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Analytic Continuation of Group Representations. VI*

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Abstract. The Gell-Mann formula for analytically continuing group representations is worked out explicitly for more cases than in previous work, and extended to certain pseudo-Riemannian symmetric spaces. The method of finding the asymptotic behavior of matrix elements of group representations introduced in Part V is developed in more detail and it is shown how it leads to new mathematical problems in the theory of dynamical systems and Hilbert space theory.

I. Introduction

We continue work on the order of ideas introduced in the earlier papers of this series [4]. The main types discussed here are: Further development of the Gell-Mann formula [3], and development of the theory of asymptotic behavior of matrix elements of group representations.

II. The Gell-Mann Formula in Terms of the Enveloping Algebra

Suppose that $\mathbf{K}$ is a Lie algebra, with a basis $Z_i (1 \leqq i, j, \ldots, \leqq n$; summation convention) such that:

$$[Z_i, Z_j] = c_{ijk} Z_k .$$

Suppose $\mathbf{P}$ is an abelian Lie algebra, with a basis X_a $(1 \leqq a, b, \ldots \leqq m)$. Suppose that $\mathfrak{G}' = \mathbf{K} + \mathbf{P}$ is a Lie algebra with $\mathbf{P}$ an ideal, i. e.,

$$[Z_i, X_a] = c_{iab} X_b .$$

Form the elements:

$$X_a^\lambda = [\Delta, X_a] + \lambda X_a$$

of $U(\mathfrak{G}')$, the universal enveloping algebra of $\mathfrak{G}'$ (Δ is the second order Casimir operator of $\mathbf{K}$). In [3] we have investigated the condition that $[X_a^\lambda, X_b^\lambda]$ be expressible in terms of the Z's, where $\mathfrak{G}$ is realized as a Lie algebra of skew-Hermitian operators on a Hilbert space H. Here, we will present a representation-independent version of this calculation, aiming to find conditions that $[X_a^\lambda, X_b^\lambda]$ be expressible within the en-

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veloping algebra of $\mathfrak{G}'$ in terms of the Z's and the Casimir operator of $\mathfrak{G}'$. (Such a calculation has been done in [4] for the case $\mathbf{K} = SO(2, R)$, $\mathbf{P}'$ 2-dimensional, i. e., $\mathfrak{G}'$ = Lie algebra of the group of rigid motions of the plane.)

Let us proceed to the calculation. Δ is of the form $g_{ij} Z_i Z_j$. For $X \in \mathbf{P}$, $X^0 = [\Delta, X] = f_i(X) Z_i + g_a(X) X_b$, where each $f_i(X)$ is a linear polynomial in the X's, and each g_a is a real number.

For $Z \in \mathbf{K}$, we have

$$\begin{aligned}[Z, X^0] &= [\Delta, [Z, X]] = f_i([Z, X]) Z_i + g_a([Z, X]) X_a \\ &= [Z, f_i(X)] Z_i + f_i(X) [Z, Z_i] + g_a(X) [Z, X_a] \\ &= [Z, f_i(X)] Z_i + f_i(X) c_{ij}(Z) Z_j + g_a(X) c_{ab}(Z) X_b\,,\end{aligned}$$

where $c_{ij}(Z)$ and $c_{ab}(Z)$ are defined by:

$$\begin{aligned}[Z, Z_i] &= c_{ij}(Z) Z_j \\ [Z, X_a] &= c_{ab}(Z) X_b\,.\end{aligned}$$

Thus, $Z \to (c_{ij}(Z))$ and $(c_{ab}(Z))$ define matrix representations of $\mathbf{K}$, that are, in fact, just the matrix representations corresponding to $A\,dK$ acting in $\mathbf{K}$ and $\mathbf{P}$, respectively. Comparing these two calculations gives the relations:

$$[Z, f_j(X)] + c_{ij}(Z) f_i(X) = f_j([Z, X]) \tag{2.1}$$

$$g_b([Z, X]) = g_a(X) c_{ab}(Z) \tag{2.2}$$

$$\text{for} \quad X \in \mathbf{P}, Z \in \mathbf{K}\,.$$

Put:

$$X' = f_i(X) Z_i \quad \text{for} \quad X \in \mathbf{P}\,.$$

For X, $Y \in \mathbf{P}$,

$$\begin{aligned}[X', Y'] &= f_i(X) Z_i f_j(Y) Z_j - f_i(Y) Z_i f_j(X) Z_j \\ &= f_i(X) f_j(Y) Z_i Z_j - f_i(Y) f_j(X) Z_i Z_j + \\ &\quad + f_i(X) [Z_i, f_j(Y)] Z_j - f_i(Y) [Z_i, f_j(X)] Z_j \\ &= f_i(X) f_j(Y) [Z_i, Z_j] + \\ &\quad + f_i(X) (f_j([Z_i, Y]) - c_{kj}(Z_i) f_k(Y)) Z_j - \\ &\quad - f_i(Y) (f_j([Z_i, X]) - c_{kj}(Z_i) f_k(X)) Z_j\,.\end{aligned}$$

Set this equal to:

$$f_j(X, Y) X_j \tag{2.3}$$

with

$$\begin{aligned}f_j(X, Y) = f_i(X) f_j([Z_i, Y]) - f_i(Y) f_j([Z_i, X]) + \\ + c_{jk}(Z_i) (f_i(Y) f_k(X) - f_i(X) f_k(Y))\,.\end{aligned}$$

Also, we have:

$$[Z, X'] = [Z, X]' \quad \text{for} \quad Z \in \mathbf{K}, X \in \mathbf{P} .$$

Hence,

$$\begin{aligned}&[[Z, X]', Y'] + [X', [Z, Y]']\\ &\quad = [Z, f_j(X, Y)] Z_j + f_j(X, Y) [Z, Z_j] , \quad \text{or}\\ &\quad f_k([Z, X], Y) + f_k X, ([Z, Y])\\ &\quad = [Z, f_k(X, Y)] + c_{jk}(Z) f_j(X, Y) .\end{aligned} \tag{2.4}$$

Let us make explicit that f_i is a second-degree polynomial:

$$[X', Y'] = A_{abj}(X, Y) X_a X_b Z_j ,$$

where (X_a) $(1 \leqq a, b, \leqq m)$ is a basis for $\mathbf{P}$.

Then, $A_{abj}(X, Y)$ can be supposed symmetric in the indices a, b, and skew-symmetric in X and Y. Suppose

$$\Delta' = g_{ab} X_a X_b$$

is a Casimir operator of $\mathfrak{G}$, i. e.,

$$[\mathbf{K}, \Delta'] = 0$$

If there are constants (c_{abi}) such that:

$$[X'_a, X'_b] = \Delta' c_{abi} Z_i , \tag{2.5}$$

then we have a relation of the following form:

$$\left[\frac{X'_a}{\sqrt{\Delta'}}, \frac{X'_b}{\sqrt{\Delta'}}\right] = c_{abi} Z_i . \tag{2.6}$$

Thus, at the expense of addition to $U(\mathfrak{G})$ elements that are more general than polynomial "functions" of the elements of $\mathfrak{G}$, we have constructed a new Lie algebra whose basis is $(Z_i, X'_a/\sqrt{\Delta'})$.

The existence of the (c_{abi}) and (g_{ab}) can be approached in two ways: Either they can be constructed explicitly in the needed special cases, or one can attempt to prove by using basic principles that conditions 2.4 imply conditions in the tensor $(A_{abj}(X_c, X_d))$ that in turn imply it must be of the form 2.5. The latter approach involves a generalization of KONSTANT's results [7] on the decomposition of the universal enveloping algebra under the action of a linear group, and will not be attempted in this paper. Note, however, that 2.6 implies that the Gell-Mann formula holds for representations of $\mathfrak{G}'$, a topic we have analyzed in [3], at least for representations in which the operators of P are "diagonalizable". Thus, the conditions presented in [3] can be regarded as necessary conditions that a Gell-Mann formula of type 2.6 hold in the enveloping algebra.

We now turn to computing some examples.

III. The Gell-Mann Formula for Rotation Groups

Let $K = SO(n, R)$, $\mathbf{P} = R^n$, with the representation of K on P just the "vector" representation of $SO(n, R)$. We shall show that the Gell-Mann formula holds within the enveloping algebra of $G' = \mathbf{K} + \mathbf{P}$, in the sense described in Section II. We will not use the technique described in Section II, but another that has interesting geometric consequences.

Regard $\mathbf{K}$ as a Lie algebra of differential operators on R^n: Z_{ij}, $1 \leqq 1$, $j, \ldots \leqq n$ summation convention in force, are the generators of K, with

$$Z_{ij} = x_i \partial_j - x_j \partial_i .$$

$\left(\text{Notation: } x_i \text{ the Euclidean coordinates on } R^n,\ \partial_j = \frac{\partial}{\partial x_j}\right).$

$\mathbf{P}$ is realized as the vector space generated by the x_i.

$\Delta = Z_{ij} Z_{ij}$ is the second degree Casimir operator of $SO(n, R)$.

Following the Gell-Mann formula prescription, we construct the operators:

$$\begin{aligned} X_k &= [Z_{ij}, x_k] Z_{ij} && (3.1)\\ &= (x_i \delta_{jk} - x_j \delta_{ik}) Z_{ij} \\ &= x_i Z_{ik} - x_j Z_{kj} \\ &= 2x_i Z_{ik} \\ &= 2x_i (x_i \partial_k - x_k \partial_i) \\ &= 2(r^2 \partial_k - x_k X) && (3.2) \end{aligned}$$

with the following notations:

$$r^2 = x_i x_i,\ X = x_i \partial_i . \tag{3.3}$$

Now,

$$\begin{aligned} X(x_k) &= [X, x_k] = x_k \\ [X, \partial_k] &= -\partial_k \\ [X, r^2] &= 2r^2 \\ [\partial_j, r^2] &= 2x_j . \end{aligned}$$

Hence,

$$\begin{aligned} [X_k, r^2] &= 2(2r^2 x_k - 2r^2 x_k) = 0 && (3.4)\\ [X, X_k] &= 2(2r^2 \partial_k - r^2 \partial_k - x_k X) \\ \frac{1}{2} [\partial_j, X_k] &= 2x_j \partial_k - \delta_{jk} X - x_k \partial_j . \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{2} [X_j, X_k] &= [X_j, r^2 \partial_k - x_k X] \\ &= 2r^2(\delta_{jk} X + x_j \partial_k - 2x_k \partial_j) - \\ &\quad - 2(r^2 \delta_{jk} - x_j x_k) X - \\ &\quad - x_k(-X_j) \\ &= 2r^2 \delta_{jk} X + 2r^2 x_j \partial_k - 4r^2 x_k \partial_j - \\ &\quad - 2r^2 \delta_{jk} X + 2x_j x_k X + 2x_k r^2 \partial_j - 2x_k x_j X \\ &= 2r^2 (x_j \partial_k - x_k \partial_j) \\ &= 2r^2 X_{kj} . && (3.5) \end{aligned}$$

Let $\mathfrak{G}' = \mathbf{K} + \mathbf{R}^n$, the Lie algebra of the group of rigid motions of R^n. Although we have calculated in terms of a realization of $U(\mathfrak{G}')$ by differential operators on R^n, the results are also true in $U(\mathfrak{G})$, since the realization of $U(\mathfrak{G}')$ by differential operators on R^n is faithful. 3.4 is then interpreted as a Gell-Mann formula giving the Lie algebra of $SO(n, 1)$ in terms of elements of $U(\mathfrak{G}')$: In fact,

$$\left[\frac{X_j}{2r}, \frac{X_k}{2r}\right] = Z_{jk} \tag{3.6}$$

which show that the Z_{jk} and $\frac{X_j}{2r}$ together generate the Lie algebra of $SO(n, 1)$.

3.4 has an interesting geometric interpretation. Interpret X_k as a first order differential operator, i. e., as a vector field on R^n. 3.4 then says that this vector field is tangent to the surfaces

$$r^2 = \text{constant}\,,$$

i. e., to the spheres in R^n. Of course, the Z_{ij} are also tangent to these spheres; the Z_{ij} and X_k when restricted to each such sphere generate a transformation group, whose Lie algebra is isomorphic with $SO(n, 1)$. One knows that $SO(n, 1)$ is just the group of conformal transformations (of the metric constant curvature) on the sphere. Now, the group generated by the Z_{ij} acts as a group of isometries of this metric. It is reasonable, then, to suspect that the one-parameter group generated by each X_j on the spheres 3.6 is a group of conformal transformations. In fact, we will now prove that this is so, using methods of differential geometry [6].

We must calculate the Lie derivative

$$\begin{aligned}
&X_k(dx_i\, dx_i)\\
&= 2d(X_k(x_i))\, dx_i\\
&= 4d(r^2\,\delta_{ki} - x_j\, x_i)\, dx_i\\
&= 4(2x_j\, dx_j\, \delta_{ki} - dx_k\, k_i - x_k\, dx_i)\, dx_i\\
&= 4(2x_j\, dx_j\, dx_k - dx_k\, x_i\, dx_i - x_k\, dx_i\, dx_i)\,.
\end{aligned}$$

On the hypersurface 3.6,

$$x_j\, dx_j = 0, \text{ hence with this relation },$$

$$X_k(dx_i\, dx_i) = -4x_k(dx_i\, dx_i)\,,$$

which shows that X_k is an infinitesimal conformal transformation on the sphere 3.6.

Thus we see that there is a close relation between the Gell-Mann formula for $SO(n, 1)$ and the geometric fact that the group acts as a group of conformal transformations on the plane.

IV. Deformation of the Gell-Mann Enveloping Algebraic Relations

Let us return to the general setting for the Gell-Mann formula, i. e., $G' = \mathbf{K}' + \mathbf{P}'$ is the semidirect sum of a Lie algebra $\mathbf{K}$ and an abelian Lie algebra $\mathbf{P}$. Let Z_u, $1 \leqq u, v, \ldots, \leqq m$, be a basis for K, X_i, $(1 \leqq 1_j, \ldots \leqq n)$ for a basis for P. Let

$$X_i^0 = [Z_u, X_i]\, Z_u \,. \tag{4.1}$$

In [5] we pointed out the following fact: If G' is realized as a Lie algebra of operators on a vector space H, if the Z_μ and X_i^0 given by 4.1 span a Lie algebra of operators, then the following operators also span the same Lie algebra, i. e., the Gell-Mann formula enables us to analytically continue representations:

$$X_i^\lambda = [Z_u, X_i]\, Z_u + \lambda\, X_i \,. \tag{4.2}$$

Now, we would like to inquire what this relation may mean in terms of the enveloping algebra interpretation of the Gell-Mann formula given in Section II. In fact, interpret 4.2 as a formula in the enveloping algebra $U(G')$. Suppose that:

$$[X_i^0, X_j^0] = \Delta'\, Z_{ij} \tag{4.3}$$

where Δ' is an element of the center of $U(G')$, and Z_{ij} are elements of K. Then,

$$\begin{aligned} [X_i^\lambda, X_j^\lambda] &= [X_i^0 + \lambda\, X_i, X_j^0 + \lambda\, X_j] \\ &= \Delta Z_{ij} + \lambda([X_i, X_j^0] + [X_i^0, X_j]) \\ &= \Delta Z_{ij} + \lambda([X_i, X_j^0] - [X_j, X_i^0]) \,. \end{aligned}$$

But,

$$[X_i^0, X_j] = [Z_u, X_i]\,[Z_u, X_j]$$

which is clearly symmetric in i and j (since $[\mathbf{K}, \mathbf{P}] \subset \mathbf{P}$, and $[\mathbf{P}, \mathbf{P}] = 0$), i. e.,

$$[\mathbf{X}_i^\lambda, \mathbf{X}_j^\lambda] = \Delta'\, Z_{ij}, \text{ hence:}$$

$$\left[\frac{X_i^\lambda}{\sqrt{\Delta'}}, \frac{X_j^\lambda}{\sqrt{\Delta'}}\right] = Z_{ij} \,. \tag{4.4}$$

We can now prove another useful fact about the Gell-Mann formula. Let us compute:

$$\Delta_K = Z_{ij}\, Z_{ij} \,. \tag{4.5}$$

Let

$$\Delta_\lambda = \frac{X_i^\lambda X_i^\lambda}{\Delta'} \,. \tag{4.6}$$

Assume that:

$$[\Delta_\lambda, X_i^\lambda] = 0 \,. \tag{4.7}$$

$$\begin{aligned}
\Delta_K &= \frac{1}{\Delta'^2}[X_i^\lambda, X_j^\lambda][X_i^\lambda, X_j^\lambda] \\
&= \frac{1}{\Delta'^2}(X_i^\lambda X_j^\lambda - X_j^\lambda X_i^\lambda)(X_i^\lambda X_j^\lambda - X_j^\lambda X_i^\lambda) \\
&= \frac{1}{\Delta'^2}(X_i^\lambda X_j^\lambda X_i^\lambda X_j^\lambda - X_j^\lambda X_i^\lambda X_i^\lambda X_j^\lambda - X_i^\lambda X_j^\lambda X_j^\lambda X_i^\lambda + X_j^\lambda X_i^\lambda X_j^\lambda X_i^\lambda) \\
&= \frac{1}{\Delta'^2}(X_i^\lambda X_i^\lambda X_j^\lambda X_j^\lambda + X_i^\lambda [X_j^\lambda, X_i^\lambda] X_j^\lambda - X_j^\lambda \Delta_\lambda \Delta' X_j^\lambda - X_i^\lambda \Delta_\lambda \Delta' X_i^\lambda + \\
&\quad + X_j^\lambda X_j^\lambda X_i^\lambda X_i^\lambda + X_j^\lambda [X_i^\lambda, X_j^\lambda] X_i^\lambda) \\
&= \frac{1}{\Delta'^2} 2 X_i^\lambda [X_j^\lambda, X_i^\lambda] X_j^\lambda = \frac{2}{\Delta} X_i^\lambda Z_{ji} X_j^\lambda \\
&= 2\left(\frac{X_i^\lambda}{\sqrt{\Delta'}} Z_{ji} \frac{X_j^\lambda}{\sqrt{\Delta'}}\right) .
\end{aligned} \tag{4.8}$$

Suppose now that $\mathbf{P}$ admits a positive definite quadratic form that is invariant under $A\,dK$. Suppose that the X_i were originally chosen to be an orthonormal basis with respect to the quadratic form. Then, it is readily verified that Δ_K is a Casimir operator of $\mathbf{K}$, i. e., is invariant under $A\,d\mathbf{K}$. Hence, so is the right-hand side of 4.8. But, this involves operators of $\mathfrak{G}^\lambda$ ($\mathfrak{G}^\lambda$ is the algebra generated by the Z_u and $X_i^\lambda/\sqrt{\Delta'}$). This is a nontrivial relation.

V. Relations between the Casimir Operators for the $SO(n, R)$ Gell-Mann Formula

Return to the situation considered in Section III, i. e., $K = SO(n, R)$, $\mathbf{P} = R^n$, $X_i = x_k$,

$$X_k^\lambda = r^2 \partial_k - x_k X + \lambda x_k \,.$$

Thus, we know from Section IV that:

$$\left[\frac{X_i^\lambda}{r}, \frac{X_j^\lambda}{r}\right] = Z_{ij} = x_i \partial_j - x_j \partial_i \,.$$

For each value of λ, let us compute the Casimir operator of the $SO(n, 1)$-algebra generated by Z_{ij}, X_i^λ/r, as a function of λ and r.

$$\begin{aligned}
X_k^\lambda X_k^\lambda &= (r^2 \partial_k - x_k X + x_k)(r^2 \partial_k - x_k X + \lambda x_k) \\
&= r^2 \partial_k r^2 \partial_k - r^2 \partial_k x_k X + \lambda r^2 \partial_k x_k - \\
&\quad - x_k X r^2 \partial_k + x_k X x_k X - \lambda x_k X x_k + \\
&\quad + \lambda(x_k r^2 \partial_k - r^2 X + \lambda r^2) \\
&= 2r^2 x_k \partial_k + r^4 \Delta - n r^2 X - r^2 X^2 + \\
&\quad + r^2 X + r^2 X^2 - \lambda r^2 - \lambda r^2 X + \\
&\quad + \lambda r^2 X - \lambda r^2 X + \lambda^2 r^2 \\
&\quad + \lambda r^2 X - \lambda r^2 X + \lambda^2 r^2 \\
&= (2r^2 - n r^2 + \lambda r^2 - 2r^2 + r^2 - \lambda r^2 + \lambda r^2 - \lambda r^2) X + r^4 \Delta - \\
&\quad - r^2 X^2 + x_k r^2 \partial_k + \\
&\quad + (-r^2 + r^2) X^2 + n \lambda r^2 + r^2(\lambda^2 - \lambda) \\
&= r^4 \Delta - r^2 X^2 + (2 - n) r^2 X + n \lambda r^2 + r^2(\lambda^2 - \lambda) \,,
\end{aligned}$$

hence:

$$\Delta_P^\lambda = \frac{X_k^\lambda}{r}\frac{X_k^\lambda}{r} = r^2\Delta - X^2 + [(2-n)X + (n\lambda + \lambda^2 - \lambda)$$
$$= \Delta_K + \lambda(n-1+\lambda)\,.$$

Now, $\Delta_K^\lambda - \Delta_K$ is the Casimir operator of the $SO(n, 1)$-algebra generated by the Z_{ij}, X_i^λ/r. Hence, we have:

Theorem 5.1. *The second degree Casimir operator of the representation of $SO(n, 1)$ defined by the Gell-Mann formula has the value:*

$$\lambda(n-1+\lambda)\,.$$

Finally, notice that there is a curious resemblance between this theory and that of the group-theoretic treatment of the hydrogen atom, by means of the Runge-Lenz vector [2].

VI. Complexification of the Gell-Mann Formula

Most of our work up to now has been concerned with semidirect product algebras $\mathbf{G}' = \mathbf{K} + \mathbf{P}$, with $[\mathbf{P}, \mathbf{P}] = 0$ and with $\mathbf{K}$ a compact Lie algebra. Suppose we consider an algebra of the form:

$$\mathbf{G}^0 = \mathbf{K}^0 + \mathbf{P}^0\,, \quad \text{with} \quad [\mathbf{P}^0, \mathbf{P}^0] = 0\,, \quad \text{and such that}$$

(a) $\mathbf{K}$ and $\mathbf{K}^0$ have the same complexification, i. e., $\mathbf{K} + i\,\mathbf{K}$ is isomorphic with $\mathbf{K}^0 + i\,\mathbf{K}^0$.

(b) $\mathbf{P}^0 + i\,\mathbf{P}^0$ is isomorphic to $\mathbf{P} + i\,\mathbf{P}$, and the isomorphism is compatible with (a).

Suppose the Gell-Mann formula holds within the enveloping algebra of $\mathbf{G}'$: Does it hold within $\mathbf{G}^0$? For example, we have proved in the last section that the enveloping-algebra Gell-Mann formula holds for $K = SO(4, R)$, $\mathbf{P}$ = vector representation. Choose $K^0 = SO(3, 1)$. $\mathbf{G}^0$ is then the Poincaré Lie algebra. A Gell-Mann formula for this would give in a way of relating the de-Sitter Lie algebra to the Poincaré Lie algebra[1].

Suppose that Δ^0 is a Casimir operator of $\mathbf{K}^0$, and that, for $X^0 \in \mathbf{P}^0$, we form:

$$X^{0\prime} = [\Delta^0, X^0]\,.$$

Suppose $\Delta^{0\prime}$ is a Casimir operator of $\mathbf{K}^0$. Suppose $T^0 : \mathbf{P}^0 \times \mathbf{P}^0 \to K^0$ is a skew-symmetric bliinear mapping that commutes with the action of $A\,dK^0$. Form:

$$[X^{0\prime}, Y^{0\prime}] - \Delta^{0\prime}\, T^0(X^0, Y^0) \quad \text{for} \quad X^0, Y^0 \in \mathbf{P}^0\,. \tag{6.1}$$

Notice that it vanishes if and only if its complexification vanishes. Thus, if Δ is a Casimir operator of $\mathbf{K}$; if $T : \mathbf{P} \times \mathbf{P} \to \mathbf{K}$ is a skew-symmetric

[1] Such a relation has been discovered by R. HWA, and one of the aims of this section is to show how this relation follows from the general theory.

bilinear mapping commuting with $A\ dK$; if Δ' is a Casimir operator of $\mathfrak{G}'$, such that:

(1) $X' = [\Delta, X]$, $[X', Y'] = \Delta'\ T(X, Y)$ for $X, Y \in P$,

(2) $\Delta = \Delta^0$, $T = T^0$, $\Delta'^0 = \Delta'$ under the isomorphism of the complexification of $\mathbf{K}^0$ and $\mathbf{K}$, $\mathbf{T}^0$ and $\mathbf{T}$,

then 6.1 does in fact vanish also, i. e., the Gell-Mann formula holds within the enveloping algebra.

There is, however, a new feature when $\mathbf{K}^0$ is not a compact Lie algebra. The Casimir operator $\Delta^{0\prime}$ of $\mathfrak{G}^0$ can have values of any sign in different representations. Thus, 6.1 is zero, we have

$$\left[\frac{X_0'}{\sqrt{\pm\Delta^{0\prime}}}\ \frac{Y_0'}{\sqrt{\pm\Delta^{0\prime}}}\right] = \pm T^0(X^0, Y^0)\,.$$

Thus, dependung on the representation chosen for $\mathfrak{G}^0$, we can realize two different Lie algebras. [For example, as is well-known, the Poincaré algebra can be approximated by $SO(3, 2)$ and $SO(4, 1)$.]

VII. Group Representations that are Linear in the Deformation Parameter

As we have indicated in [4], Part V (following Nijenhuis and Richardson [8]), there is a relation between deformations of group and Lie algebra deformations. Such relations are important, for example, in problems concerning the integral representation and asymptotic behavior of matrix elements of group representations. In [4], Part V, these relations were worked out in detail for the simplest example, $SL(2, R)$. In this section we present several further general remark, preparing the way for applications to representations satisfying the Gell-Mann formula in the following section. Let $\mathfrak{G}$ be a Lie algebra, ϱ a representation of $\mathfrak{G}$ by linear transformations on a vector space. H Let V be the space of linear operators: $H \to H$, and let Φ be the following representation of G in V:

$$\Phi(X)\,(A) = [\varrho(X), A] \quad \text{for} \quad A \in V, X \in G\,.$$

Suppose ϱ_λ is a one-parameter family of such representations, reducing to the given one at $\lambda = 0$, of the form:

$$\varrho_\lambda(X) = \varrho(X) + \lambda\,\omega(X)\,, \tag{7.1}$$

where ω is a linear mapping $\mathfrak{G} \to V$, i. e., a 1-cochain in $C^1(\Phi)$. We know that ω must satisfy the two conditions:

(a) $d\omega = 0$.

(b) $[\omega(X, \omega(Y)] = 0$ for $X, Y \in \mathfrak{G}$.

Theorem 7.1. *Suppose X is a fixed element of $\mathfrak{G}$, and A is an element of V such that:*

$$[\varrho(X), A] = \omega(X)\,, \tag{7.2}$$

$$[A, \omega(X)] = 0\,. \tag{7.3}$$

The operator

$B^\lambda = \exp(\lambda A)$, *exists, i. e., through the usual power series expansion. Then,*

$$B^\lambda \varrho_\lambda(X) = \varrho(X) B^\lambda . \tag{7.4}$$

Proof: If the usual power sense expansion for B^λ holds, then

$$\begin{aligned}[\varrho(X), B^\lambda] &= \lambda B^{\lambda-1}[\varrho(X), B] \\ &= \lambda B^{\lambda-1}[\varrho(X), A] B \\ &= \lambda B \omega(X) .\end{aligned}$$

This proves 7.4.

Note that B^λ is an intertwining operator between $\varrho_\lambda(X)$ and $\varrho(X)$. The physical interpretation of B^λ is then that it is the "S-matrix" relating $\varrho(X)$ to $\varrho_\lambda(X)$. For if $\varrho_\lambda(X)$ and $\varrho(X)$ were Hamilton operators on the Hilbert space H that were the Hamiltonians of physical systems, then 7.4 is the characteristic property of the "S-matrix".

Note also that 7.4 implies (at least formally) that

$$B^\lambda \exp(t \varrho_\lambda(X)) = \exp(t \varrho(X)) B^\lambda . \tag{7.5}$$

This relation was our starting point in [4], Part V, and we saw there how it could be used (in the case $G = SL(2, R)$) to derive results about the asymptotic behavior of the matrix elements of its representations.

Now, we turn to consideration of a class of representations for which one can find this intertwining operator B^λ explicitly. However, we must change our emphasis from algebra to geometry.

VIII. Continuations and Cocycles Determined by Tensor Fields

In this and the following sections, we will need the theory of differentiable manifolds and transformation groups, for which we refer to [1] and [6].

Let M be a manifold, with $F(M)$ its ring of real-valued, C^∞ functions. (All manifolds, maps, tensor fields, etc. will be of differentiability class C^∞ unless mentioned otherwise.)

A vector field, X, is a derivation of the ring $F(M)$, i. e., a linear map $f \to X(f)$ such that

$$X(f_1 f_2) = X(f_1) f_2 + f_1 X(f_2) \quad \text{for} \quad f_1, f_2 \in F(M) .$$

$V(M)$ denotes the set of vector fields. It is a Lie algebra, under the Jacobi bracket operation:

$$f \to [X, Y](f) = X(Y(f)) - Y(X(f)) .$$

If T is a tensor-field on M, $X \in V(M)$, $X(T)$ denotes the Lie-derivative of T by X, a tensor-field of the same algebraic type as ω. For example, if T

is an r-fold covariant tensor field, i. e., an $F(M)$ multilinear map

$$(X_1, \ldots, X_r) \to T(X_1, \ldots, X_r), \quad \in F(M), \quad \text{for} \quad X_1, \ldots, X_r \in V(M),$$

$$X(T)(X_1, \ldots, X_r) = X(T(X_1, \ldots, X_r)) - T([X, X_1], X_2, \ldots, X_r) - \cdots$$
$$- T(X_1, \ldots, [X, X_r]).$$

Lie derivative acts as a derivation on tensor-products of tensor fields.

$$X(T_1 \otimes T_2) = X(T_1) \otimes T_2 + T_1 \otimes X(T_2).$$

Suppose that G is a Lie algebra of vector fields on M, and T is a tensor-field such that:

$$X(T) = \omega(X)\, T \text{ for each } X \in \mathrm{G}.$$

$\omega(X)$ is to be an element of $F(M)$.

Now, for $X, Y \in G$,

$$\begin{aligned}
[X, Y](T) &= \omega([X, X])\, T \\
&= X(\omega(Y)\, T) - Y(\omega(X)\, T) \\
&= X(\omega(Y))\, T + \omega(Y)\, \omega(X)\, T - Y(\omega(X))\, T - \omega(X)\, \omega(Y)\, T \\
&= X(\omega(Y))\, T - Y(\omega(X))\, T, \text{ or}
\end{aligned}$$

$$X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = 0. \tag{8.1}$$

Let V be the space of linear mapping: $V(M) \to V(M)$. For each $A \in V(M)$, $X \in G$, define $\Phi(X)(A)$ as the commutator $[X, A]: f \to X A(f) - A X(f)$. Interpret each $\omega(X)$ as an element of V:

$$f \to \omega(X) f.$$

Then, ω can be interpreted as 1-cochain of G with coefficients on V, i. e., an element of $C^1(\Phi)$. 8.1 then says that this is a cocycle, since:

$$[X, \omega(Y)](f) = X(\omega(Y))(f).$$

Since further $[\omega(X), \omega(Y)] = 0$, we know from our earlier work that defining

$$\varrho_\lambda(X) = X + \lambda\, \omega(X) \quad \text{for} \quad X \in G$$

gives a one-parameter family of representations of G by operators on $F(M)$.

Let us see how ω changes when T us changed in the following way:

$$T' = f\, T, \text{ for a function } f \in F(M).$$

Then,

$$\begin{aligned}
X(T') &= \omega'(X)\, T \\
&= X(f)\, T + f\, \omega(X)\, T, \quad \text{or}
\end{aligned}$$

$$\omega'(X) = \left(\frac{X(f)}{f} + \omega(X)\right), \quad \text{or}$$

$$\omega'(X) = X(\log f) + \omega(X). \tag{8.2}$$

Thus, if $\log f \in F(M)$, ω' differs from ω by a coboundary.

Suppose that X is a fixed element of $\mathfrak{G}$, and we want to satisfy the hypotheses of Theorem 7.1, i. e., we want to find an $A \in V$ such that:

$$[X, A] = \omega(X)$$
$$[A, \omega(X)] = 0 .$$

We can satisfy the second of these conditions by demanding that A result from multiplication by a fixed function g_X. Then, the first condition requires that:

$$X(g_X) = \omega(X) . \tag{8.3}$$

Suppose, in local coordinates $(x_1, \ldots, x_n)$ for M, $X = A_1\, \partial/\partial x_1 + \cdots + A_n\, \partial/\partial x_n$. Then, g_X is a solution of the differential equation:

$$A_1 \frac{\partial g_X}{\partial x_1} + \cdots + A_n \frac{\partial g_X}{\partial x_n} = \omega(X) .$$

Let us examine the case where T is a differential form of the same degree as the dimension of M, i. e., a volume-element differential form for M.

Suppose local coordinates $(x_1, \ldots, x_n)$ chosen so that:

$$T = dx_1 \wedge \cdots \wedge dx_n ,$$

and

$$X = A_1\, \partial/\partial x_1 + \cdots + A_n\, \partial/\partial x_n .$$

Then,

$$\begin{aligned} X(T) &= X(dx_1) \wedge dx_2 \wedge \cdots \wedge dx_n + dx_1 \wedge X(dx_2) \wedge \cdots \wedge dx_n + \\ &\quad + \cdots dx_1 \wedge \cdots \wedge X(dx_n) \\ &= \left(\frac{\partial A_1}{\partial x_1} + \ldots + \frac{\partial A_n}{\partial x_n}\right) dx_1 \wedge \cdots \wedge dx_n, \quad \text{or} \end{aligned}$$

$$\omega(X) = \frac{\partial A_1}{\partial x_1} + \ldots + \frac{\partial A_n}{\partial x_n} . \tag{8.4}$$

Solving 8.3 means solving

$$A_1 \frac{\partial g}{\partial x_1} + \cdots + A_n \frac{\partial g}{\partial x_n} = \left(\frac{\partial A_1}{\partial x_1} + \ldots + \frac{\partial A_n}{\partial x_n}\right) . \tag{8.5}$$

One case where the solution can be written down can be immediately suggested. Suppose A_1 is a function $A_1(x_1)$ of x_1 alone, $A_2(x_2)$, etc. Then, g can be taken as:

$$g = \log A_1 + \log A_2 + \cdots + \log A_n . \tag{8.6}$$

IX. Calculation of the Intertwining Operator B^λ for Certain Representations

Let G be a non-compact, connected semisimple Lie group with finite center, K be its maximal compact sunbroup, $\mathfrak{G} = \mathbf{K} + \mathbf{P}$ its Cartan decomposition, i. e.,

$$[\mathbf{K}, \mathbf{P}] \subset \mathbf{P},\ [\mathbf{P}, \mathbf{P}] \subset \mathbf{K} .$$

Let X_0 be an element of $\mathbf{P}$. Then, $\operatorname{Ad} X_0$ has real eigenvalues and is completely reducible [2]. Let $\mathbf{H}(X_0)$ be the subspace of G spanned by the eigenvectors of $\operatorname{Ad} X_0$ with non-negative eigenvalues. $\mathbf{H}(X_0)$ is a subalgebra of $\mathfrak{G}$: Let $H(X_0)$ be the connected subgroup of G generated by $\mathbf{H}(X_0)$. Let M' be the coset space G/H, and let p_0 be the coset of the identity elements. Let $\mathbf{N}^-(X_0)$ be the subalgebra of $\mathfrak{G}$ spanned by the eigenvectors of $\operatorname{Ad} X_0$ for negative eigenvalues. Thus $\mathfrak{G}$, as a vector space, is the direct sum $\mathbf{H}(X_0) + \mathbf{N}^-(X_0)$. Let $\mathbf{N}^-(X_0)$ be the connected subgroup of G generated by the subalgebra $\mathbf{N}^-(X_0)$. Let M be the orbit $N^-(X_0) \cdot p_0$. It is known that it is an open subset of M, and the complement of M in M' is a set of measure zero. (Typically, it is these spaces M that are used by Gelfand and Neumark to construct representations of the classical groups [2].) Now, G acts a as global transformation group on $M' = G/H$. Hence, the Lie algebra $\mathfrak{G}$ acts on $F(M')$ as a subalgebra of $V(M')$:

$$X(f)(p) = \frac{\partial}{\partial t}\big(f(\exp(-X\,t)\,p)\big)\big|_{t=0}$$

for

$$X \in G,\ f = F(M'),\ p \in M'\ .$$

Since M is an open subset of M', $\mathfrak{G}$ also acts as a Lie algebra of vector fields on M, i. e., $\mathfrak{G}$ can be identified with a subalgebra of $V(M)$.

In this section we will use a volume element-differential form dx on M that is invariant under $N^-(X_0)$. Using this, we will, following the pattern described in the last section, define $\omega(X)$, for $X \in \mathfrak{G}$, as the function in $F(M)$ such that:

$$X(dx) = \omega(X)\,dx \quad \text{for} \quad X \in \mathfrak{G}\,,$$

and define

$$\varrho_\lambda(X)(f) = X(f) + \lambda\,\omega(X)\,f \quad \text{for} \quad f \in F(M),\ X \in \mathfrak{G}\ .$$

Notice then that:

$$\omega(\mathbf{N}^-(X_0)) = 0\ .$$

Given $X \in \mathfrak{G}$, our problem is to find the intertwining operator B^λ such that:

$$\varrho_\lambda(X) = B^{-\lambda}\,\varrho_0(X)\,B^\lambda\ .$$

We shall first deal with the following case:
X belongs to $\mathbf{A}$, a maximal abelian subalgebra of $\mathbf{P}$ which contains X_0 also.

Now, the elements of $\operatorname{Ad}\mathbf{A}$ can be simultaneously diagonalized, and have real eigenvalues. Let $\sigma_1, \ldots, \sigma_n$ be the non-zero, real-valued forms on $\mathbf{A}$ resulting from this diagonalization. (The $\sigma_1, \ldots, \sigma_n$ are not necessarily distinct as linear forms on $\mathbf{A}$.) For each σ_i, $1 \leqq i \leqq n$, there are elements

W_i, $W_{-i} \in \mathbf{G}$ such that:

$$[X, W_i] = \sigma_i(X)\, W\,,$$
$$[X, W_{-i}] = -\sigma_i(X)\, W_i \quad \text{for} \quad X \in \mathbf{A}\,.$$

For each i, there is a decomposition:

$$W_i = Z_i + Y_i\,,$$
$$W_{-i} = Z_i - Y_i$$

with $Z_i \in \mathbf{K}$, $Y_i \in \mathbf{P}$, $B(Z_i, Z_i) + -1$, $B(Y_i, Y_i) = 1$,

$$[X, Z_i] = \sigma_i(X)\, Y_i\,, \quad [X, Y_i] = \sigma_i(X)\, Z_i \quad \text{for} \quad X \in A\,.$$

[$B(\ldots)$ is the Killing form on $\mathbf{G}$: it is negative definite on $\mathbf{K}$, positive definite on $\mathbf{P}$.]

Suppose the ordering of the σ's is chosen so that: $\sigma_1, \ldots, \sigma_m$ are the forms that are non-zero on X_0, while $\sigma_{m+1}(X_0) = 0 = \cdots \sigma_n(X_0)$. Then, $\mathbf{N}^-(X_0)$ is spanned by $W_{-1}, \ldots, W_{-m}$. Hence the bracket $[W_{-i} \times \times W_{-j}]$ is, if non-zero, an eigenvector of $\operatorname{Ad}\mathbf{A}$. We see that $\mathbf{N}^-(X_0)$ is a nilpotent subalgebra of $\mathbf{G}$. M then admits a coordinate system $(x_1, \ldots, x_m)$ such that:

$$x_i(\exp(t_1 W_{-1}) \exp(t_2 W_{-2}) \ldots \exp(t_m W_{-m}) \cdot p_0) = t_i$$
$$\text{for} \quad 1 \leqq i \leqq m\,.$$

In terms of this coordinate system for M, the vector field on M generated by an element $X \in A$ takes the form

$$X = \sum_{i=1}^{m} -\sigma_i(X)\, x_i \frac{\partial}{\partial x_i}\,. \tag{9.1}$$

The volume element-differential form dx on M that is invariant under $N^-(X_0)$ takes the form:

$$dx = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m\,.$$

Hence, if $\omega(X)$ is defined by 8.4, we have:

$$\omega(X) = -(\sigma_1(X) + \cdots + \sigma_m(X))\,.$$

We see that everything is set up so that 8.6 applies:

$$X(g) = \omega(X), \text{ where } g = \log((-1)^m \sigma_1(X) \ldots \sigma_m(X)\, x_1 \ldots x_m)\,.$$

This, combined with our previous remarks, proves the following

Theorem 9.1. *For $X \in A$, the following intertwining operator links $\varrho_\lambda(X)$ and $\varrho_0(X)$:*

$$\begin{aligned} &\varrho_\lambda(X)\, (\sigma_1(X)\, x_1 \ldots \sigma_m(X)\, x_m)^\lambda f \\ &= (\sigma_1(X)\, x_1 \ldots \sigma_m(X)\, x_m)^\lambda\, \varrho_0(X)\,(f) \quad \textit{for} \quad f \in F(M)\,. \end{aligned} \tag{9.2}$$

The important qualitative point to keep in mind is that the coordinate system $(x_1, \ldots, x_m)$ valid in M' is that for which the vector fields of **A** take the form 9.1.

So far, we have used one property of the space M, namely that the orbit $N^-(X^0) \cdot p$ is as open, dense subset which admits a coordinate system having nice properties.

Now, this method of writing down the operators is not the most convenient for the purposes of physics: One wants the decomposition under K to be more explicit. This can be remedied by using a volume element that is invariant under K. In fact, one knows that K acts transitively on M. Let dp be a volume element-differential form for M that is invariant under the action of K. Suppose:

$$X(dp) = \omega'(X)\, dp \quad \text{for} \quad X \in G\,.$$

Then,

$$\varrho_\lambda'(X)(f) = X(f) + \lambda\, \omega'(X)\, f \quad \text{for} \quad f \in F(M')$$

defines another deformation of the representation ϱ_0. Let us calculate ϱ_λ' in terms of ϱ_λ: Suppose that

$$dp = h\, dx, \quad \text{where} \quad h \in F(M')\,.$$

Then, for $X \in \mathrm{G}$

$$\begin{aligned} X(dp) &= \omega'(X)\, dp \\ &= X(h)\, dx + h\, \omega(X)\, dx \\ &= X(\log h)\, dp + \omega(X)\, dp, \quad \text{or} \\ \omega'(X) &= X(\log h) + \omega(X), \quad \text{hence:} \\ \varrho_\lambda'(X) &= \varrho_\lambda(X) + \lambda\, X(\log h)\,. \end{aligned}$$

The key fact is that $X(\log h)$, as an operator, commutes with the intertwining operator between $\varrho_\lambda(X)$ and $\varrho_0(X)$. Thus, we have:

$$\begin{aligned} &(\sigma_1(X)\, x_1 \ldots \sigma_m(X)\, x_m)^{-\lambda}\, \varrho_\lambda'(X)(f) \\ &= (\sigma_1(X)\, x_1 \ldots \sigma_m(X)\, x_m)^{-\lambda}(X(f) + \lambda\, X(\log h)) \quad \text{for} \quad f \in F(M')\,. \end{aligned}$$

Also,

$$h^\lambda(X + \lambda\, X(\log h)\, h)^{-\lambda} = X \tag{9.3}$$

hence,

$$\begin{aligned} \varrho'(X) &= (\sigma_1(X)\, x_1 \ldots \sigma_m(X)\, x_m)^\lambda\, \varrho_0(X)\, (\sigma_1(X) \ldots x_m)^{-\lambda} \\ &= (\sigma_1(X) \ldots x_m)^\lambda\, (\varrho_0(X) + \lambda\, X\, (\log h))\, (\sigma_1(X) \ldots x_m)^{-\lambda} \\ &= (\sigma_1(X)\, x_1 \ldots x_m\, h)^\lambda\, (\varrho_0(X))\, (\sigma_1(X)\, x_1 \ldots x_m\, h)^{-\lambda}\,. \end{aligned} \tag{9.4}$$

This is the most useful form of the indentity for the application to group representation theory. Let $F(M', C)$ be the complex-valued, C^∞ functions on M', i. e., $F(M'\, C) = F(M') + i\, F(M')$. Let us make

$F(M', C)$ into an (incomplete) Hilbert space by adopting the following inner product:

$$\langle\psi|\psi'\rangle = \int_{M'} \psi(p)^* \, \psi'(p) \, dp$$

$$\text{for} \quad \psi, \psi' \in F(M, C) \, .$$

Consider $F(M, C) = F(M) + i\,F(M)$ as a subspace of $F(M', C)$, i. e., $F(M, C)$ consists of those C^∞ functions that can be extended smoothly from M' to M. Since the complement of M' in M is (relative to the measure defined by dp) a set of measure zero, $F(M, C)$ is (relative to the Hilbert space topology) dense in $F(M', C)$, and

$$\langle\psi|\psi'\rangle = \int_M \psi(p)^* \, \psi'(p) \, dp \quad \text{for} \quad \psi, \psi' \in F(M, C) \, .$$

Thus, for $X \in G$, $\psi, \psi' \in F(M, C)$,

$$\begin{aligned}
\langle \varrho'_\lambda(X)\,\psi|\psi'\rangle &= \int_M (X(\psi)^* + \lambda^*\,\omega'(X)\,\psi^*)\,\psi'\,dp \\
&= \int_M (-\psi^*\,X(\psi') - \psi^*\,\psi'\,\omega'(X) + \lambda^*\,\omega'(X)\,\psi^*\,\psi')\,dp \\
&= \langle\psi|(-X + (\lambda^* - 1)\,\omega'(X))\,\psi'\rangle \, .
\end{aligned}$$

We see that: $\varrho'_\lambda(G)$, acting the domain $F(M, C)$, is skew-Hermitian if and only if

$$\lambda^* + \lambda = 1, \quad \text{or} \tag{9.5}$$

λ is the form $1/2 + i\,b$, with real b.

We are now in position to show how to calculate the asymptotic behavior of matrix elements:

$$\langle\psi'|\varrho_\lambda(\exp(t\,X))\,\psi'\rangle \quad \text{as} \quad t \to \infty \, .$$

X. Asymptotic Behavior of Matrix Elements

Suppose the group G acts on a manifold M' as a transformation group. Let ϱ'_λ be a representation of G by operators in ad (incomplete) Hilbert space H. Suppose, in fact, that H is just $F(M', C)$, the space of complex-valued, C^∞ functions on M', with the inner product given by:

$$\langle\psi|\psi'\rangle = \int_M \psi(p)^* \, \psi'(p) \, dp \, ,$$

where dp is a volume element-differential form on M'. Suppose $\varrho'_0(X)$ is just the action $\psi \to X(\psi)$ of $X \in G$ by derivations of $F(M, C)$, describing the infinitesimal action of the one-parameter group $t \to \exp(t\,X)$ on M.

Let X be a fixed element of G. Suppose h_X is a function on M (possibly with singularities lying on submanifolds of M') such that:

$$\varrho_\lambda(X)\,(\psi) = h_X^\lambda\,\varrho_0(X)\,(h_X^{-\lambda}\,\psi) \, . \tag{10.1}$$

In terms of 9.4, h_X can be identified with

$$\sigma_1(X)\, x_1 \ldots \sigma_m(X)\, x_m\, h\,.$$

Then,

$$\exp(t\, \varrho_\lambda(X)) = h_X^\lambda \exp(t\, \varrho_0(X))\, h_X^{-\lambda}\,.$$

Now, for $\psi \in H$, $p \in M'$,

$$\exp(t\, \varrho_0(X))\, (\psi)\, (p) = \psi(\exp(-t\, X) \cdot p)\,,$$

where $p \to \exp(t\, X) \cdot p$ is the given action of the one-parameter subgroup $t \to \exp(t\, X)$ on M.

Thus, for $\psi, \psi' \in H$,

$$\alpha(t, \lambda) = \langle \psi | \exp(t\, \varrho_\lambda(X))\, \psi' \rangle$$
$$= \int_M \psi(p)^*\, h_X(p)^\lambda\, h(\exp(-t\, X)\, p)^{-\lambda}\, \psi'(\exp(-t\, X)\, p)\, dp\,.$$

As we have seen in [4], $P\, t \cdot V$, for the case $G = SL(2, R)$, there are two immediate interesting asymptotic problems:

(a) Asymptotic behavior as $t \to \infty$, with λ held fixed.

(b) Asymptotic behavior as λ goes to infinity. (For example, for the case $G = SL(2, R)$, $\lim_{\lambda \to \infty} \alpha(t/\lambda, \lambda)$ exists).

In turn, 10.1 shows that these are reduced to various geometric questions concerning the asymptotic behavior of the orbits $\exp(-t\, X) \cdot p$ as $t \to \infty$ hence are closely related to the problems of the modern theory of dynamical systems. We will deal with these problems in full technical details in a paper that will be published in a mathematics journal. We will present here various heuristic remarks.

The general problem we face can be described as follows: Suppose M' is a space with a measure dp, such that the total measure of M' is finite. Suppose $t \to g(t)$, defined for $t \geqq 0$, a sa one-parameter semigroup acting on M'. Suppose f_1 and f_2 are measurable functions on M'. Does there exist a number a such that:

$$\lim_{t \to \infty} t^a \int_{M'} f_1(p)\, f_2(g(t)\, p)\, dp \text{ exists ?}$$

For example, suppose f_1, f_2 are bounded, continuous functions on M', space such that continuous functions are measurable. Suppose the following condition is satisfied:

There is a point $p' \in M'$ such that

$$\lim_{t \to \infty} g(t) \cdot p = p' \quad \text{for all} \quad p \in M' \tag{10.2}$$

except possibly for a set of points of zero measure.

Then, the sequence of functions,

$$t \to f_3^t(p)$$

with

$$f_3^t(p) = f_1(p)\, f_2(g(t)\, p)$$

converges as $t \to \infty$ to the function

$$f_4(p) = f_1(p)\, f_2(p')\,,$$

with the convergence taking place for all but a set of measure zero of points p. Thus, by the Lesbesgue bounded convergence theorem,

$$\lim_{t\to\infty} \int_{M'} f_3^t(p)\, dp = \int_{M'} f_4(p)\, dp, \quad \text{or}$$

$$\lim_{t\to\infty} \int_{M'} f_1(p)\, f_2(g(t)\, p)\, dp = f_2(p') \int_{M'} f_1(p)\, dp\,.$$

Let us now consider a more general such problem. Suppose that 10.2 continues to be satisfied, and the problem is to find the limit as $t \to \infty$ of:

$$\int_{M'} f_1(p)\, f_2(g(t)\, p)\, dp\,,$$

as before. However, we do not assume that $f_2(p)$ is everywhere continuous, but assume that it has poles. For example, suppose it has a pole at $p = p'$. To have no trouble with the convergence of the integral, let us suppose that:

$$g(t) \cdot p' = p' \quad \text{for all} \quad t \geqq 0\,, \quad \text{and} \quad p \to f_1(p)\, f_2(g(t)\, p) \qquad (10.3)$$

is continuous in a neighborhood of p', i. e., $f_1(p)$ has a zero at p' sufficiently strong to cancel out the pole at $f_2(p')$.

Now, our assumtion 10.2 is that, for al. p except possibly for a set of measure zero, $\lim\limits_{t\to\infty} g(t)\, p = p'$.

Let us assume that

$$f_2(g(t)\, p) \sim c\, e^{a\, t} \quad \text{as} \quad t \to \infty\,.$$

Then, using the Lebesgue bounded convergence theorem as before, we see that:

$$e^{-a\, t} \int_{M'} f_1(p)\, f_2(g(t)\, p)\, dp \to c \int_{M'} f_1(p)\, dp \quad \text{at} \quad t \to \infty\,.$$

This, then, is a sketch of our "geometric" method for finding the asymptotic behavior of matrix elements of certain types of group representations.

XI. An Abstract Approach to the Problem of Asymptotic Behavior of Matrix Elements of Representations

There is an abstract pattern to the preceding work that is worth discussing separately. Suppose G is a Lie group, realized or a group of operators on a Hilbert space H. Let $t \to g(t)$ be a one-parameter subgroup of G, and let ψ be an element of H. Let $t \to c(t)$ be a curve in C, the complex numbers. Suppose that ψ is an element of H. Suppose that:

$$\frac{g(t)\psi}{c(t)} \tag{11.1}$$

approaches ψ_∞ via weak convergences as $c \to \infty$. (Recall this means that:

$$\lim_{t\to\infty} \frac{\langle\psi'|g(t)\psi\rangle}{c(t)} = \langle\psi'|\psi_\infty\rangle \quad \text{for each} \quad \psi' \in H.)$$

We will symbolize this relation as follows:

$$g(t)\,(\psi) \sim c(t)\,\psi_\infty\,. \tag{11.2}$$

We will understand that H is not necessarily a complete Hilbert space: In fact, much of the same set of ideas can be applied to the case where H is a general set of topological vector space, and the elements ψ' are taken from a given family of complex-valued continuous linear functions on H.

Our main concern in this section will be to consider (as far as possible without making further specific assumptions) what one can say about the action of the elements of G on limiting element ψ_∞.

First, suppose that g is an element of G. Let g^* be the adjoint transformation of g, i. e.,

$$\langle g^*\,\psi'\,|\,\psi\rangle = \langle\psi'|g\,\psi\rangle \quad \text{for} \quad \psi, \psi' \in H\,.$$

We will, in fact, suppose that g^* is defined on H also. (This is why we want H to be non-complete.) Then:

$$\langle\psi'|g\,\psi_\infty\rangle = \langle g^*\,\psi'|\psi_\infty\rangle = \lim_t \frac{\langle g^*\psi'|g(t)\psi\rangle}{c(t)} = \lim_t \langle\psi'|g\,g(t)\,\psi\rangle\,c(t). \tag{11.3}$$

We can read off immediately the following facts:

Theorem 11.1. *If g commutes with each $g(t)$, then*

$$g(t)\,(g\,\psi) \sim c(t)\,g\,\psi_\infty\,.$$

Suppose now that $g = g(s)$ for some real s.

Then:

$$\begin{aligned}\langle\psi'|\psi_\infty\rangle &= \lim_t \langle\psi'|g(t+s)\,\psi\rangle/c(t)\\ &= \lim_t \langle\psi'|g(t)\,\psi\rangle/c(t-s)\\ &= \lim_t \frac{\langle\psi'\mid g(t)\psi\rangle}{c(t)}\,\frac{c(t)}{c(t-s)}\,.\end{aligned}$$

Suppose that:

$$\lim_{t\to\infty} \frac{c(t)}{c(t-s)} = b(s)\,. \tag{11.4}$$

Then, we have:

Theorem 11.2. *If* 11.4 *is satisfied, then*

$$g(t)\,\psi_\infty = b(t)\,\psi_\infty \tag{11.5}$$

i. e., ψ_∞ is an eigenvector for each $g(t)$.

Now, suppose X is the infinitesimal generator for the one-parameter group $g(t)$, i. e.,

$$g(t) = \exp(t X) .$$

Suppose Y is an operator on H such that:

$$[X, Y] = \sigma Y . \tag{11.6}$$

Then:

$$\mathrm{Ad} \exp(t X) (y) = e^{\sigma t} Y, \quad \text{or}$$

$$\exp(t X) Y \exp(-t X) = e^{\sigma t} Y, \quad \text{or}$$

$$g(t) Y g(t)^{-1} = e^{\sigma t} Y .$$

Then,

$$g(t) e^{-\sigma t} Y g(t)^{-1} = Y .$$

Suppose:

$$h(s) = \exp(s Y) .$$

Then,

$$g(t) h(e^{-\sigma t} s) = h(s) g(t) . \tag{11.7}$$

We will, in fact, use 11.7 the "global" form of 11.6. Then, using 11.3, we have:

$$\langle \psi' | h(s) \psi_\times \rangle = \lim_t \frac{\langle \psi' \mid h(s) g(t) \psi \rangle}{c(t)} = \lim_t \frac{\langle \psi' \mid g(t) h(e^{-\sigma t} s) \psi \rangle}{c(t)} .$$

Now,

$$g(t) h(e^{-\sigma t} s) \psi - g(t) \psi = g(t) (h(e^{-\sigma t} s) \psi - \psi) .$$

Theorem 11.3. *Suppose that*

(a) $\sigma > 0$.

(b) *The representation of G by operators on H is continuous.*

(c) *The operators $g(t)/c(t)$ on H have a common bound B. Then,*

$$h(s) \psi_\infty = \psi_\infty \quad \textit{for all} \quad s .$$

Proof:

Hypotheses (a) and (b) tell us that:

$$\|h(e^{-\sigma t} s) \psi - \psi\| \to 0 \quad \text{as} \quad t \to \infty .$$

Hence,

$$\|(g(t)/c(t)) h(e^{-\sigma t} s) \psi - g(t)/c(t) \psi\|$$
$$\leqq B \|h(e^{-\sigma t} s) \psi - \psi\| .$$

In particular, we see that it converges strongly to zero as $t \to \infty$, hence also converges weakly to zero.

We know that $\lim_t \dfrac{\langle \psi' \mid g(t) h(e^{-\sigma t} s) \psi \rangle}{c(t)}$ exists, hence, by the above argument, it equals

$$\lim_t \frac{\langle \psi' \mid g(t) \psi \rangle}{c(t)}, \quad \text{i. e.},$$

$h(s) \psi_\infty = \psi_\infty$, since $h(s) \psi_\infty - \psi_\infty$ is perpendicular to all of H.

This simple argument enables us to say that ψ_∞ is left fixed by a whole subgroup of G determined by X. Let $\mathbf{N}^+(X)$ be the subalgebra of G spanned by the eigenvector of $\mathrm{Ad}\,X$ for positive eigenvalues. Let $N^+(X)$ be the connected subgroup of G. It is nilpotent, hence every element is a product of exponentials of the $\mathrm{Ad}\,X$-eigenvector generators of $\mathbf{N}^+(X)$, hence, using Theorem 11.3, we have:

$$N^+(X) \cdot \psi_\infty = \psi_\infty .$$

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