Commun. math. Phys. 6, 128—150 (1967)

Scattering and Bound State Solutions for a Class of Nonlocal Potentials (S-wave)

M. Bertero
Institut für Theoretische Kernphysik der Universität, Bonn

G. Talenti
Istituto Matematico dell'Università, Genova

G. A. Viano
Istituto di Fisica dell'Università, Genova

Received May 31, 1967

Abstract. The s-wave scattering solution is discussed for a class of nonlocal (non-separable) potentials. Existence and uniqueness theorems are given and the analyticity domain in the k-variable (k = wave number in the C.M. system) is determined. Furthermore it is proved that solutions of the bound state problem exist and a discussion of the square-integrable solutions, which can occur for a real positive value of the energy, is given. In this last case the scattering solution also exists but it is not unique. Finally the S-matrix is introduced and it is proved that it is unambigously defined even if the scattering solution is not unique.

1. Introduction

In a previous paper [1] the Born expansion of the scattering solution for a class of nonlocal potentials was considered. The analysis was restricted to the s-wave Schroedinger equation

$$y''(r) + k^2 y(r) = g \int_0^{+\infty} V(r, s)\, y(s)\, ds \tag{1.1}$$

where g is a real quantity and the following assumptions are made on $V(r, s)$:

a) $V(r, s)$ is a real and symmetric function

$$V(r, s) = V^*(r, s) = V(s, r) \tag{1.2}$$

in order to have a time-reversal invariant and hermitian interaction;

b) $V(r, s)$ is a measurable function of both variables, $0 \leqq r < +\infty$, $0 \leqq s < +\infty$, and a real constant $\alpha > 0$ exists such that:

$$C = \int_0^{+\infty} e^{\alpha r}\, dr \int_0^{+\infty} s e^{\alpha s} |V(r, s)|\, ds < +\infty\,. \tag{1.3}$$

The scattering solution is the solution of eq. (1.1) satisfying the following boundary conditions:

$$\begin{aligned} \psi(k, r) &= \sin kr + \Phi(k, r) \\ \Phi(k, 0) &= 0, \quad \lim_{r \to +\infty} [\Phi'(k, r) - ik\Phi(k, r)] = 0 . \end{aligned} \tag{1.4}$$

In ref. [1] the problem expressed by the integrodifferential equation (1.1) with the boundary condition (1.4) was reduced to the problem of solving the linear integral equation

$$\begin{aligned} v(k, r) &= v_0(k, r) + g \int_0^{+\infty} L(k; r, s)\, v(k, s)\, ds \\ v_0(k, r) &= \int_0^{+\infty} V(r, s) \sin ks\, ds \\ L(k; r, s) &= \int_0^{+\infty} V(r, t)\, G(k; t, s)\, dt \\ G(k; t, s) &= -\frac{1}{k} \sin[k \min(s, t)] \exp[ik \max(s, t)] \end{aligned} \tag{1.5}$$

in the Banach space

$$X = \left\{ x(r) : \|x\| = \int_0^{+\infty} e^{\alpha r} |x(r)|\, dr < +\infty \right\} . \tag{1.6}$$

Of course we can write eq. (1.5) as a linear functional equation in X:

$$[1 - gL(k)]\, v(k, \cdot) = v_0(k, \cdot) . \tag{1.7}$$

When eq. (1.5) is solved, the scattering solution $\psi(k, r)$ is obtained by means of $v(k, r)$ as follows:

$$\psi(k, r) = \sin kr + g \int_0^{+\infty} G(k; r, s)\, v(k, s)\, ds . \tag{1.8}$$

In [1] we found that, if $|g|\, C < 1$, then the Born expansion converges, it is holomorphic in $|\mathrm{Im}\, k| < \alpha$ and gives the solution of the integrodifferential equation (1.1) with the boundary conditions (1.4).

In the present paper we prove that, for the class of nonlocal potentials characterized by conditions a) and b) above, $L(k)$ is a compact operator in X for each k in $\mathrm{Im}\, k \geqq -\alpha$ and therefore we can use the well-known Riesz-Schauder theory [2] to discuss the solutions of eq. (1.7).

In fact this theory allows us to prove that, for an arbitrary fixed value of g, $\psi(k, r)$ exists, it is unique (in a class of functions to be specified later) and is analytic in k in the strip $|\mathrm{Im}\, k| < \alpha$, except at those k-points where a nonzero solution of the homogeneous equation

$$[1 - gL(k)]x = 0 \tag{1.9}$$

exists.

On the other hand we prove that, if a nonzero solution $x(r)$ of eq. (1.9) exists at $k = k_0$, $\operatorname{Im} k_0 \geqq 0$, then

$$\chi(r) = \int_0^{+\infty} G(k_0; r, s)\, x(s)\, ds \tag{1.10}$$

is a solution of eq. (1.1) which satisfies the conditions

$$\chi(0) = 0, \quad \int_0^{+\infty} |\chi(r)|^2\, dr < +\infty . \tag{1.11}$$

We obtain also some informations about the distribution of the points (in $\operatorname{Im} k \geqq 0$) where a nonzero solution of eq. (1.9) occurs: these points are contained in a circle of finite radius (for fixed g) and they can lie only on the imaginary or on the real axis[1]. Furthermore they have no cluster point except perhaps $k = 0$.

We can conclude that the nonzero solutions of eq. (1.9) are related to bound states with negative binding energy if the value of k is pure imaginary or to "spurious" bound states of positive energy if the value of k is real[2] [3].

We prove also that, when a nonzero solution of eq. (1.9) exists for a real value of k, then also eq. (1.7) admits a solution; in other words, when a "spurious" bound state exists, then necessarily the scattering solution exists too[3]. However the scattering solution is not unique. In spite of this fact, it is very easy to show that the S-matrix is uniquely defined.

Nonlocal potentials have been discussed elsewhere in the framework of the Lipmann-Schwinger equation in momentum space [5]. On the other hand our method works directly in configuration space, as is usually done in the theory of local potentials [6].

In Sec. 2 we prove the main properties of the operator $L(k)$. In Sec. 3 the existence, uniqueness and analyticity in k of the scattering solution is derived, whereas in Sec. 4 the bound states are discussed. Finally Sec. 5 is devoted to the definition and properties of the S-matrix.

2. Properties of the Operator $L(k)$

The main properties of the operator $L(k)$, defined by eqs. (1.7) and (1.5) above, are contained in the following theorems.

---

[1] This last fact is a consequence of the reality of g and of condition a) above.

[2] One can find very easily examples of nonlocal potentials satisfying condition a) and b), and such that "spurious" bound states exist. Such an example is the "nonlocal square well", i.e. the potential: $V(r, s) = 1,\ r, s \leqq r_0;\ V(r, s) = 0$ otherwise. If g takes one of the following values: $g = (2\pi n)^2/r_0^3;\ n = 1, 2, \ldots,$ then one "spurious" bound state exists.

[3] This fact was already conjectured by A. Martin [4].

Theorem 2.1. *$L(k)$ is a bounded operator in X for each k in $\operatorname{Im} k \geqq -\alpha$; furthermore $\|L(k)\| \to 0$ if $|k| \to +\infty$ in $\operatorname{Im} k \geqq -\alpha$.*

Proof. From the inequality

$$|\sin z| \leqq \frac{2|z|}{1+|z|} \exp(|\operatorname{Im} z|) \tag{2.1}$$

the following bound on $G(k; r, s)$ (defined in eq. (1.5)) is obtained

$$|G(k; r, s)| \leqq \frac{2r}{1+|k|r} e^{\alpha(r+s)}, \quad \operatorname{Im} k \geqq -\alpha. \tag{2.2}$$

Now, from eq. (2.2) we get

$$\|L(k)\| = \sup_{x \in X} \frac{\|L(k)x\|}{\|x\|} \leqq 2 \int_0^{+\infty} e^{\alpha r} dr \int_0^{+\infty} \frac{s e^{\alpha s}}{1+|k|s} |V(r,s)| ds, \operatorname{Im} k \geqq -\alpha. \tag{2.3}$$

As a consequence of condition (1.3), the integral is convergent and therefore $L(k)$ is a bounded operator if k is in $\operatorname{Im} k \geqq -\alpha$.

If we change the order of integration and we write

$$\|L(k)\| \leqq 2 \int_0^a s e^{\alpha s} ds \int_0^{+\infty} e^{\alpha r} |V(r,s)| dr + \frac{2}{|k|} \int_a^{+\infty} e^{\alpha s} ds \int_0^{+\infty} e^{\alpha r} |V(r,s)| dr \tag{2.4}$$

both integrals are clearly convergent. For any $\varepsilon > 0$ we can choose a such that

$$\int_0^a s e^{\alpha s} ds \int_0^{+\infty} e^{\alpha r} |V(r, s)| dr < \frac{\varepsilon}{4} \tag{2.5}$$

and k such that

$$|k| > \frac{4}{\varepsilon} \int_a^{+\infty} e^{\alpha s} ds \int_0^{+\infty} e^{\alpha r} |V(r, s)| dr. \tag{2.6}$$

From eqs. (2.4), (2.5) and (2.6) it follows that

$$\|L(k)\| < \varepsilon, \quad \operatorname{Im} k \geqq -\alpha \tag{2.7}$$

and the Theorem is proved.

Theorem 2.2. *$L(k): X \to X$ is a compact operator for each k in $\operatorname{Im} k \geqq -\alpha$.*

Proof. Let $\{x_n\}$ be any bounded sequence in X, i.e. $\|x_n\| \leqq \gamma$ (γ is a constant independent of n).

We write $y_n = L(k)x_n$. We have to prove the existence of a subsequence $\{y_{n_j}\}$ ($n_j < n_{j+1}$) and of an element $y \in X$ such that

$$\|y_{n_j} - y\| \to 0, \quad j \to +\infty. \tag{2.8}$$

We prove at first the following statements

$$\begin{aligned}
&\text{(i)}\quad \|y_n\| \leqq \gamma\|L(k)\| \\
&\text{(ii)}\quad \sup_n \int_0^{+\infty} |e^{\alpha(r+h)} y_n(r+h) - e^{\alpha r} y_n(r)|\, dr \to 0, \quad h \to 0 \\
&\text{(iii)}\quad \sup_n \int_R^{+\infty} e^{\alpha r} |y_n(r)|\, dr \to 0, \quad R \to +\infty .
\end{aligned} \tag{2.9}$$

Inequality (i) is a trivial consequence of the hypothesis $\|x_n\| \leqq \gamma$ and of Theorem 2.1.

For what concerns (ii), by means of the following inequality

$$|\sin z| \leqq |z| \exp(|\mathrm{Im}\, z|) \tag{2.10}$$

one obtains:

$$|G(k; r, s)| \leqq r e^{\alpha(r+s)}, \quad \mathrm{Im}\, k \geqq -\alpha \tag{2.11}$$

and therefore

$$\begin{aligned}
&\sup_n \int_0^{+\infty} |e^{\alpha(r+h)} y_n(r+h) - e^{\alpha r} y_n(r)|\, dr \leqq \\
&\qquad \leqq \gamma \int_0^{+\infty} dr \int_0^{+\infty} t e^{\alpha t} |e^{\alpha(r+h)} V(r+h, t) - e^{\alpha r} V(r, t)|\, dt .
\end{aligned} \tag{2.12}$$

The r.h.s. of eq. (2.12) tends to zero when $h \to 0$. This fact is a consequence of condition (1.3) on $V(r, s)$ and of the continuity, under the integral sign, of the operation of translation.

For what concerns (iii), from the inequality (2.11) we have

$$\sup_n \int_R^{+\infty} e^{\alpha r} |y_n(r)|\, dr \leqq \gamma \int_R^{+\infty} e^{\alpha r}\, dr \int_0^{+\infty} t e^{\alpha t} |V(r, t)|\, dt \tag{2.13}$$

and also in this case condition (1.3) implies that the r.h.s. tends to zero, when $R \to +\infty$.

Therefore (i), (ii) and (iii) are proved.

Now, if we write

$$w_n(r) = e^{\alpha r} y_n(r) \Rightarrow w_n(r) \in L^1(0, +\infty) \tag{2.14}$$

from the statements (i), (ii) and (iii) and from the M. Riesz theorem [7] the sequence $w_n(r)$ is a relatively compact set in $L^1(0, +\infty)$. — Therefore a subsequence $\{w_{n_j}(r)\}$ and a function $w(r) \in L^1(0, +\infty)$ exist such that

$$\int_0^{+\infty} |w_{n_j}(r) - w(r)|\, dr \to 0, \quad j \to +\infty . \tag{2.15}$$

If we consider the subsequence $\{y_{n_j}(r)\} = \{e^{-\alpha r} w_{n_j}(r)\}$ and the function $y(r) = e^{-\alpha r} w(r)$, then eq. (2.15) implies eq. (2.8) and the Theorem is proved.

Theorem 2.3. *$k \to L(k)$ is a holomorphic operator valued function in the half-plane* $\mathrm{Im}\, k > -\alpha$.

Proof. We have to prove that, for each k in $\operatorname{Im} k > -\alpha$, a bounded operator $\dot{L}(k) : X \to X$ exists such that

$$\left\| \frac{L(k+h) - L(k)}{h} - \dot{L}(k) \right\| \to 0, \quad |h| \to 0. \tag{2.16}$$

By means of the Cauchy Theorem and of the inequality (2.11) we get[4]

$$|\dot{G}(k; r, s)| = \frac{1}{2\pi} \left| \oint_{\mathbb{C}} \frac{G(k'; r, s)}{(k' - k)^2} dk' \right| \leqq \frac{r}{R} e^{\alpha (r+s)} \tag{2.17}$$

where the path of integration $\mathbb{C}$ is the circle with center in k and radius $R = \alpha + \operatorname{Im} k$; furthermore, if $|h| < R$, we also have

$$\begin{aligned} &\left| \frac{G(k+h; r, s) - G(k; r, s)}{h} - \dot{G}(k; r, s) \right| = \\ &= \frac{|h|}{2\pi} \left| \oint_{\mathbb{C}} \frac{G(k'; r, s)}{(k' - k)^2 (k' - k - h)} dk' \right| \leqq \frac{|h|}{R(R - |h|)} r e^{\alpha (r+s)}. \end{aligned} \tag{2.18}$$

Now, let us define

$$\begin{aligned} \dot{L}(k) x(r) &= \int_0^{+\infty} \dot{L}(k; r, s) x(s) ds \\ \dot{L}(k; r, s) &= \int_0^{+\infty} V(r, t) \dot{G}(k; t, s) dt. \end{aligned} \tag{2.19}$$

From eq. (2.17) we easily obtain

$$\|\dot{L}(k)x\| \leqq \frac{1}{R} \int_0^{+\infty} e^{\alpha r} dr \int_0^{+\infty} ds \int_0^{+\infty} dt |V(r, t)| t e^{\alpha (t+s)} |x(s)| = \frac{C}{R} \|x\|. \tag{2.20}$$

It follows that $\dot{L}(k)$ is a bounded operator in $X (\operatorname{Im} k > -\alpha)$ and that

$$\|\dot{L}(k)\| \leqq \frac{C}{\alpha + \operatorname{Im} k}. \tag{2.21}$$

On the other hand, from eq. (2.18) we have

$$\begin{aligned} &\left\| \frac{L(k+h) - L(k)}{h} - \dot{L}(k) \right\| \leqq \\ &\leqq \sup_{x \in X} \frac{1}{\|x\|} \int_0^{+\infty} e^{\alpha r} dr \int_0^{+\infty} ds \int_0^{+\infty} |V(r, t)| \left| \frac{G(k+h; r, s) - G(k; r, s)}{h} - \right. \\ &\left. - \dot{G}(k; r, s) \right| |x(s)| dt \leqq \frac{|h| C}{R(R - |h|)} \to 0, \quad |h| \to 0 \end{aligned} \tag{2.22}$$

and the Theorem is proved.

---

[4] We indicate with an upper dot differentiation with respect to k.

3. The Scattering Solution

From Theorem 2.2 and the Riesz-Schauder theory [2] it follows that, for fixed k in $\mathrm{Im}\, k \geqq -\alpha$, the resolvent

$$R(k, g) = [1 - g L(k)]^{-1} \tag{3.1}$$

is a bounded operator in X for every value of g in the complex g-plane, except at most for a countable set of values, say $g_n(k)$. Each $g_n(k)$ is an eigenvalue of $L(k)$ of finite multiplicity, i.e. the number of linearly independent solutions of the homogeneous equation (1.9) (with $g = g_n(k)$) is finite and nonzero.

However the main interest does not lie in the properties of $R(k, g)$ as a function of g for fixed k, but in the properties of $R(k, g)$ as a function of k for fixed g.

Let us suppose that g is a fixed real quantity.

We shall call Ω the set of k-points in $\mathrm{Im}\, k \geqq -\alpha$ such that g belongs to the resolvent set of $L(k)$; and Ω' the set of k-points in $\mathrm{Im}\, k \geqq -\alpha$ such that g belongs to the spectrum of $L(k)$. Ω and Ω' are disjoint sets and their union is the half-plane $\mathrm{Im}\, k \geqq -\alpha$. $R(k, g)$ exists and it is bounded in X if $k \in \Omega$, whereas it does not exist if $k \in \Omega'$.

We shall analyze later the structure of the set Ω'. We observe only that Ω' is certainly contained in a circle of finite radius. In fact from Theorem 2.1 it follows that, for fixed g, there exists a $k_0(g)$ such that for $|k| > k_0(g)$, $\mathrm{Im}\, k \geqq -\alpha$ we have

$$|g| \, \|L(k)\| < 1 \tag{3.2}$$

i.e. $g L(k)$ is a contraction in X. Therefore $R(k, g)$ certainly exists and it is bounded in X for $|k| > k_0(g)$, $\mathrm{Im}\, k \geqq -\alpha$.

The main Theorem of this Section is the following.

Theorem 3.1. *Let Ω_0 be the intersection of the set Ω with the strip $|\mathrm{Im}\, k| < \alpha$. For each $k \in \Omega_0$ there exists one and only one function $\psi(k, r)$, $r \geqq 0$ such that:*

(i) *$\psi(k, r)$ has an absolutely continuous first derivative,*

(ii) *$\psi(k, r)$ is a solution of the integrodifferential equation* (1.1) *with the boundary conditions* (1.4)

(iii) *the following condition*

$$\int_0^{+\infty} e^{\alpha r} |\psi''(k, r) + k^2 \psi(k, r)| \, dr < +\infty \tag{3.3}$$

is satisfied.

Furthermore $\psi(k, r)$ is holomorphic, for fixed $r \geqq 0$, in Ω_0.

In order to prove this Theorem we need the following Lemmas.

Lemma 3.1. *Ω is an open set. $k \to R(k, g)$ is a holomorphic operator valued function in Ω.*

Proof. Let $k_0 \in \Omega$ and let δ be given by

$$\delta = \frac{\alpha + \operatorname{Im} k_0}{1 + C|g| \, \|R(k_0, g)\|} \tag{3.4}$$

We prove that the open circle $|k - k_0| < \delta$ is contained in Ω.

As is well known, if A is a bounded operator which has an inverse A^{-1}, then any operator B such that

$$\|B - A\| < \frac{1}{\|A^{-1}\|} \tag{3.5}$$

has an inverse B^{-1} which can be represented by the series (convergent in norm)

$$B^{-1} = \sum_{n=0}^{+\infty} [A^{-1}(A - B)]^n A^{-1} . \tag{3.6}$$

We write:

$$A = 1 - gL(k_0), \quad B = 1 - gL(k_0 + h) \tag{3.7}$$

where $|h| < \delta$. Then, by Theorem 2.3 and inequality (2.21) it follows that

$$\begin{aligned} \|B - A\| = |g| \, \|L(k_0 + h) - L(k_0)\| &\leqq \frac{|g| \, |h| \, C}{\alpha + \operatorname{Im} k_0 - |h|} < \frac{|g| \, \delta C}{\alpha + \operatorname{Im} k_0 - \delta} = \\ &= \frac{1}{\|R(k_0, g)\|} = \frac{1}{\|A^{-1}\|} \end{aligned} \tag{3.8}$$

and condition (3.5) is satisfied. Therefore $B^{-1} = R(k_0 + h, g)$ exists for any h such that $|h| < \delta$ and, as a consequence, Ω is open.

Furthermore, from eqs. (3.6) and (3.7) it follows

$$\begin{aligned} &\frac{R(k_0 + h, g) - R(k_0, g)}{h} - gR(k_0, g) \frac{L(k_0 + h) - L(k_0)}{h} R(k_0, g) = \\ &= \frac{1}{h} \sum_{n=2}^{+\infty} \{R(k_0, g) \, [L(k_0 + h) - L(k_0)]\}^n g^n R(k_0, g) . \end{aligned} \tag{3.9}$$

The norm of the r.h.s. of eq. (3.9) is bounded by

$$\begin{aligned} &\left\| \frac{1}{h} \sum_{n=2}^{+\infty} g^n \{R(k_0, g) \, [L(k_0 + h) - L(k_0)]\}^n R(k_0, g) \right\| \leqq \\ &\leqq \frac{1}{|h|} \sum_{n=2}^{+\infty} |g|^n \, \|R(k_0, g)\|^{n+1} \, \|L(k_0 + h) - L(k_0)\|^n < \\ &< \frac{|h|}{\delta^2} \sum_{n=2}^{+\infty} |g|^n \, \|R(k_0, g)\|^{n+1} \left(\frac{|h|}{\delta}\right)^{n-2} \left(\frac{C\delta}{\alpha + \operatorname{Im} k_0 - \delta}\right)^n = \\ &= \frac{|h|}{\delta^2} \|R(k_0, g)\| \sum_{n=2}^{+\infty} \left(\frac{|h|}{\delta}\right)^{n-2} = \frac{|h|}{\delta^2} \|R(k_0, g)\| \left(1 - \frac{|h|}{\delta}\right)^{-1} \end{aligned} \tag{3.10}$$

and therefore it tends to zero as $|h| \to 0$. It follows

$$\left\| \frac{R(k_0 + h, g) - R(k_0, g)}{h} - gR(k_0, g) \, \dot{L}(k_0) \, R(k_0, g) \right\| \to 0, \quad |h| \to 0 \tag{3.11}$$

and the Lemma is proved.

Lemma 3.2. *$k \to v_0(k, \cdot)$ (defined in eq. (1.5)) is a function with values in X, holomorphic in the strip $|\mathrm{Im}\, k| < \alpha$.*

Proof. We have to prove the existence of a function $\dot{v}_0(k, \cdot) \in X$, $|\mathrm{Im}\, k| < \alpha$, such that

$$\left\| \frac{v_0(k+h, \cdot) - v_0(k, \cdot)}{h} - \dot{v}_0(k, \cdot) \right\| \to 0, \quad |h| \to 0 . \tag{3.12}$$

Let us define

$$\dot{v}_0(k, r) = \int_0^{+\infty} V(r, s) s \cos ks \, ds . \tag{3.13}$$

Then, if $|\mathrm{Im}\, k| < \alpha$

$$\|\dot{v}_0(k, \cdot)\| \leqq \int_0^{+\infty} e^{\alpha r} dr \int_0^{+\infty} s |V(r, s)| \, e^{|s \, \mathrm{Im}\, k|} ds < C \tag{3.14}$$

and therefore $\dot{v}_0(k, \cdot) \in X$.

Now we have

$$\begin{aligned} &\left\| \frac{v_0(k+h, \cdot) - v_0(k, \cdot)}{h} - \dot{v}_0(k, \cdot) \right\| \leqq \\ &\leqq \int_0^{+\infty} e^{\alpha r} dr \int_0^{+\infty} s |V(r, s)| \left| \frac{\sin(k+h)s - \sin ks}{hs} - \cos ks \right| ds \end{aligned} \tag{3.15}$$

and from the following relation

$$\frac{\sin(k+h)s - \sin ks}{hs} - \cos ks = -2 \int_0^1 \sin\left(ks + \frac{1}{2} hsy\right) \sin\left(\frac{1}{2} hsy\right) dy \tag{3.16}$$

we get

$$\left| \frac{\sin(k+h)s - \sin ks}{hs} - \cos ks \right| \leqq |h| \, s \exp[s(|\mathrm{Im}\, k| + |\mathrm{Im}\, h|)] . \tag{3.17}$$

Inequality (3.17), modified as follows

$$s \exp[s(|\mathrm{Im}\, k| + |\mathrm{Im}\, h|)] < \frac{e^{\alpha s}}{\alpha - |\mathrm{Im}\, k| - |\mathrm{Im}\, h|} \, ; \, |\mathrm{Im}\, k| + |\mathrm{Im}\, h| < \alpha \tag{3.18}$$

substituted in eq. (3.15) gives

$$\left\| \frac{v_0(k+h, \cdot) - v_0(k, \cdot)}{h} - \dot{v}_0(k, \cdot) \right\| \leqq \frac{|h| C}{\alpha - |\mathrm{Im}\, k| - |\mathrm{Im}\, h|} \tag{3.19}$$

which tends to zero as $|h| \to 0$, and the Lemma is proved. The Riesz-Schauder theory, Lemma 3.1 and Lemma 3.2 imply:

Lemma 3.3. *A solution $v(k, \cdot)$ of the inhomogeneous equation*

$$[1 - g L(k)] \, v(k, \cdot) = v_0(k, \cdot) \tag{3.20}$$

exists and is unique in X for every $k \in \Omega_0$. Furthermore $k \to v(k, \cdot)$ is a function holomorphic in Ω_0.

We are now able to prove Theorem 3.1.

Proof of Theorem 3.1. Let us consider the following function

$$\psi(k, r) = \sin kr + g \int_0^{+\infty} G(k; r, s)\, v(k, s)\, ds \tag{3.21}$$

where $v(k, \cdot)$, $k \in \Omega_0$, is the solution in X of the inhomogeneous integral equation (3.20).

We can easily check that $\psi'(k, r)$ is absolutely continuous, and that $\psi(k, r)$ is a solution of the inhomogeneous differential equation

$$\psi''(k, r) + k^2 \psi(k, r) = g v(k, r)\,. \tag{3.22}$$

From this equation it follows that $\psi(k, r)$ satisfies condition (3.3). Furthermore, from the integral equation (1.5), inverting the order of integration, we have

$$\begin{aligned} v(k, r) &= \int_0^{+\infty} V(r, s)\, [\sin ks + g \int_0^{+\infty} G(k; s, t)\, v(k, t)\, dt]\, ds \\ &= \int_0^{+\infty} V(r, s)\, \psi(k, s)\, ds\,. \end{aligned} \tag{3.23}$$

Eqs. (3.22) and (3.23) imply that $\psi(k, r)$ is a solution of the integro-differential equation (1.1).

$\psi(k, r)$ satisfies also the boundary conditions (1.4). In fact, writing explicitely the obtained representation of $\Phi(k, r)$

$$\Phi(k, r) = -g \left[e^{ikr} \int_0^r \frac{\sin ks}{k}\, v(k, s)\, ds + \frac{\sin kr}{k} \int_r^{+\infty} e^{iks} v(k, s)\, ds \right] \tag{3.24}$$

it is clear that $\Phi(k, 0)$ is zero; furthermore, computing the first derivative, we have

$$\begin{aligned} |\Phi'(k, r) - ik\Phi(k, r)| &= \left| \int_r^{+\infty} e^{ik(s-r)} v(k, s)\, ds \right| \leqq \\ &\leqq \int_r^{+\infty} e^{\alpha s} |v(k, s)|\, ds \to 0,\; r \to +\infty \end{aligned} \tag{3.25}$$

since $v(k, \cdot) \in X$. Therefore the existence is proved.

Now, let $\psi_1(k, r)$ and $\psi_2(k, r)$ be two solutions of the equation (1.1) with the boundary conditions (1.4) and let them satisfy condition (3.3); their difference

$$y(k, r) = \psi_1(k, r) - \psi_2(k, r) \tag{3.26}$$

is such that

$$y''(k, r) + k^2 y(k, r) = g \int_0^{+\infty} V(r, s)\, y(k, s)\, ds \tag{3.27a}$$

$$y(k, 0) = 0, \quad \lim_{r \to +\infty} [y'(k, r) - iky(k, r)] = 0 \tag{3.27b}$$

$$\int_0^{+\infty} e^{\alpha r} |y''(k, r) + k^2 y(k, r)|\, dr < +\infty\,. \tag{3.27c}$$

If we write

$$u(k, r) = y''(k, r) + k^2 y(k, r) \tag{3.28}$$

then eqs. (3.27a) and (3.27c) imply

$$u(k, r) = g \int_0^{+\infty} V(r, s)\, y(k, s)\, ds \tag{3.29a}$$

$$\int_0^{+\infty} e^{\alpha r} |u(k, r)|\, dr < +\infty \tag{3.29b}$$

i.e. $u(k, \cdot) \in X$.

Solving eq. (3.28) by means of the usual method of variation of constants, from conditions (3.27b) we obtain

$$y(k, s) = \int_0^{+\infty} G(k; s, t)\, u(k, t)\, dt \tag{3.30}$$

and substituting in eq. (3.29a) we have

$$u(k, r) = g \int_0^{+\infty} ds \int_0^{+\infty} V(r, s)\, G(k; s, t)\, u(k, t)\, dt\,. \tag{3.31}$$

It is straightforward to verify that the double integral of the r.h.s. of eq. (3.31) exists as a consequence of eq. (3.29b); then we can change the order of integration and write (3.31) as follows

$$u(k, \cdot) = g L(k)\, u(k, \cdot)\,. \tag{3.32}$$

By hypothesis $k \in \Omega$ and therefore nonzero solutions of the homogeneous equation do not exist. We conclude that $u(k, \cdot) \equiv 0$; it follows $y(k, r) \equiv 0$ and the uniqueness is proved.

We still have to prove that $\psi(k, r)$ is holomorphic in Ω_0.

Let us define, if $k \in \Omega_0$

$$\dot{\psi}(k, r) = r \cos kr + g \int_0^{+\infty} \dot{G}(k; r, s)\, v(k, s)\, ds + g \int_0^{+\infty} G(k; r, s)\, \dot{v}(k, s)\, ds \tag{3.33}$$

where $\dot{v}$ is the derivative of $k \to v(k, \cdot)$ (see Lemma 3.3).

We have to prove that the quantity

$$\begin{aligned} \frac{\psi(k+h, r) - \psi(k, r)}{h} - \dot{\psi}(k, r) = \left[\frac{\sin(k+h)\, r - \sin kr}{h} - r \cos kr\right] + \\ + g \int_0^{+\infty} \left[\frac{G(k+h; r, s)\, v(k+h, s) - G(k; r, s)\, v(k, s)}{h} - \right. \\ \left. - \dot{G}(k; r, s)\, v(k, s) - G(k; r, s)\, \dot{v}(k, s)\right] ds \end{aligned} \tag{3.34}$$

tends to zero as $|h| \to 0$, for any $r \geqq 0$.

For what concerns the first term in the r.h.s. of eq. (3.34), it is trivial to see that it tends to zero. The second term can be written as the sum of the three quantities

$$\int_0^{+\infty} \left[\frac{G(k+h;r,s) - G(k;r,s)}{h} - \dot{G}(k;r,s)\right] v(k,s)\, ds \tag{3.35a}$$

$$\int_0^{+\infty} \left[\frac{v(k+h,s) - v(k,s)}{h} - \dot{v}(k,s)\right] G(k;r,s)\, ds \tag{3.35b}$$

$$\int_0^{+\infty} \left[\frac{G(k+h;r,s) - G(k;r,s)}{h}\right] [v(k+h,s) - v(k,s)]\, ds\,. \tag{3.35c}$$

The first quantity (3.35a) tends to zero as a consequence of eq. (2.18); the second quantity (3.35b) tends to zero because of the bound (2.11) and of the definition of derivative of the function $k \to v(k,\cdot)$; the third quantity (3.35c) also tends to zero as a consequence of the bound

$$|G(k+h;r,s) - G(k;r,s)| = \frac{|h|}{2\pi}\left|\oint_{\mathfrak{C}} \frac{G(k';r,s)\,dk'}{(k'-k)(k'-k-h)}\right| \leqq$$

$$\leqq \frac{|h|}{R-|h|}\, r e^{\alpha(r+s)}$$

(R is defined in Theorem 2.3) and of the continuity of $v(k,\cdot)$. Therefore also the analyticity of $\psi(k,r)$ is proved.

4. The Bound State Solutions

The problem studied in this Section is the analysis of the solutions of the homogeneous equation (1.9) and the investigation of the structure of the set Ω'.

The main Theorem of this Section is the following:

Theorem 4.1. *Let* $\chi(r)$ *be a solution of the integrodifferential equation* (1.1), *with* $\operatorname{Im} k \geqq 0$, $k \neq 0$, *satisfying the following conditions:*

(i) $\chi(0) = 0$

(ii) $\sup\limits_{0 \leqq r < +\infty} |e^{-\alpha r}\chi(r)| < +\infty$

(iii) $\int\limits_0^{+\infty} |\chi(r)|^2\, dr < +\infty$

then

$$x(r) = \int_0^{+\infty} V(r,s)\,\chi(s)\, ds \tag{4.1}$$

is a solution, belonging to X, *of the homogeneous integral equation* (1.9).

Conversely, if $x \in X$ *is a solution of eq.* (1.9) *with* $\operatorname{Im} k \geqq 0$, *then*

$$\chi(r) = g \int_0^{+\infty} G(k; r, s)\, x(s)\, ds \tag{4.2}$$

is a solution of the integrodifferential equation (1.1) *and satisfies conditions* (i), (ii), *and* (iii) *above.*

If k *is real, then one has*

$$\int_0^{+\infty} \sin kr\, dr \int_0^{+\infty} V(r, s)\, \chi(s)\, ds = 0\,. \tag{4.3}$$

Remark. One can easily prove that

$$X' = \{y(r) : \|y\| = \underset{0 \leqq r < +\infty}{\text{ess.sup.}} |e^{-\alpha r} y(r)|\} \tag{4.4}$$

is the dual space of X[5]. — Condition (ii) of Theorem 4.1 implies that $\chi(r) \in X'$.

We need the following Lemma:

Lemma 4.1. *Let* $y_1(r)$ *and* $y_2(r)$ *be two solutions of the integrodifferential equation* (1.1), *respectively with* $k = k_1$ *and* $k = k_2$; *let* $y_1(r)$ *and* $y_2(r)$ *satisfy conditions* (i), (ii), *and* (iii) *of Theorem* 4.1. — *Then*

$$\lim_{r \to +\infty} [y_1(r)\, y_2^{*\prime}(r) - y_1'(r)\, y_2^*(r)] = (k_1^2 - k_2^{*2}) \int_0^{+\infty} y_1(r)\, y_2^*(r)\, dr\,. \tag{4.5}$$

If $k = k_1 = k_2$ *is real and if* $y(r) = y_1(r) = y_2(r)$ *satisfies conditions* (i) *and* (ii) *of Theorem* 4.1. *then*

$$\lim_{r \to +\infty} [y(r)\, y^{*\prime}(r) - y'(r)\, y^*(r)] = 0\,. \tag{4.6}$$

Proof. Multiplying the integrodifferential equation (1.1) in $y_1(r)$ by $y_2^*(r)$ and the equation (1.1) in $y_2^*(r)$ by $y_1(r)$ and subtracting we have [condition (1.2) on $V(r, s)$ has to be used]:

$$\begin{aligned} &\frac{d}{dr} [y_1(r)\, y_2^{*\prime}(r) - y_1'(r)\, y_2^*(r)] + (k_2^{*2} - k_1^2)\, y_1(r)\, y_2^*(r) \\ &= g \int_0^{+\infty} V(r,s)\, [y_1(r)\, y_2^*(s) - y_1(s)\, y_2^*(r)]\, ds\,. \end{aligned} \tag{4.7}$$

Integrating over $(0, +\infty)$, as a consequence of condition (iii) the second term in the l.h.s. of eq. (4.7) is integrable; the r.h.s. is also integrable because of conditions (i), (ii) and the double integral is zero as a consequence of condition (1.2) on $V(r, s)$.

Therefore eq. (4.5) is proved.

If $k = k_1 = k_2$ is real and $y(r) = y_1(r) = y_2(r)$, the second term in the l.h.s. of eq. (4.7) is identically zero. Integrating over $(0, +\infty)$, the r.h.s.

[5] ess. supp. means the essential least upper bound.

of eq. (4.7) is integrable as a consequence of conditions (i), (ii) and the integral is zero because of condition (1.2) on $V(r, s)$.

Therefore eq. (4.6) is also proved.

Proof of Theorem 4.1. Let $x \in X$ be a solution of the homogeneous equation (1.9), $\operatorname{Im} k \geqq 0$; we prove first that $\chi(r)$ defined by eq. (4.2) is a solution of the integrodifferential equation (1.1). In fact, by means of two differentiations we have that $\chi(r)$ is a solution of the inhomogeneous differential equation:

$$\chi''(r) + k^2 \chi(r) = g x(r) . \tag{4.8}$$

Now, from the homogeneous equation (1.9), written explicitely as follows,

$$x(r) = g \int_0^{+\infty} x(s)\, ds \int_0^{+\infty} V(r, t)\, G(k; t, s)\, dt \tag{4.9}$$

by means of a change of the order of integration we have

$$x(r) = \int_0^{+\infty} V(r, t)\, \chi(t)\, dt . \tag{4.10}$$

Eqs. (4.8) and (4.10) imply that $\chi(r)$ is a solution of the eq. (1.1).

We prove now that $\chi(r)$ satisfies conditions (i), (ii) and (iii).

For what concerns (i) and (ii), from the bound on $G(k; r, s)$, which holds in $\operatorname{Im} k \geqq 0$:

$$|G(k; r, s)| \leqq r < \frac{1}{\alpha} e^{\alpha r} \tag{4.11}$$

one obtains that $\chi(r)$, given by eq. (4.2), satisfies these conditions.

For what concerns (iii) we distinguish two cases: $\operatorname{Im} k > 0$ and $\operatorname{Im} k = 0$.

A. $\operatorname{Im} k > 0$. By means of the representation (4.2) and inequality (2.10) we have:

$$|\chi(r)| \leqq |g|\, r \left[e^{-r \operatorname{Im} k} \int_0^r e^{s \operatorname{Im} k} |x(s)|\, ds + e^{r \operatorname{Im} k} \int_r^{+\infty} e^{-s \operatorname{Im} k} |x(s)|\, ds\right] . \tag{4.12}$$

At this point we consider the two subcases: $\operatorname{Im} k < \alpha$ and $\operatorname{Im} k \geqq \alpha$. If $\operatorname{Im} k < \alpha$, then

$$\begin{aligned} |\chi(r)| &\leqq |g|\, r \left[e^{-r \operatorname{Im} k} \int_0^r e^{\alpha s} |x(s)|\, ds + e^{-r \operatorname{Im} k} \int_r^{+\infty} e^{s \operatorname{Im} k} |x(s)|\, ds\right] \leqq \\ &\leqq |g| \|x\|\, r e^{-r \operatorname{Im} k} . \end{aligned} \tag{4.13}$$

On the other hand, if $\operatorname{Im} k \geqq \alpha$, then

$$\begin{aligned} |\chi(r)| &\leqq |g|\, r \left[\int_0^r e^{-(r-s) \operatorname{Im} k} |x(s)|\, ds + \int_r^{+\infty} e^{-(s-r) \operatorname{Im} k} |x(s)|\, ds\right] \leqq \\ &\leqq |g|\, r \left[\int_0^r e^{-\alpha(r-s)} |x(s)|\, ds + \int_r^{+\infty} e^{-\alpha(s-r)} |x(s)|\, ds\right] \leqq \\ &\leqq |g|\, \|x\|\, r e^{-\alpha r} . \end{aligned} \tag{4.14}$$

The inequalities (4.14) and (4.13) imply condition (iii).

B. $\mathrm{Im}\, k = 0$. We observe first that, as a consequence of eqs. (4.2) and (4.10), $\chi(r)$ satisfies the following integral equation

$$\chi(r) = g \int_0^{+\infty} G(k; r, s)\, ds \int_0^{+\infty} V(s, t)\, \chi(t)\, dt \,. \tag{4.15}$$

By means of this equation and of Lemma 4.1 (eq. (4.6)), it is easy to prove that $\chi(r)$ satisfies relation (4.3), which can be written (eq. (4.10) has to be used) as

$$\int_0^{+\infty} \sin ks\, x(s)\, ds = 0 \,. \tag{4.16}$$

Eq. (4.16) implies that

$$\chi(r) = g e^{ikr} \int_r^{+\infty} \frac{\sin ks}{k} x(s)\, ds - g \frac{\sin kr}{k} \int_r^{+\infty} e^{iks} x(s)\, ds = \\ = g \int_r^{+\infty} \frac{\sin k(s-r)}{k} x(s)\, ds \tag{4.17}$$

and therefore

$$|\chi(r)| \leqq |g| \int_r^{+\infty} \left|\frac{\sin k(s-r)}{k}\right| |x(s)|\, ds \leqq \frac{|g|}{|k|} \int_r^{+\infty} |x(s)|\, ds \leqq \\ \leqq |g| \frac{\|x\|}{|k|} e^{-\alpha r} \,. \tag{4.18}$$

It follows that condition (iii) is satisfied, except if $k = 0$. Viceversa we prove now that, if $\chi(r)$ is a solution of eq. (1.1) which satisfies conditions (i), (ii) and (iii), then $x(r)$, defined by eq. (4.1), is a solution of the homogeneous equation (1.9).

Condition (ii) implies that $x \in X$. Furthermore, by definition we have

$$\chi''(r) + k^2 \chi(r) = g x(r) \tag{4.19}$$

and the general solution of this inhomogeneous differential equation can be written as

$$\chi(r) = c_1 \sin kr + c_2 \cos kr + g \int_0^{+\infty} G(k; r, s)\, x(s)\, ds \,. \tag{4.20}$$

If $\mathrm{Im}\, k > 0$, we can prove, as in the first part of the Theorem that the third term in the r.h.s. of eq. (4.20) belongs to $L^2(0, +\infty)$. Therefore condition (i) implies $c_2 = 0$ and condition (iii) implies $c_1 = 0$.

Furthermore, if $\mathrm{Im}\, k = 0$, condition (i) implies once more $c_2 = 0$. By means of eq. (4.20) (with $c_2 = 0$) and Lemma 4.1 (eq. (4.6)) one easily obtains

$$\operatorname{Im}\left[c_1^* \int_0^{+\infty} \sin ks\, x(s)\, ds\right] = -\frac{g}{k}\left|\int_0^{+\infty} \sin ks\, x(s)\, ds\right|^2$$

$$|\chi(r)|^2 = |c_1|^2 \sin^2 kr - \frac{g}{k} \operatorname{Im}\left[e^{2ikr} c_1^* \int_0^{+\infty} \sin ks\, x(s)\, ds\right] + \tag{4.21}$$

$$+ O(e^{-\alpha r}); \quad r \to +\infty$$

and from condition (iii) it follows $c_1 = 0$.

Therefore in both cases $\operatorname{Im} k > 0$ and $\operatorname{Im} k = 0$, we have

$$\chi(r) = g \int_0^{+\infty} G(k; r, s)\, x(s)\, ds\,. \tag{4.22}$$

Eq. (4.22), substituted in eq. (4.1), shows that $x(r)$ is a solution of the homogeneous equation (1.9) and the Theorem is completely proved.

Remark 1. We observe that eq. (4.15) (which holds for $\operatorname{Im} k \geqq 0$) means that $\chi(r)$ is an eigenfunction of the adjoint of $L(k)$, say $L'(k)$, whose kernel is given by

$$L'(k; r, s) = \int_0^{+\infty} G(k; r, t)\, V(t, s)\, dt = L(k; s, r)\,. \tag{4.23}$$

We can write

$$\chi = g L'(k) \chi, \quad \chi \in X' \tag{4.24}$$

the adjoint equation of eq. (1.9).

Remark 2. We observe also that if $\chi_1(r)$ and $\chi_2(r)$ are two solution of eq. (1.1), satisfying conditions (i), (ii), (iii), then necessarily $\chi_1(r)$ and $\chi_2(r)$ (and also their derivatives) are bounded as in eqs. (4.13) or (4.14) or (4.18) and the l.h.s. of eq. (4.5) (Lemma 4.1) is zero; it follows that

$$(k_1^2 - k_2^{*2}) \int_0^{+\infty} \chi_1(r)\, \chi_2^*(r)\, dr = 0\,. \tag{4.25}$$

Now, if $k_1 = k_2 = k$ and $\chi_1 = \chi_2$, eq. (4.25) shows that $\operatorname{Im} k^2 = 0$, i.e. k^2 is real which implies that k is either real or pure imaginary. Therefore solutions of the homogeneous equation (1.9) or (4.24) occur only on the real or on the imaginary axis in the half-plane $\operatorname{Im} k \geqq 0$.

Furthermore, if $k_1 \neq k_2$ and both k_1^2 and k_2^2 are real, eq. (4.25) implies the orthogonality relation

$$\int_0^{+\infty} \chi_1(r)\, \chi_2^*(r)\, dr = 0\,. \tag{4.26}$$

In conclusion we can say that:

1. If at $k = ib$ ($b > 0$) a solution of the homogeneous equation (1.9) exists, then a bound state of energy $E = -b^2$ exists and $\chi(r)$ is the corresponding wave function (within a multiplicative normalization constant).

2. If at $k = b$ (b real) a solution of eq. (1.9) exists, then a "spurious" bound state of energy $E = b^2$ exists and $\chi(r)$ is the corresponding wave function.

The degeneracy of the bound states and of the "spurious" bound states is always finite.

The next Theorem[6] contains some information about the distribution of the bound states.

Theorem 4.2. *The set of the bound-states and of the "spurious" bound-states has no limit point except perhaps $k = 0$.*

Proof. Let us assume that a limit point $k \neq 0$ exists. Then we can find a sequence of points $\{k_n\}$, $k_n \in \Omega'$, such that $|k_n - k| \to 0$. To each k_n we associate one solution of the homogeneous equation (1.9), say $x_n \in X$, and we normalize these functions in such a way that $\|x_n\| = 1$. As a consequence of Theorem 2.3 we have

$$\|L(k_n)\,x_n - L(k)x_n\| \leqq \|L(k_n) - L(k)\| \to 0\,, \quad n \to +\infty \tag{4.27}$$

so that

$$\|x_n - gL(k)x_n\| \to 0\,, \quad n \to +\infty\,. \tag{4.28}$$

Since $L(k)$ is compact, from the sequence $\{gL(k)x_n\}$ we can extract a subsequence $\{gL(k)x_{n_j}\}$ which converges to an element $x \in X$

$$\|x - gL(k)x_{n_j}\| \to 0\,, \quad j \to +\infty\,. \tag{4.29}$$

From the two inequalities

$$\|x_{n_j} - x\| \leqq \|gL(k)\,x_{n_j} - x\| + \|x_{n_j} - gL(k)x_{n_j}\| \tag{4.30a}$$

$$\|x - gL(k)x\| \leqq \|x - gL(k)x_{n_j}\| + \|gL(k)\,(x_{n_j} - x)\| \tag{4.30b}$$

and from eqs. (4.28) and (4.29) we have

$$\|x_{n_j} - x\| \to 0\,, \quad j \to +\infty \tag{4.31a}$$

$$x = gL(k)x\,, \quad \|x\| = 1\,. \tag{4.31b}$$

Let $\chi_{n_j}(r)$ and $\chi(r)$ be the functions obtained from $x_{n_j}(r)$ and $x(r)$ by means of eq. (4.2). We want to prove that the sequence $\{\chi_{n_j}(r)\}$ converges to $\chi(r)$ uniformly in r, $0 \leqq r < +\infty$. For this purpose we distinguish two cases: k pure imaginary and k real.

---

[6] The method of the proof is essentially the method used in ref. [8].

A. *k pure imaginary.* Since

$$\sup_{0 \leqq r < +\infty} |\chi_{n_j}(r) - \chi(r)| \leqq \sup_{0 \leqq r < +\infty} \Bigg[|g| \int_0^{+\infty} |G(k_{n_j}; r, s) - G(k; r, s)| \cdot |x(s)| \, ds + |g| \int_0^{+\infty} |G(k; r, s)| \cdot |x_{n_j}(s) - x(s)| \, ds + |g| \int_0^{+\infty} |G(k_{n_j}; r, s) - G(k; r, s)| \cdot |x_{n_j}(s) - x(s)| \, ds \Bigg] \tag{4.32}$$

and

$$|G(k; r, s)| \leqq s < \frac{1}{\alpha} e^{\alpha s} \tag{4.33a}$$

$$|G(k_{n_j}; r, s) - G(k; r, s)| = \frac{|k_{n_j} - k|}{2\pi} \left| \oint_{\mathbb{C}'} \frac{G(k'; r, s)}{(k' - k)(k' - k_{n_j})} dk' \right| \leqq \frac{|k_{n_j} - k|}{\alpha} \frac{e^{\alpha s}}{R - |k_{n_j} - k|}, \quad |k - k_{n_j}| < R \tag{4.33b}$$

(where $\mathbb{C}'$ is the circle with center in k and radius $R = \operatorname{Im} k$) it follows that:

$$\sup_{0 \leqq r < +\infty} |\chi_{n_j}(r) - \chi(r)| \leqq \frac{|g|}{\alpha} \frac{|k_{n_j} - k|}{R - |k_{n_j} - k|} (1 + \|x_{n_j} - x\|) + \frac{|g|}{\alpha} \|x_{n_j} - x\| \to 0 \, . \tag{4.34}$$

B. *k-real and positive.* We call $k_0 = \min(k_{n_j}, k) > 0$. In this case the representation (4.17) holds for both $\chi_{n_j}(r)$ and $\chi(r)$. It follows

$$\sup_{0 \leqq r < +\infty} |\chi_{n_j}(r) - \chi(r)| \leqq |g| \sup_{0 \leqq r < +\infty} \Bigg[\int_r^{+\infty} \left| \frac{\sin k_{n_j}(s - r)}{k_{n_j}} - \frac{\sin k(s - r)}{k} \right| \cdot |x(s)| \, ds + \int_r^{+\infty} \left| \frac{\sin k(s - r)}{k} \right| \cdot |x_{n_j}(s) - x(s)| \, ds + \int_r^{+\infty} \left| \frac{\sin k_{n_j}(s - r)}{k_{n_j}} - \frac{\sin k(s - r)}{k} \right| \cdot |x_{n_j}(s) - x(s)| \, ds \Bigg] \tag{4.35}$$

and from the inequalities

$$\left| \frac{\sin k(s - r)}{k} \right| \leqq (s - r) \leqq s < \frac{e^{\alpha s}}{\alpha} \tag{4.36a}$$

$$\left| \frac{\sin k_{n_j}(s - r)}{k_{n_j}} - \frac{\sin k(s - r)}{k} \right| \leqq |k_{n_j} - k| \max_{k'} \left| \frac{d}{dk'} \frac{\sin k'(s - r)}{k'} \right| \tag{4.36b}$$

$$\left| \frac{d}{dk'} \frac{\sin k'(s - r)}{k'} \right| \leqq \left| \frac{(s - r) \cos k'(s - r)}{k'} \right| + \left| \frac{\sin k'(s - r)}{k'^2} \right| \leqq \frac{2}{k'} (s - r) \leqq \frac{2s}{k'} < \frac{2}{k' \alpha} e^{\alpha s} \leqq \frac{2}{k_0 \alpha} e^{\alpha s}$$

we have

$$\sup_{0 \leq r < +\infty} |\chi_{n_j}(r) - \chi(r)| \leq \frac{2|g|}{k_0 \alpha} |k_{n_j} - k| (1 + \|x_{n_j} - x\|) + + \frac{|g|}{\alpha} \cdot \|x_{n_j} - x\| \to 0. \tag{4.37}$$

Eqs. (4.34) and (4.37) imply the uniform convergence of the sequence $\{\chi_{n_j}(r)\}$ to $\chi(r)$.

Now, $\chi_{n_j}(r)$ and $\chi(r)$ are orthogonal (see eq. (4.26)) and they satisfy condition (iii) of Theorem 4.1. Therefore we can write

$$\int_0^{+\infty} |\chi(r)|^2 \, dr = \int_0^{+\infty} \chi^*(r) \, [\chi(r) - \chi_{n_j}(r)] \, dr \leq \leq \sup_{0 \leq r < +\infty} |\chi_{n_j}(r) - \chi(r)| \cdot \int_0^{+\infty} |\chi(r)| \, dr \tag{4.38}$$

and the integral in the r.h.s. of eq. (4.38) is convergent because of the bounds (4.13), (4.14) and (4.18). Eq. (4.38) can be written

$$\sup_{0 \leq r < +\infty} |\chi_{n_j}(r) - \chi(r)| \geq \int_0^{+\infty} |\chi(r)|^2 \, dr \left[\int_0^{+\infty} |\chi(r)| \, dr \right]^{-1} > 0 \tag{4.39}$$

and clearly contradicts eq. (4.34) or eq. (4.37). Therefore $k \neq 0$ cannot be a limit point. The argument breaks down for $k = 0$.

Remark. A consequence of this Theorem is the fact that there exists at most a countable set of bound states and of "spurious" bound states.

Theorem 4.3. *If for k real, $k \neq 0$ a "spurious" bound state exists, then also the scattering solution exists.*

Proof. This Theorem is a direct consequence of eq. (4.3) which can be written

$$\int_0^{+\infty} \chi(s) \, v_0(k, s) \, ds = 0. \tag{4.40}$$

But according to the Riesz-Schauder Theory, condition (4.40) is precisely the condition which guarantees the simultaneous existence of a solution both of the homogeneous equation (1.9) and of the inhomogeneous equation (1.7); of course, in this case, the solution of the inhomogeneous equation is no longer unique.

5. The Scattering Amplitude

The following Theorem defines the scattering amplitude.

Theorem 5.1. *For each $k \in \Omega_0$*

$$\lim_{r \to +\infty} |e^{-ikr} \Phi(k, r) - T(k)| = 0 \tag{5.1}$$

where

$$T(k) = -g \int_0^{+\infty} \frac{\sin ks}{k} v(k, s)\, ds\,. \tag{5.2}$$

Proof. By means of the representation (3.24) for $\Phi(k, r)$ we have

$$e^{-ikr}\Phi(k, r) - T(k) = \frac{g}{k} e^{-ikr} \int_r^{+\infty} \sin k(s - r)\, v(k, s)\, ds \tag{5.3}$$

and therefore ($|\mathrm{Im}\, k| \leqq \alpha$)

$$\begin{aligned} |e^{-ikr}\Phi(k, r) - T(k)| &\leqq \left|\frac{g}{k}\right| e^{-r\,\mathrm{Imk}} \int_r^{+\infty} e^{(s-r)\,|\mathrm{Imk}|}\, |v(k, s)|\, ds \leqq \\ &\leqq \left|\frac{g}{k}\right| \int_r^{+\infty} e^{\alpha s} |v(k, s)|\, ds \to 0, \quad r \to +\infty \end{aligned} \tag{5.4}$$

since $v(k, \cdot) \in X$. On the other hand, if $k = 0 \in \Omega_0$, then $v(0, \cdot) \equiv 0 \Rightarrow \Rightarrow \Phi(0, r) \equiv 0$, $T(0) = 0$. The Theorem is proved.

Remark 1. From Lemma 4.1, eq. (4.6), it follows for real values of k (observe that $\psi(k, \cdot) \in X'$)

$$\begin{aligned} \lim_{r \to +\infty} & [\psi^*(k, r)\, \psi'(k, r) - \psi(k, r)\, \psi^{*\prime}(k, r)] \\ & = 2ik|T(k)|^2 - 2ik\, \mathrm{Im}\, T(k) = 0 \end{aligned} \tag{5.5}$$

i.e.

$$\mathrm{Im}\, T(k) = |T(k)|^2 \tag{5.6}$$

and $T(k)$ satisfies the unitarity condition. If we introduce the quantity

$$S(k) = 1 + 2i\, T(k) \tag{5.7}$$

we have

$$|S(k)| = 1 \Rightarrow S(k) = e^{2i\delta(k)}, \quad T(k) = e^{i\delta(k)} \sin \delta(k) \tag{5.8}$$

where $\delta(k)$ is a real function of k. Therefore Theorem 5.1 and eq. (5.8) imply the following asymptotic behaviour for $\psi(k, r)$, k real

$$\psi(k, r) \sim e^{i\delta(k)} \sin[kr + \delta(k)], \quad r \to +\infty \tag{5.9}$$

and $\delta(k)$ is the usual s-wave scattering phase-shift.

Remark 2. $T(k)$ is uniquely defined for every real value of k. In fact, if a "spurious" bound state exists, then the scattering solution exists but it is not unique and its general expression is given by

$$\psi(k, r) = \psi^{(0)}(k, r) + \sum_{j=1}^{n} c_j \chi_j(r) \tag{5.10}$$

where $\psi^{(0)}(k, r)$ is any solution of the scattering problem, the χ_j's are n linearly independent solutions of the adjoint homogeneous equation (if n is the multiplicity of the eigenvalue) and the c_j's are n arbitrary constants.

For each $\chi_j(r)$ eq. (4.40) holds, so that

$$T(k) = -g \int_0^{+\infty} \frac{\sin ks}{k} v(k, s)\, ds = -\frac{g}{k} \int_0^{+\infty} v_0(k, s)\, \psi(k, s)\, ds =$$
$$= -\frac{g}{k} \int_0^{+\infty} v_0(k, s)\, \psi^{(0)}(k, s)\, ds - \frac{g}{k} \sum_{j=1}^{n} c_j \int_0^{+\infty} v_0(k, s)\, \chi_j(s)\, ds = \tag{5.11}$$
$$= -\frac{g}{k} \int_0^{+\infty} v_0(k, s)\, \psi^{(0)}(k, s)$$

and $T(k)$ is independent of the c_j's.

Theorem 5.2. *$T(k)$ is holomorphic in Ω_0.*

Proof. Let us define

$$Z(k) = k\, T(k) = -g \int_0^{+\infty} \sin kr\, v(k, r)\, dr \tag{5.12}$$

and

$$\dot{Z}(k) = -g \int_0^{+\infty} r \cos kr\, v(k, r)\, dr - g \int_0^{+\infty} \sin kr\, \dot{v}(k, r)\, dr \tag{5.13}$$

where $\dot{v}$ is the derivative of $k \to v(k, \cdot)$. We have

$$\left|\frac{Z(k+h) - Z(k)}{h} - \dot{Z}(k)\right| \leqq |g| \int_0^{+\infty} \left|\frac{\sin(k+h) r - \sin kr}{h} - r \cos kr\right| |v(k, r)|\, dr +$$
$$+ |g| \int_0^{+\infty} |\sin kr| \left|\frac{v(k + h + r) - v(k, r)}{h} - \dot{v}(k, r)\right| dr + \tag{5.14}$$
$$+ \left|\frac{g}{h}\right| \int_0^{+\infty} |\sin(k+h) r - \sin kr| \cdot |v(k+h, r) - v(k, r)|\, dr\,.$$

From inequality (3.17) and inequality

$$r^2 \exp[r(|\mathrm{Im}\, k| + |\mathrm{Im}\, h|)] < \frac{e^{\alpha r}}{(\alpha - |\mathrm{Im}\, k| - |\mathrm{Im}\, h|)^2}\,, \quad |\mathrm{Im}\, k| + |\mathrm{Im}\, h| < \alpha \tag{5.15}$$

it follows that the first term in the r.h.s. if eq. (5.14) is bounded by

$$|g h| \frac{\|v(k, \cdot)\|}{(\alpha - |\mathrm{Im}\, k| - |\mathrm{Im}\, h|)^2} \to 0, \quad |h| \to 0\,. \tag{5.16}$$

For what concerns the second term, it is bounded by

$$|g| \left\|\frac{v(k+h, \cdot) - v(k, \cdot)}{h} - \dot{v}(k, \cdot)\right\| \tag{5.17}$$

and it tends to zero, as a consequence of the definition of $\dot{v}$.

For the third term, by means of the following inequality

$$|\sin(k+h) r - \sin kr| \leqq |h|\, r e^{r(|\mathrm{Im}\, k| + |\mathrm{Im}\, h|)} \tag{5.18}$$

and of inequality (3.18), we have that it is bounded by

$$\frac{|g|}{\alpha - |\mathrm{Im}\, k| - |\mathrm{Im}\, h|} \|v(k+h, \cdot) - v(k, \cdot)\| \tag{5.19}$$

and it tends to zero, since $v(k, \cdot)$ is holomorphic and therefore continuous in Ω_0. It follows that $Z(k)$ is holomorphic in Ω_0 and clearly the same result holds for $T(k)$.

6. Conclusions

One of the main results of this paper is the proof that the scattering solution exists, is unique and holomorphic in a nonvoid open subset of the strip $|\mathrm{Im}\, k| < \alpha$ for the class of nonlocal potentials satisfying conditions a) and b), Sec. 1. In particular the solution exists always for real values of k.

Of course the strip can be enlarged to the whole k-plane for the nonlocal potentials such that the condition (1.3) is satisfied for unrestricted values of α. Examples of such potentials are given by functions $V(r, s)$ which vanish outside some finite region in the (r, s)-plane or which are of the type

$$V(r, s) = P(r, s)\, e^{-a(r^2 + s^2)}, \quad a > 0 \tag{6.1}$$

where $P(r, s)$ is a bounded, measurable function of both variable r, s.

We have also proved that, with our method, we obtain all the bound states and the "spurious" bound states which satisfy the additional condition (ii) of Theorem 4.1.

Some questions are still open.

The first question is that we cannot decide whether the number of bound states and of "spurious" bound states is finite or infinite.

That the number of bound states is finite has been proved for another class of nonlocal potentials [9]. Of course the intersection of the class here considered and the class of ref. [9] in nonvoid and therefore, at least for the potentials which belong to this subclass, we can say that all the results of our work hold true and also that the number of bound states is finite. However the method of ref. [9] gives no informations about the number of the "spurious" bound states.

Always concerning the bound states, we have proved that they appear as isolated singularities of the scattering solution but we have said nothing about the nature of these singularities.

Another important question concerns the singularities in the strip $-\alpha < \mathrm{Im}\, k < 0$. In the case of a local potential satisfying the condition

$$\int_0^{+\infty} r e^{\alpha r} |V(r)|\, dr < +\infty, \quad \alpha > 0 \tag{6.2}$$

these singularities are isolated and are related to the resonances or to the antibound states of the system. For the class of nonlocal potentials here considered we have not yet such a nice picture. In fact we have no informations about the distribution of the singularities in the strip $-\alpha < \operatorname{Im} k < 0$, except that they are all contained in a bounded domain (as a consequence of Theorem 2.1).

Acknowledgment. One of us (M. B.) wishes to thank Prof. K. BLEULER for the warm hospitality extended to him at the Institut für Theoretische Kernphysik.

References

1. BERTERO, M., G. TALENTI, and G. A. VIANO: Nuovo Cimento **46**, 337 (1966).
2. YOSIDA, K.: Functional analysis. Berlin-Heidelberg-New York: Springer 1965.
3. GOURDIN, M., and A. MARTIN: Nuovo Cimento **6**, 757 (1957); 8, 699 (1958).
4. MARTIN, A.: Nuovo Cimento **7**, 607 (1958).
5. TANI, S.: Ann. Phys. **37**, 411; **37**, 451 (1966) and other papers there quoted.
6. NEWTON, R. G.: J. Math. Phys. **1**, 319 (1960).
7. DUNFORD, N., and J. I. SCHWARTZ: Linear operators, Part I, p. 298—301. New York: Interscience Publishers 1958.
8. HUNZIKER, W.: Helv. Phys. Acta **34**, 593 (1961).
9. GHIRARDI, G. C., and A. RIMINI: J. Math. Phys. **6**, 40 (1965).