Commun. math. Phys. 7, 51—54 (1968)

A Geometrical Model Showing the Independence of Locality and Positivity of the Energy

S. Doplicher*
Palmer Physical Laboratory, Princeton University, N. J.

T. Regge
Institute for Advanced Study, Princeton, N. J.

and I. M. Singer**
M. I. T., Mathematics Department, Cambridge, Mass.

Received June 1, 1967

Abstract. A model of local rings system is constructed which satisfies all the usual postulates but has no representation where the energy momentum spectrum lies in the forward light cone.

1. Introduction

We consider local rings systems described by a C^*-algebra $\mathfrak{A}$, the algebra of quasilocal observables [1], a mapping $O \to \mathfrak{A}(O)$ which assigns to each bounded open region in space-time the C^*-subalgebra of $\mathfrak{A}$ of the observables localized in O, and a representation $x \to \alpha_x$ of the space-time translation group T in the $*$-automorphisms group of $\mathfrak{A}$ such that

$\mathfrak{A}$ is simple with unit; (1)

$\mathfrak{A}(O_1) \subseteq \mathfrak{A}(O_2)$ if $O_1 \subseteq O_2$ (isotony),
$AB = BA$ all $A \in \mathfrak{A}(O_1)$, $B \in \mathfrak{A}(O_2)$ if $O_1 - O_2$ is spacelike (locality), $\bigcup_O \mathfrak{A}(O)$ is dense in $\mathfrak{A}$; (2)

$\alpha_x(\mathfrak{A}(O)) = \mathfrak{A}(O + x)$, $x \in T$; (3)

$x \to \alpha_x(A)$, $A \in \mathfrak{A}$, is continuous from T into $\mathfrak{A}$ [1]; (4)

if Δ is the region between any two assigned equal time hyperplanes, then (5)
$\bigcup_{O \subseteq \Delta} \mathfrak{A}(O)$ generates $\mathfrak{A}$ (time slice axiom) [4].

We say that a representation π of $\mathfrak{A}$ on the Hilbert space $\mathfrak{H}$ satisfies the spectrum condition if there exists a unitary continuous representa-

---

* On leave from Istituto di Fisica G. Marconi, Università di Roma.

** Work supported by the National Science Foundation (Contract GP-6166).

1 For a discussion and motivation of (4) see ref. [2]; for some special consequences of it see [3]. Though (4) was not explicitly stated in [5], some implications of it were used there.

tion $\mathbf{U}$ of T on $\mathfrak{H}$ such that

$$\pi(\alpha_x(A)) = \mathbf{U}(x)\,\pi(A)\,\mathbf{U}(x)^{-1} \quad \text{if} \quad x \in T,\ A \in \mathfrak{A};$$
$$\mathbf{U}(x) = \int_{\bar{V}^+} e^{ipx}\, dE(p)$$

where $\bar{V}^+$ denotes the closed forward light cone. An algebraic condition for the existence of such a representation is the following [5]: there is a closed left ideal $\mathscr{I}$ in $\mathfrak{A}$ such that, if $\mathfrak{A}$ satisfies (1)—(4), it has a representation satisfying the spectrum condition if and only if

$$\mathscr{I} \neq \mathfrak{A}\,. \tag{6}$$

By the Reeh-Schlieder lemma, it follows from (1)—(4) that only the following two cases are possible [5]:

(i) $\mathscr{I} \cap \mathfrak{A}(O) = \{0\}$ all $O \subsetneqq \mathbb{R}^4$;

(ii) $\mathscr{I} = \mathfrak{A}$.

In this note we construct a model satisfying (1)—(5) but not (6) showing that (ii) is also possible and (6) is an independent axiom.

2. The Model

We call $\mathscr{F}$ the set of all timelike or lightlike straight lines in Minkowski space, and define a map $O \to \mathscr{F}_O$ which assignes to each region O in $\mathbb{R}^4$ the set of all elements in $\mathscr{F}$ which intersect O. This map has the properties

(i) $\mathscr{F}_{O_1} \cap \mathscr{F}_{O_2} = \emptyset$ if and only if $O_1 - O_2$ is spacelike;

(ii) $\mathscr{F}_O = \mathscr{F}_{\hat{O}}$, where $\hat{O} = \{y/u \cap O \neq \emptyset \text{ if } y \in u,\, u \in \mathscr{F}\}$

(iii) $\delta_x \mathscr{F}_O = \mathscr{F}_{O+x}$, where $\delta_x u \in \mathscr{F}$ is the line obtained translating $u \in \mathscr{F}$ by $x \in T$.

Given a reference frame in $\mathbb{R}^4$, elements $u \in \mathscr{F}$ can be parametrized by the cohordinate $\mathbf{a}$ of the intersection of u with the hyperplane $t = 0$ and by the velocity v corresponding to the world line u, and $\mathscr{F}$ is accordingly identified with the subset of $\mathbb{R}^6$ of all pairs $(\mathbf{a}, \mathrm{v})$ with $\mathbf{a}, \mathrm{v} \in \mathbb{R}^3$ and $|\mathrm{v}| < 1$. If $u = (\mathbf{a}, \mathrm{v})$, $\delta_x u = (\mathbf{a} + \mathrm{x} - \mathrm{v}t, \mathrm{v})$ for $x = (\mathrm{x}, t) \in T$, and the Lebesgue measure du on $\mathscr{F}$ is evidently *translation invariant.*

Let K be the Hilbert space $\mathscr{L}^2(\mathscr{F}, du)$; with $f \in K$, define $U(x) f \in K$ by

$$(U(x)f)(u) = f(\delta_{-x} u), \quad x \in T,\ u \in \mathscr{F}$$

$x \to U(x)$ is a strongly continuous unitary representation of T in K, whose infinitesimal operators are

$$(-i\nabla_{\mathbf{a}},\, i\mathrm{v} \cdot \nabla_{\mathbf{a}});$$

the spectrum of U is accordingly the set of all spacelike or lightlike points, the spectral measure being equivalent to Lebesgue measure.

Let K_O be the subspace of K of all functions in K with support in $\mathscr{F}_O$, with O a region in spacetime, $\tilde{\mathfrak{A}}$ the C^*-algebra of Canonical Anticommutation Relations on K, $\mathfrak{A}(\mathfrak{A}(O))$ the norm closure in $\tilde{\mathfrak{A}}$ of the set of all *even* polynomials in the field operators $\psi(f)$ with $f \in K$ $(f \in K_O)$. We have that

a) $\mathfrak{A}$ *is a simple C^*-algebra with unit* [6]; moreover, from (i) and (ii) above,

b) *the map* $O \to \mathfrak{A}(O)$ *satisfies* (2) *and* (5) *of the preceeding paragraph.*

We define now the action of the translation group on $\mathfrak{A}$; let $x \to \tilde{\alpha}_x$ be the *-automorphisms of $\tilde{\mathfrak{A}}$ induced by $U(x)$, $x \in T$, namely such that

$$\tilde{\alpha}_x(\psi(f)) = \psi(U(x)f), \quad f \in K;$$

$\tilde{\alpha}_x$ leaves $\mathfrak{A}$ invariant; call α_x its restriction to $\mathfrak{A}$. From (iii) and the strong continuity of $U(x)$ we have that

c) *the map* $x \to \alpha_x$ *is a representation of* T *in* $^*\mathrm{Aut}\,\mathfrak{A}$ *satisfying* (3) *and* (4).

We now show that

d) $\mathfrak{A}$ *has no representation satisfying the spectrum condition.*

It suffices to show that there exists no such representation π of $\mathfrak{A}$ with a vector Ω invariant under $\mathbf{U}(x)$, $x \in T$ [5]. Assume $\|\Omega\| = 1$.

With $f \in K$ and $\bar{f}$ the complex conjugate function, define $Q_f = \psi(f) + \psi(\bar{f})^* \in \tilde{\mathfrak{A}}$; with $x \in T$, $\tilde{\alpha}_x(Q_f)\, Q_f \in \mathfrak{A}$ and the vector

$$\pi(\tilde{\alpha}_x(Q_f)\, Q_f)\,\Omega \tag{7}$$

has spectrum in $\mathscr{S} + \mathscr{S}$ if $\mathscr{S}$ is the spectrum of $f \in K$ for the representation $x \to U(x)$. Moreover the vector (7) is zero for all $x \in T$ only if $\langle \Omega\, \pi(Q_f^*\, Q_f)\, \Omega \rangle = 0$[2]. However the expression $\langle f, g \rangle_1 = \langle \Omega, \pi(Q_f^*\, Q_g)\, \Omega \rangle$, $f, g \in K$, is a bounded inner product in K which is invariant under $U(x)$, $x \in T$, and non zero, because the anti-commutation relations give $\langle f, f \rangle_1 = \langle f, f \rangle$ if $f = \bar{f}$. With U_1 the restriction of U to the orthogonal complement in K of the null space of $\langle \cdot, \cdot \rangle_1$ and p a point in the spectrum of U_1, we can choose a neighborhood $\mathscr{S}$ of p such that $\mathscr{S} + \mathscr{S}$ is spacelike and $f \in K$ with spectrum in $\mathscr{S}$ such that $\langle f, f \rangle_1 \neq 0$, whence the vector (7) is non zero for some $x \in T$ and d) is proved.

Remark. The model is actually invariant under the inhomogeneous Lorentz group G. If $u \to \delta_g u$ is the action of $g \in G$ on the straight line $u \in \mathscr{F}$, an easy computation shows that for all $g \in G$

$$\frac{d\,\delta_g u}{d\,u} = (1 - \boldsymbol{\beta}^2)^{3/2}\,(1 + \boldsymbol{\beta}\,\mathbf{v})^{-3}$$

(where $u = (\mathbf{a}, \mathbf{v})$ and $g = (x, \Lambda_{\boldsymbol{\beta}} R)$ with R a space rotation and $\Lambda_{\boldsymbol{\beta}}$ the pure Lorentz transformation to the velocity $\boldsymbol{\beta}$) which is a continuous

[2] This follows from a computation similar to the one in Theorem 3b, ref. [7].

positive function on $\mathscr{F}$, so that du is *quasi invariant.* A unitary continuous representation U of G in K which extends $x \to U(x)$, is defined by

$$f \in K \to U(g)\, f \in K \quad \text{with} \quad (U(g) f)(u) = \sqrt{\frac{d\,\delta_{g^{-1}} u}{d u}}\; f(\delta_{g^{-1}} u);$$

we have clearly $U(g) K_O = K_{gO}$ for all regions O, and $U(g)$ induces a representation of G in $^*\mathrm{Aut}\,\mathfrak{A}$ which satisfies the obvious generalizations of (3) and (4)[3].

Acknowledgments. We would like to thank professor A. S. WIGHTMAN for his interest in this note; one of us (S. D.) would like to thank him also for the hospitality received in Princeton University.

References

1. HAAG, R., and D. KASTLER: J. Math. Phys. **5**, 848 (1964).
2. DOPLICHER, S., D. KASTLER, and D. W. ROBINSON: Commun. Math. Phys. **3**, 1 (1966).
3. MONTVAY, I.: Nuovo Cimento **40**, 121 (1965).
4. HAAG, R., and B. SCHROER: J. Math. Phys. **3**, 248 (1962).
5. DOPLICHER, S.: Commun. Math. Phys. **1**, 1 (1965).
6. —, and R. T. POWERS: On the simplicity of the even CAR algebra and free field models. To appear in Commun. Math. Phys.
7. ROBINSON, D. W., and D. RUELLE: Extremal invariant states, to appear in Ann. Inst. H. Poincaré.
8. LURÇAT, F.: Physics, **1**, N° 2, 95 (1964).

[3] The model so obtained is remindful of Lurçat's theory [8].