(h_i, t), \psi(h_j, 0)\} \tag{16}$$

$$G_i(a, t) = \{\hat{\psi}_a^*(h_i, t), \psi(\alpha)\} \tag{17}$$

(denoting $e^{-iH_0 t} \psi(\cdot) e^{iH_0 t}$ by $\hat{\psi}(\cdot, t)$). Then each monomial in the integrand of (15) is of the form

$$\int da_1 \int \dots \int da_n G(a_1 - a_{j_1}, t_1 - t_{j_1}) \dots G(a_{n-1} - a_{j_{n-1}}, t_{n-1} - t_{j_{n-1}}) \\ G(a_n, t_n) \prod_m \hat{\psi}_{a_m}^{(*)}(h_m, t_m) \tag{18}$$

where m runs over some finite set, and $\psi^{(*)}$ means either ψ or ψ^* . The indices $j_1 \dots j_n$ are chosen from $1, \dots, n$ with at most $k - 1$ repetitions. Since $e^{iH_0 t}$ and $U(a)$ are unitary, the norm of $\prod_m \hat{\psi}_{a_m}^{(*)}(h_m, t_m)$ is unity (assuming the test-functions are normalized). Therefore the question reduces to the convergence of

$$\Sigma K^n n! \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \int |G(a_1 - a_{j_1}, t_1 - t_{j_1})| \dots \\ \dots |G(a_n, t_n)| da_1 \dots da_n. \tag{19}$$

If we choose H_0 as in (5) or (6), then the G 's are respectively smooth solutions of Schrodinger's equation and the Klein-Gordon equation. It is easily shown by the method of JOST [8] that there exists a constant C (depending on bounds on the various h_j and the first few derivatives) such that

$$\int |G_{ij}(a, t)| d^3a < C \quad \text{if} \quad |t| \leq 1.$$

Bounding successively the integrals $\int da_1, \int da_2$ etc. by C we can bound the sum (15) by the geometric series $\Sigma (KCt)^n$, thus completing the proof of theorem 2.

Corollary. For our models, $U(t) \mathfrak{Q} U^{-1}(t) = \mathfrak{Q}$.

Proof. The relation $\|\psi(f)\| = \|f\|_2$ implies that \mathfrak{Q} is separable and since the momentum operator is self-adjoint, the action of space-translation on \mathfrak{Q} is continuous, i.e., $\|A_a - A\| \rightarrow 0$ as $a \rightarrow 0$ in R^3 . (See [9] for a proof of this.) We may therefore regard the space integrals in (18) in the Riemann-Bochner sense (rather than as weak integrals), and since they converge, the term (18) lies in \mathfrak{Q} . The same remarks apply to the integrations over time, and so every term in Guenin's series lies in \mathfrak{Q} , and therefore the sum does. But the matrix elements of this series between vectors

in D coincides with the series for the matrix elements of $\psi(\alpha, t)$. Since $\psi(\alpha, t)$ is bounded, the series converges to $\psi(\alpha, t)$, which therefore lies in \mathfrak{A} (for small t). Since $\psi(\alpha, 0)$ generate \mathfrak{A} , $A(t)$ lies in \mathfrak{A} for all $A \in \mathfrak{A}$. The group law $U(t_1)U(t_2) = U(t_1 + t_2)$ then ensures $A(t) \in \mathfrak{A}$ for all t .

Remarks. 1. For small times the Wightman functions are analytic in λ .

2. As in [5] one can show that the series defines an automorphism of \mathfrak{A} without recourse to the existence of H as a self-adjoint operator on \mathcal{H} , and this includes the case where V causes vacuum polarization. If $V\Omega = 0$, Ω is invariant under the automorphism showing that $H_0 + V$ has a self-adjoint extension in Fock space (not positive in general).

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9. Ref. [1], Proposition on p. 4.