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Tempered Distributions in Infinitely Many Dimensions

III. Linear Transformations of Field Operators

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Abstract. In the present paper we continue investigating spaces of tempered distributions in infinitely many dimensions. In particular, we prove that those linear homogeneous transformations of the canonical pair of field operators, which preserve the commutation relations, can be implemented by an essentially unique intertwining operator. The dependence of this operator on the transformation is studied.

1. Introduction

Summary of results

In two previous papers [6] and [7] (in the sequel quoted as I and II) we have studied certain spaces of tempered distributions in infinitely many dimensions, in particular the space $\tilde{\mathfrak{S}}$, which is essentially identical with the space Σ considered by Borchers [1].

In the present work we investigate linear homogeneous transformations of the canonical pair of field operators; in particular linear transformations induced by the real symplectic group Σ over Schwartz's space $\mathscr{S}$[1]. This group we define as the family of all matrices

$$\mathbf{U} = \begin{pmatrix} U & V \\ \overline{V} & \overline{U} \end{pmatrix}$$

with matrix elements in $L(\mathscr{S}, \mathscr{S}) \cap L(\mathscr{S}^*, \mathscr{S}^*)$ and satisfying

$$\mathbf{U}^* \mathbf{J} \mathbf{U} = \mathbf{U} \mathbf{J} \mathbf{U}^* = \mathbf{J}\,, \tag{1}$$

where

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The linear transformation induced by $\mathbf{U}$ is then defined as the mapping

$$a(\overline{\varphi}) \rightsquigarrow a_{\mathbf{U}}(\overline{\varphi}) \equiv a(\overline{U}^* \overline{\varphi}) + a^*(\overline{V}^* \overline{\varphi})$$

$$a^*(\varphi) \rightsquigarrow a^*_{\mathbf{U}}(\varphi) \equiv a^*(U^* \varphi) + a(V^* \varphi)\,.$$

[1] In the study of spaces of type $\mathfrak{S}$ in I and II we assumed $\mathscr{S} = \mathscr{S}(\mathbf{R}^1)$. In case $\mathscr{S} = \mathscr{S}(\mathbf{R}^n)$, $n > 1$, spaces of type $\mathfrak{S}$ should be modified in the obvious way.

In Section 3 we prove that such a linear transformation can be represented by an essentially unique operator $\Omega(\mathbf{U}) \in L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$, which intertwines the pairs (a, a^*) and $(a_{\mathbf{U}}, a_{\mathbf{U}}^*)$ in the sense that

$$a_{\mathbf{U}}(\bar{\varphi})\,\Omega(\mathbf{U}) = \Omega(\mathbf{U})\,a(\bar{\varphi}),$$
$$a_{\mathbf{U}}^*(\varphi)\,\Omega(\mathbf{U}) = \Omega(\mathbf{U})\,a^*(\varphi),$$

if and only if the image $\bar{U}^*(\mathscr{S})$ is dense in $\mathscr{S}$ (here – in contrast to I and II – the bar denotes complex conjugation). The subset of Σ having this property is denoted by Σ_0.

In Section 4 we prove that the suitably normalized operator $\Omega(\mathbf{U})$, $\mathbf{U} \in \Sigma_0$, is continuous and differentiable when considered as a function of $(V\,\bar{U}^{-1},\, U^{-1},\, \bar{V}^*\,\bar{U}^{*-1}) \in L(\mathscr{S}, \mathscr{S}^*)^3$ into $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$.

Finally, in Section 5 we study the conditions under which the operator $\Omega(\mathbf{U})$ can be extended to a unitary operator in the Fock-Cook Hilbert space $\mathfrak{H}$. These conditions are essentially well known and are for example formulated in Friedrichs [3]. A detailed proof has been given by Shale [8], who considers the Weyl operators as the basic objects. We consider instead the creation- and annihilation-operators, and thus, our line of reasoning is closer in spirit to that of Friedrichs.

The quantization of certain differential equations

The results mentioned above have an application to the problem of quantization of certain types of differential equations including linear wave equations of the form

$$\left(\frac{\partial^2}{\partial t^2} + H_0^2 + B(t)\right) u(t) = 0 \tag{2}$$

where H_0 is a positive definite operator (e. g. $H_0 = (-\Delta + m^2)^{\frac{1}{2}}$), and where $B(t)$ is a real symmetric operator for each value of t. Both H_0 and $B(t)$ are operators in some space of complex valued functions over $\mathbf{R}^n$.

It is well known that if the Cauchy problem for (2) has a unique solution then the propagation in time is given in terms of a two parameter family of transformations $\mathbf{U}(t_2, t_1)$ satisfying

$$\mathbf{U}(t_2, t_3)\,\mathbf{U}(t_3, t_1) = \mathbf{U}(t_2, t_1)\,. \tag{3}$$

The types of differential equations to which our results apply may be obtained in the following way:

Let $\mathbf{U}(t_2, t_1)$, $-\infty < t_2, t_1 < \infty$, be a family of symplectic transformations over $\mathscr{S}$, for which (3) is satisfied. If $U(\cdot, t_1)$ and $V(\cdot, t_1)$ are differentiable from the real axis into $L(\mathscr{S}, \mathscr{S}) \cap L(\mathscr{S}^*, \mathscr{S}^*)$, then in virtue of (3), $\mathbf{U}(t, t_1)$ satisfies an equation

$$i\,\dot{\mathbf{U}}(t, t_1) = \mathbf{A}(t)\,\mathbf{U}(t, t_1)\,, \tag{4}$$

where $\mathbf{A}(t)$ is independent of t_1. Moreover, it follows from the real symplectic character of $\mathbf{U}$ that $\mathbf{A}$ is of the form

$$\mathbf{A}(t) = \begin{pmatrix} a(t) & i b(t) \\ i \bar{b}(t) & -\bar{a}(t) \end{pmatrix} \tag{5}$$

where a and b are in $L(\mathscr{S}, \mathscr{S})$ for each value of t, and where $a^* = a$ and $b^* = \bar{b}$.

Hence it follows that if the Cauchy problem for the system

$$\begin{aligned} i\,\dot{v} &= a\,v + i\,b\,w \\ i\,\dot{w} &= i\,\bar{b}\,v - \bar{a}\,w \end{aligned} \tag{6}$$

has a unique solution, then $\mathbf{U}(t_2, t_1)$ is the family of propagators associated with the system. In (6) v and w denote functions of t with values in $\mathscr{S}$.

Consider again the wave equation (2). If the operators a and b of the equation (6) can be represented in the form

$$a(t) = e^{iH_0 t}\, H_0^{-\frac{1}{2}}\, B(t)\, H_0^{-\frac{1}{2}}\, e^{-iH_0 t}$$

$$b(t) = -i\, e^{iH_0 t}\, H_0^{-\frac{1}{2}}\, B(t)\, H_0^{-\frac{1}{2}}\, e^{iH_0 t}\,,$$

and we define

$$u(t) = (2H_0)^{-\frac{1}{2}}\, (e^{-iH_0 t}\, v(t) + e^{iH_0 t}\, w(t))\,,$$

then $u(t)$ satisfies the wave equation (2) in an appropriate sense.

If the family $\mathbf{U}(t_2, t_1)$ is in Σ_0 then $\Omega(\cdot)$ – normalized as in Section 4 – maps $\mathbf{U}(t_2, t_1)$ into a two parameter family

$$S(t_2, t_1) = \Omega(\mathbf{U}(t_2, t_1)) \tag{7}$$

of elements of $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$. In this context we call $\Omega(\cdot)$ a quantization of the family $\mathbf{U}(t_2, t_1)$ or, alternatively, of the system of equations (6).

We assume now that the Cauchy problem for (6) has a unique solution, so that $\mathbf{U}(t_2, t_1)$ exists. Consider the Feynmann boundary value problem

$$\begin{pmatrix} \varphi \\ w(t_2) \end{pmatrix} = \mathbf{U}(t_2, t_1) \begin{pmatrix} v(t_1) \\ 0 \end{pmatrix} \tag{8}$$

where $\varphi \in \mathscr{S}$ is given. It follows from the results of Section 3 that $\mathbf{U}(t_2, t_1) \in \Sigma_0$ if and only if (8) has a solution $(v(t_1), w(t_2)) \in \mathscr{S} \oplus \mathscr{S}$ for a set of φ's which is dense in $\mathscr{S}$, or, equivalently, if and only if the homogeneous Feynmann problem ($\varphi = 0$) has only the null solution in $\mathscr{S}^* \oplus \oplus\, \mathscr{S}^*$.

Assume that this condition is fulfilled for all t_2, t_1. Then it follows from the results of Section 4 that if the map $(t_2, t_1) \to U(t_2, t_1)^{-1} \in L(\mathscr{S}, \mathscr{S}^*)$ is continuous, then the operators $S(t_2, t_1)$ given in (7) depend continuously in $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$ on t_2, t_1. Also, if the limit $\mathbf{U}(\infty, -\infty)$ exists in Σ_0, then the associated S-matrix $S(\infty, -\infty)$ exists as the limit of $S(t_2, t_1)$ in $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$.

Let us remark that since $\mathscr{S}$ is a perfect space (bounded closed sets are compact), the continuity of $U(t_2, t_1)^{-1}$ in $L(\mathscr{S}, \mathscr{S}^*)$ is equivalent to the continuity of $\langle\varphi, U(t_2, t_1)^{-1}\psi\rangle$ for all fixed φ, ψ in $\mathscr{S}$. A similar remark applies to $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$.

From the results of Section 5 it follows that $S(\infty, -\infty)$ is unitary if and only if the map $v(-\infty) \rightharpoonup \varphi = U(\infty, -\infty)\, v(-\infty)$ defined by (8) is norm bounded, and the map $v(-\infty) \rightharpoonup w(\infty) = \overline{V}(\infty, -\infty)\, v(-\infty)$ is of Hilbert-Schmidt class.

As a further consequense of Section 4 we note: If $U(t_2, t_1)^{-1} \in L(\mathscr{S}, \mathscr{S}^*)$ is differentiable as a function of t_2, so is $S(t_2, t_1)$, and

$$i\,\frac{\partial S(t_2, t_1)}{\partial t_2} = :\hat{H}_{t_1}(t_2)\, S(t_2, t_1): \tag{9}$$

with $\hat{H}$ a homogeneous polynomial of the second degree in a and a^* to be read off from Theorem 4. However, it is not always possible to write (9) in the form of a Schrödinger equation.

If the conventional quantization procedure is applicable then $S(t_2, t_1)$ is the unitary transformation which in the interaction picture maps from initial time t_1 to t_2. Since $S(t_2, t_1)$ is in general not unitary, $\Omega(\cdot)$ is a generalization of the conventional quantization scheme.

Notation

Our notation is that of I and II (cf. II, Section 2) except that in this paper $\bar{\varphi}$ and not φ^* denotes the complex conjugate of φ in $\mathscr{S}^n$ or in $\mathscr{S}^{n*}$. Further we shall make use of the notation $\overline{A}$ for the operator defined by

$$\overline{A}\,\varphi = \overline{A\,\bar{\varphi}}$$

and A^{T} for the transpose of the operator A,

$$A^{\mathsf{T}} = \overline{A}^* .$$

For reference purposes we cite here a particular case of Schwartz' nuclear theorem (cf. for instance, EHRENPREIS [2]).

Lemma 1. *If $\gamma \in L(\mathscr{S}, \mathscr{S}^*)$, then there exists a unique distribution $\gamma_{\mathsf{K}} \in \mathscr{S}^{2*}$, called the kernel of the operator γ, such that*

$$\langle\bar{\varphi}, \gamma\,\psi\rangle = \langle\bar{\varphi} \otimes \bar{\psi}, \gamma_{\mathsf{K}}\rangle$$

for all $\varphi, \psi \in \mathscr{S}$.

If γ has the kernel γ_{K}, then the kernel of $\bar{\gamma}$ is $\bar{\gamma}_{\mathsf{K}}$ and the kernel of γ^{T} is characterized by

$$\langle\varphi \otimes \psi, (\gamma^{\mathsf{T}})_{\mathsf{K}}\rangle = \langle\psi \otimes \varphi, \gamma_{\mathsf{K}}\rangle \,.$$

In particular, $\gamma = \gamma^{\mathsf{T}}$ if and only if γ_{K} is symmetric, i. e. $\gamma_{\mathsf{K}} \in \mathscr{S}^{2*}_{+}$.

Continuous linear mappings $U \in L(\mathscr{S}^*, \mathscr{S}^*)$ are applied to distributions with values in an arbitrary space by duality in the usual way, i. e. we define

$$U\,a(\varphi) = a(U^{\mathsf{T}}\,\varphi) \quad \varphi \in \mathscr{S} .$$

2. Gaussian Elements

Definition 1. *If* $\gamma \in L(\mathscr{S}, \mathscr{S}^*)$, *then an element* $\Gamma \in \tilde{\mathfrak{S}}^*$ *is called a Gaussian element associated with* γ *if and only if*

$$a(\varphi)\,\Gamma = a^*(\gamma\,\varphi)\,\Gamma \quad \textit{for all} \quad \varphi \in \mathscr{S}\,. \tag{2.1}$$

If Γ is a Gaussian element associated with γ, then

$$\begin{aligned} a(\varphi)\,a(\psi)\,\Gamma &= a(\varphi)\,a^*(\gamma\,\psi)\,\Gamma = a^*(\gamma\,\psi)\,a(\varphi)\,\Gamma + \langle\bar{\varphi}, \gamma\,\psi\rangle\,\Gamma \\ &= a^*(\gamma\,\psi)\,a^*(\gamma\,\varphi)\,\Gamma + \langle\bar{\varphi}, \gamma\,\psi\rangle\,\Gamma \end{aligned}$$

for all $\varphi, \psi \in \mathscr{S}$, and since $[a(\varphi), a(\psi)] = [a^*(\gamma\,\varphi), a^*(\gamma\,\psi)] = 0$, we have

$$\langle\bar{\varphi}, \gamma\,\psi\rangle\,\Gamma = \langle\bar{\psi}, \gamma\,\varphi\rangle\,\Gamma\,,$$

i. e. $\gamma = \gamma^{\mathsf{T}}$ is a necessary condition for the existence of a non-zero Gaussian element associated with γ.

Theorem 1. *Let* $\gamma = \gamma^{\mathsf{T}} \in L(\mathscr{S}, \mathscr{S}^*)$. *Then every Gaussian element* Γ *associated with* γ *is of the form* $\Gamma = c\,\Gamma(\gamma)$, $c \in \mathbf{C}$, *where*

$$\Gamma(\gamma) = \sum_{n=0}^{\infty} ((2n)!)^{-\frac{1}{2}}\, a^{*2n\otimes}\left(\operatorname{sym} 2^{-n}\binom{2n}{n}^{\frac{1}{2}}\gamma_{\mathsf{K}}^{n\otimes}\right)\Psi_0\,, \tag{2.2}$$

and, conversely, $\Gamma(\gamma)$ *is a Gaussian element associated with* γ.

Proof. Assume that

$$\Gamma = \sum_{k=0}^{\infty} (k!)^{-\frac{1}{2}}\, a^{*k\otimes}(\Gamma_k)\,\Psi_0 \in \tilde{\mathfrak{S}}^*$$

is a Gaussian element associated with γ, and let $\varphi \in \mathscr{S}$. Then we get, replacing φ by $\bar{\varphi}$ in Definition 1:

$$\langle\varphi, \Gamma_1\rangle = 0\,,$$

$$(k+1)^{\frac{1}{2}}\,\langle\varphi, \Gamma_{k+1}\rangle_{(1)} = k^{\frac{1}{2}}\operatorname{sym}(\gamma\,\bar{\varphi}\otimes\Gamma_{k-1}) \quad \text{for} \quad k \geqq 1\,,$$

and hence

$$(k+1)^{\frac{1}{2}}\,\langle\varphi^{(k+1)\otimes}, \Gamma_{k+1}\rangle = k^{\frac{1}{2}}\,\langle\varphi^{(k+1)\otimes}, \gamma_{\mathsf{K}}\otimes\Gamma_{k-1}\rangle\,.$$

Since $\varphi^{n\otimes}$ span a dense subset of $\mathscr{S}_+^n$ (II, Lemma 1) we get $\Gamma_k = 0$ for k odd, while

$$\Gamma_{2n} = c\,2^{-n}\binom{2n}{n}^{\frac{1}{2}}\operatorname{sym}(\gamma_{\mathsf{K}}^{n\otimes})\,, \quad \text{for} \quad n = 1, 2, \ldots,$$

with $c = \Gamma_0$, so that in fact $\Gamma = c\,\Gamma(\gamma)$.

On the other hand, straightforward computation shows that $\Gamma(\gamma)$ is a Gaussian element associated with γ, and the theorem is proved.

It is obvious that a necessary condition that $\Gamma(\gamma) \in \mathfrak{H}$ is that $\gamma_{\mathsf{K}} \in \mathscr{H}^2$, i. e. that the kernel of γ be a Hilbert-Schmidt kernel. A necessary and sufficient condition is given in Theorem 2, in the proof of which we need:

Lemma 2. *If an element $\gamma_K \in \mathscr{H}^2$ is symmetric, then it has a representation of the form*

$$\gamma_K = \sum_n \gamma_n \varphi_n^{2\otimes}$$

where $\{\varphi_n\}$ is an orthonormal system in $\mathscr{H}$ and $\gamma_n \geqq 0$.

Proof. Assume that $\gamma_K \in \mathscr{H}^2$, and let γ be the Hilbert-Schmidt operator in $\mathscr{H}$ with kernel γ_K. The symmetry of γ_K is equivalent to the equation $\gamma^* = \bar{\gamma}$.

Since γ is compact, it has a representation of the form

$$\gamma = \Sigma \gamma_n P_n ,$$

where $\{\gamma_n\}$ is the set of non-zero eigenvalues of $|\gamma| = (\gamma^* \gamma)^{\frac{1}{2}}$ while P_n is a partial isometry (elementary operator) with initial domain equal to the eigenspace for $|\gamma|$ corresponding to the eigenvalue γ_n (cf. Al. Ghika [4] théorème 4.2, see also Hestenes [5] Theorem 11.1).

We then have

$$\Sigma \gamma_n P_n^* = \gamma^* = \bar{\gamma} = \Sigma \gamma_n \overline{P}_n ,$$

where P_n^* and $\overline{P}_n$ are again partial isometries. From the uniqueness of the representation considered in [4] and [5] it follows that $P_n^* = \overline{P}_n$, and in order to prove the result it is obviously sufficient to establish it for each of the partial isometries P_n.

Thus, assume that $P^* = \overline{P}$ for a partial isometry P of finite rank. Let φ be any non-zero vector in the initial domain of $\overline{P}$, and define $\psi = \overline{P}\varphi = P^*\varphi$. Then $\varphi = P\psi$, and the two vectors $\chi' = \varphi + \bar{\psi}$, $\chi'' = i(\varphi - \bar{\psi})$ are such that $P\bar{\chi} = \chi$.

Since at least one of the vectors χ', χ'' is non-zero, there exists a normalized vector χ satisfying $P\bar{\chi} = \chi$. Let Q be the operator with kernel $\chi^{2\otimes}$, then $P = Q + R$, where $R^* = \overline{R}$ and R is a partial isometry whose initial domain has smaller dimension than that of P. The proof is completed by finite induction.

In order to compute the norm of $\Gamma(\gamma)$, we first consider an operator γ of finite rank with a kernel

$$\gamma_K = \sum_{j=1}^{k} \gamma_j \varphi_j^{2\otimes} ,$$

where $\{\varphi_1, \ldots, \varphi_k\}$ are orthonormal and $\gamma_j \geqq 0$. For such a kernel we have

$$\gamma_K^{n\otimes} = \sum_{j_1=1}^{k} \cdots \sum_{j_n=1}^{k} \gamma_{j_1} \cdots \gamma_{j_n} \varphi_{j_1}^{2\otimes} \otimes \cdots \otimes \varphi_{j_n}^{2\otimes} .$$

Rearranging the tensor factors of each term according to the ordering determined by the indices, and denoting the number of factors of the form $\gamma_j\, \varphi_j^{2\otimes}$ in a given term by ν_j, we get

$$\operatorname{sym}(\gamma_{\mathsf{K}}^{n\otimes}) = \sum \frac{n!}{\nu_1!\,\nu_2!\ldots\nu_k!}\, \gamma_1^{\nu_1} \cdots \gamma_k^{\nu_k} \operatorname{sym}(\varphi_1^{2\nu_1\otimes} \otimes \cdots \otimes \varphi_k^{2\nu_k\otimes})$$

where the sum is extended over the set of all representations of n in the form $n = \nu_1 + \nu_2 + \cdots + \nu_k$, where $\nu_1, \nu_2, \ldots, \nu_k$ are non-negative integers.

Since the φ_j are orthonormal, we have

$$\langle \varphi_1^{2\nu_1\otimes} \otimes \cdots \otimes \varphi_k^{2\nu_k\otimes},\ \operatorname{sym} \varphi_1^{2\mu_1\otimes} \otimes \cdots \varphi_k^{2\mu_k\otimes} \rangle = 0$$

if $(\nu_1, \ldots, \nu_k) \neq (\mu_1, \ldots, \mu_k)$, and

$$\langle \varphi_1^{2\nu_1\otimes} \otimes \cdots \otimes \varphi_k^{2\nu_k\otimes},\ \operatorname{sym} \varphi_1^{2\nu_1\otimes} \otimes \cdots \otimes \varphi_k^{2\nu_k\otimes} \rangle = \frac{(2\nu_1)!\ldots(2\nu_k)!}{(2n)!}\,,$$

and since sym is an orthogonal projection in $\mathscr{H}^{2n}$, we obtain

$$\|\operatorname{sym} \gamma_{\mathsf{K}}^{n\otimes}\|^2 = \sum_{\nu_1 + \cdots + \nu_k = n} \frac{(n!)^2}{(2n)!} \prod_{j=1}^{k} \frac{(2\nu_j)!}{(\nu_j!)^2}\, |\gamma_j|^{2\nu_j}\,.$$

Now let $\varphi_1, \varphi_2, \ldots$ be an orthonormal sequence in $\mathscr{H}$, let

$$\gamma_{\mathsf{K}} = \sum_{j=1}^{\infty} \gamma_j\, \varphi_j^{2\otimes} \in \mathscr{H}^2\,,$$

and define

$$\gamma_{\mathsf{K}}^k = \sum_{j=1}^{k} \gamma_j\, \varphi_j^{2\otimes}\,.$$

Then $\gamma_{\mathsf{K}}^k \to \gamma_{\mathsf{K}}$ in $\mathscr{H}^2$, and since the mapping, which sends $K \in \mathscr{H}^2$ into $\operatorname{sym} K^{n\otimes} \in \mathscr{H}^{2n}$ is continuous, we get

$$\|\operatorname{sym}((\gamma_{\mathsf{K}}^k)^{n\otimes})\| \to \|\operatorname{sym}(\gamma_{\mathsf{K}}^{n\otimes})\|\,.$$

Since all terms are non-negative, it follows that the unordered infinite sum

$$\sum_{\Sigma \nu_j = n} \frac{(n!)^2}{(2n)!} \prod_{j=1}^{\infty} \frac{(2\nu_j)!}{(\nu_j!)^2}\, |\gamma_j|^{2\nu_j}$$

is convergent and has the value $\|\operatorname{sym}(\gamma_{\mathsf{K}}^{n\otimes})\|^2$. The summation is to be extended over all sequences $\{\nu_1, \nu_2, \ldots\}$ of non-negative integers with $\Sigma\, \nu_j = n$, and hence, in particular, the product appearing in each term is in reality finite, and, furthermore, the sum is extended over a countable set. The convergence follows from the observation that the terms can be ordered into a series in such a way that certain partial sums have the values $\|\operatorname{sym}(\gamma_{\mathsf{K}}^k)^{n\otimes}\|^2$.

We shall now prove:

Theorem 2. *Let $\gamma = \gamma^{\mathsf{T}} \in L(\mathscr{S}, \mathscr{S}^*)$. A necessary and sufficient condition that the Gaussian element $\Gamma(\gamma)$ belong to $\mathfrak{H}$ is that γ be a Hilbert-Schmidt operator in $\mathscr{H}$ and that $\|\gamma\|_{op} < 1$, where $\|\gamma\|_{op}$ denotes the operator norm of γ in $\mathscr{H}$, i. e. $\|\gamma\|_{op} = \sup\{\|\gamma\, \varphi\| \mid \|\varphi\| \leq 1\}$.*

If these conditions are fulfilled, then $|||\Gamma(\gamma)||| = \prod_j (1-\gamma_j^2)^{-\frac{1}{4}}$, *where* $\{\gamma_j\}$ *denotes the eigenvalues of* $|\gamma| = (\gamma^* \gamma)^{1/2}$.

Proof. It has already been mentioned that a necessary condition in order that $\Gamma(\gamma)$ belong to $\mathfrak{H}$ is that γ be a Hilbert-Schmidt operator. Assume this to be the case, then, since

$$\binom{2\nu}{\nu} = 2^{2\nu}(-1)^\nu \binom{-1/2}{\nu},$$

we get

$$\begin{aligned}|||\Gamma(\gamma)|||^2 &= \sum_{n=0}^{\infty} \|\mathrm{sym}\,(2^{-n}\binom{2n}{n}^{1/2} \gamma_{\mathsf{K}}^{n\otimes})\|^2 \\ &= \sum_{n=0}^{\infty} \sum_{\Sigma\nu_j = n} \prod_j \binom{-1/2}{\nu_j} (-\gamma_j^2)^{\nu_j} \\ &= \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} \cdots \prod_j \binom{-1/2}{\nu_j} (-\gamma_j^2)^{\nu_j}\end{aligned}$$

where the multiple sum is only extended over sets $\{\nu_j\}$ with finitely many ν_j different from 0. Since all terms are positive we get

$$|||\Gamma(\gamma)|||^2 = \prod_j \left(\sum_{\nu=0}^{\infty} \binom{-1/2}{\nu} (-\gamma_j^2)^\nu \right)$$

which is infinite if $\|\gamma\|_{op} = \max \gamma_j \geqq 1$.

This proves the necessity of the conditions. Conversely, if these are fulfilled, then

$$|||\Gamma(\gamma)|||^2 = \prod_j (1-\gamma_j^2)^{-\frac{1}{2}} < \infty,$$

and the theorem follows.

3. Linear Transformations of Pairs of Field Operators

Definition 2. *By a pair of field operators we understand a pair* $\mathbf{a} = (a, a^*)$ *of continuous linear mappings from* $\mathscr{S}$ *into* $L(\mathfrak{S}^?, \mathfrak{S}^?)$ *(where* $\mathfrak{S}^?$ *is a space of type* $\mathfrak{S}$*) such that* $a^*(\varphi)$ *and* $a(\bar\varphi)$ *are dual for all* $\varphi \in \mathscr{S}$, *and the canonical commutation relations hold.*

Let (a, a^*) be a pair of field operators and let $\varepsilon, \omega \in L(\mathscr{S}, \mathscr{S})$. Consider the pair of operators a_1, a_1^* defined by

$$\begin{aligned} a_1(\bar\varphi) &= a(\bar\varepsilon\,\bar\varphi) + a^*(\bar\omega\,\bar\varphi) \\ a_1^*(\varphi) &= a^*(\varepsilon\,\varphi) + a(\omega\,\varphi)\,. \end{aligned} \tag{3.1}$$

Straightforward computation shows that $\mathbf{a}_1 = (a_1, a_1^*)$ is a pair of field operators iff

$$\begin{aligned} \omega^{\mathsf{T}}\,\varepsilon - \varepsilon^{\mathsf{T}}\,\omega &= 0 \\ \varepsilon^*\,\varepsilon - \omega^*\,\omega &= 1\,. \end{aligned} \tag{3.2}$$

Let in particular $\mathbf{a}$ be the canonical pair in $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}})$. We then ask whether there exists an intertwining operator $\Omega \in L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$ such that

$$\begin{aligned} a_1(\varphi)\,\Omega &= \Omega\, a(\varphi) \\ a_1^*(\varphi)\,\Omega &= \Omega\, a^*(\varphi) \end{aligned} \tag{3.3}$$

for $\varphi \in \mathscr{S}$.

If Ω is such an intertwining operator, and if we define

$$\Psi_1 = \Omega\, \Psi_0\,,$$

then obviously

$$a_1(\varphi)\,\Psi_1 = 0 \quad \text{for all} \quad \varphi \in \mathscr{S}\,.$$

Conversely, if $\Psi_1 \in \tilde{\mathfrak{S}}^*$ satisfies $a_1(\varphi)\,\Psi_1 = 0$ for all $\varphi \in \mathscr{S}$, and if we define Ω by

$$\Omega\, a^{*n\otimes}(\psi_n)\,\Psi_0 = a_1^{*n\otimes}(\psi_n)\,\Psi_1\,, \tag{3.4}$$

then it is easily verified that Ω is an intertwining operator. Here $a_1^{*n\otimes}$ is the unique continuous extension to $\mathscr{S}_+^n$ of the mapping from $\mathscr{S}_+^{n\otimes}$ into $L(\tilde{\mathfrak{S}}^*, \tilde{\mathfrak{S}}^*)$ defined by $a_1^{*n\otimes}(\varphi^{n\otimes}) = a_1^*(\varphi)^n$ (cf. I, p. 203). Thus, to study existence and uniqueness of intertwining operators we need only study the equations

$$a(\bar{\varepsilon}\,\varphi)\,\Psi = -\,a^*(\bar{\omega}\,\varphi)\,\Psi\,, \quad \varphi \in \mathscr{S}\,. \tag{3.5}$$

From (3.2) follows that

$$\|\varepsilon\,\varphi\|^2 = \|\omega\,\varphi\|^2 + \|\varphi\|^2$$

so that ε^{-1} exists from $\varepsilon(\mathscr{S})$ onto $\mathscr{S}$ and $\|\omega\,\varphi\| < \|\varepsilon\,\varphi\|$. Consequently there exists a unique mapping γ' on $\varepsilon(\mathscr{S})$ such that

$$-\bar{\omega}\,\varphi = \gamma'\,\bar{\varepsilon}\,\varphi \tag{3.6}$$

for $\varphi \in \mathscr{S}$. Since $\|\bar{\omega}\,\varphi\| < \|\bar{\varepsilon}\,\varphi\|$, the mapping γ' has an extension to an operator in $L(\mathscr{H}, \mathscr{H})$ of norm at most 1. In particular γ' has an extension $\gamma \in L(\mathscr{S}, \mathscr{S}^*)$, and it can be shown that it follows from (3.2) that γ can be chosen so that $\gamma^{\mathsf{T}} = \gamma$.

The equation (3.5) can now be written

$$a(\varphi)\,\Psi = a^*(\gamma\,\varphi)\,\Psi\,, \tag{3.7}$$

for $\varphi \in \bar{\varepsilon}(\mathscr{S})$.

A Gaussian element associated with γ is of course a solution to these equations. If $\bar{\varepsilon}(\mathscr{S})$ is dense in $\mathscr{S}$, then γ is unique, and a slight modification of the proof of Theorem 1 shows that the complete solution to (3.5) is $c\,\Gamma(\gamma)$, $c \in \mathbf{C}$. In case $\bar{\varepsilon}(\mathscr{S})$ is not dense in $\mathscr{S}$, it can be shown that the manifold of solutions to (3.7) is infinite-dimensional.

Definition 3. *The (real) symplectic group Σ over $\mathscr{S}$ is the family of matrices $\mathbf{U}$ of the form*

$$\mathbf{U} = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix},$$

such that $U, V, U^*, V^* \in L(\mathscr{S}, \mathscr{S})$ *and*

$$\mathbf{U}^* \mathbf{J}\, \mathbf{U} = \mathbf{U}\, \mathbf{J}\, \mathbf{U}^* = \mathbf{J} \tag{3.8}$$

where

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

1 *denoting the identity operator in* $\mathscr{S}$, *and*

$$\mathbf{U}^* = \begin{pmatrix} U^* & V^{\mathsf{T}} \\ V^* & U^{\mathsf{T}} \end{pmatrix}.$$

In this definition, U^* and V^* denote adjoints of U and V. In the sequel we also use the symbols U, V, U^*, V^* to denote the unique continuous extensions in $L(\mathscr{S}^*, \mathscr{S}^*)$ of $U, V, U^*, V^* \in L(\mathscr{S}, \mathscr{S})$. Obviously $U \in L(\mathscr{S}^*, \mathscr{S}^*)$ is the dual of $U^* \in L(\mathscr{S}, \mathscr{S})$, etc.

To each element $\mathbf{U}$ of Σ we may associate the transformation

$$\mathbf{a} = \begin{pmatrix} a \\ a^* \end{pmatrix} \frown \mathbf{U}\,\mathbf{a} = \begin{pmatrix} U & V \\ \overline{V} & \overline{U} \end{pmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix} = \begin{pmatrix} a_{\mathbf{U}} \\ a^*_{\mathbf{U}} \end{pmatrix} \tag{3.9}$$

in the class of all pairs of field operators.

It is clear that this is a transformation of the type (3.1) with

$$\begin{aligned} \varepsilon &= U^* \\ \omega &= V^*. \end{aligned} \tag{3.10}$$

Moreover, Σ consists exactly of the invertible transformations (3.1).

Of particular interest is the case, where there exists a unique ray of intertwining operators $\Omega = \Omega(\mathbf{U}) \in L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$ associated with $\mathbf{U}$. As remarked above, this is the case if and only if the range of $\bar{\varepsilon} = U^{\mathsf{T}}$ on $\mathscr{S}$ is dense in $\mathscr{S}$. Another case of interest (cf. § 5) is that, where $U(\mathscr{S})$ and $U^{\mathsf{T}}(\mathscr{S})$ are dense in $\mathscr{H}$.

Definition 4. *The set of elements* $\mathbf{U} \in \Sigma$, *for which* $U^{\mathsf{T}}(\mathscr{S})$ *is dense in* $\mathscr{S}$ *is denoted by* Σ_0. *The set of elements* $\mathbf{U} \in \Sigma$, *for which* $U(\mathscr{S})$ *and* $U^{\mathsf{T}}(\mathscr{S})$ *are dense in* $\mathscr{H}$ *is denoted by* Σ_1.

The following lemma contains a number of facts about transformations $\mathbf{U} \in \Sigma$.

Lemma 3. *A necessary and sufficient condition in order that a transformation*

$$\mathbf{U} = \begin{pmatrix} U & V \\ \overline{V} & \overline{U} \end{pmatrix}$$

with $U, V \in L(\mathscr{S}, \mathscr{S}) \cap L(\mathscr{S}^*, \mathscr{S}^*)$ *belong to* Σ *is that the following hold in* $\mathscr{S}$ *(and then by continuity also in* $\mathscr{S}^*$*)*:

$$\begin{aligned} U\, U^* &= 1 + V\, V^* \\ U^*\, U &= 1 + V^{\mathsf{T}}\, \overline{V} \\ U\, V^{\mathsf{T}} &= V\, U^{\mathsf{T}} \\ U^*\, V &= V^{\mathsf{T}}\, \overline{U}, \end{aligned} \tag{3.11}$$

and then

$$\mathbf{U}^{-1} = \begin{pmatrix} U^* & -V^\mathsf{T} \\ -V^* & U^\mathsf{T} \end{pmatrix}. \tag{3.12}$$

The subsets Σ_0 and Σ_1 of the group Σ satisfy

$$\Sigma_0^{-1} = \Sigma_0 \subset \Sigma_1 = \Sigma_1^{-1} \subset \Sigma .$$

If $\mathbf{U} \in \Sigma_1$, then $U(\mathscr{H}) \supset \mathscr{H}$ and $U^*(\mathscr{H}) \supset \mathscr{H}$, and U^{-1}, U^{*-1}, $\gamma = -V^\mathsf{T} U^{\mathsf{T}-1}$, and $\gamma_1 = V \overline{U}^{-1}$ are bounded operators in $\mathscr{H}$ satisfying the relations

$$\begin{aligned} U^{-1} U^{*-1} + \gamma \bar{\gamma} &= 1 \\ U^{*-1} U^{-1} + \gamma_1 \bar{\gamma}_1 &= 1 \\ \gamma &= \gamma^\mathsf{T} \\ \gamma_1 &= \gamma_1^\mathsf{T} . \end{aligned} \tag{3.13}$$

Proof. The relations (3.11) are merely the equation (3.8) spelled out in detail, and (3.12) is equivalent with the equation

$$\mathbf{U}^{-1} = \mathbf{J}\, \mathbf{U}^*\, \mathbf{J} ,$$

which is still another formulation of (3.8).

To prove that $\Sigma_0^{-1} \subset \Sigma_0$ (and hence $= \Sigma_0$), assume that $\mathbf{U} \in \Sigma_0$. Then by duality, U is one-to-one on $\mathscr{S}^*$. Now consider an element $f \in \mathscr{S}^*$ such that

$$U^* f = 0 .$$

From the third equation in (3.11) it follows that

$$U\, V^\mathsf{T}\, \bar{f} = V\, U^\mathsf{T}\, \bar{f} = V(U^* f)^- = 0 ,$$

and hence that

$$V^* f = (V^\mathsf{T}\, \bar{f})^- = 0$$

since U is $1-1$.

From the first equation in (3.11) it then follows that

$$f = (U\, U^* - V\, V^*)\, f = 0 ,$$

and hence U^* is one-to-one on $\mathscr{S}^*$. By duality once more, $U(\mathscr{S})$ is dense in $\mathscr{S}$, and hence $\mathbf{U}^{-1} \in \Sigma_0$.

It is now trivial that $\Sigma_0 \subset \Sigma_1$, and in view of (3.12) it is clear that $\Sigma_1^{-1} = \Sigma_1$.

Assume next that $\mathbf{U} \in \Sigma_1$. From the second equation in (3.11) we get

$$\|U\, \varphi\| \geqq \|\varphi\| \tag{3.14}$$

for $\varphi \in \mathscr{S}$. Now, for each $f \in \mathscr{H}$ there exists a sequence $\{\varphi_n\} \subset \mathscr{S}$ such that $U\, \varphi_n \to f$ in $\mathscr{H}$. In view of (3.14) the sequence $\{\varphi_n\}$ converges in $\mathscr{H}$, and hence also in $\mathscr{S}^*$, to some element $g \in \mathscr{H}$, and since $U \in L(\mathscr{S}, \mathscr{S}^*)$ it follows that $U\, \varphi_n \to U\, g$ in $\mathscr{S}^*$. Consequently, $f = U\, g$ for some $g \in \mathscr{H}$, so that $U(\mathscr{H}) \supset \mathscr{H}$, and, by symmetry, also $U^*(\mathscr{H}) \supset \mathscr{H}$. Since

$U(\mathscr{S})$ is dense in $\mathscr{H}$, it follows by duality that U^* is one-to-one as a mapping of $\mathscr{H}$ into $\mathscr{S}^*$, and similarly for U. Thus, U^{-1} and U^{*-1} are well-defined as mappings of $U(\mathscr{H})$ resp. $U^*(\mathscr{H})$ onto $\mathscr{H}$, and hence as mappings of $\mathscr{H}$ into $\mathscr{H}$.

The final statements of the lemma are now easily verified.

We now collect the main results of the above discussion in:

Theorem 3. *If* $\mathbf{U} \in \Sigma_0$, *then there exists a unique ray* $\Omega = c\,\Omega(\mathbf{U})$ *of intertwining operators associated with* $\mathbf{U}$, *i. e. satisfying* (3.3), *where*

$$\begin{pmatrix} a_1 \\ a_1^* \end{pmatrix} = \begin{pmatrix} a_{\mathrm{U}} \\ a_{\mathrm{U}}^* \end{pmatrix} = \mathbf{U} \begin{pmatrix} a \\ a^* \end{pmatrix}. \tag{3.15}$$

The operator $\Omega(\mathbf{U})$ *is given by*

$$\Omega(\mathbf{U})\,\Psi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, a_{\mathrm{U}}^{*\,n\otimes}(\psi_n)\,\Gamma(\gamma) \tag{3.16}$$

for all $\Psi = \{\psi_n\} \in \tilde{\mathfrak{S}}$, *where the operator* $\gamma = \gamma(\mathbf{U})$ *associated with* $\mathbf{U}$ *is defined by* $\gamma = -V^{\mathsf{T}}\, U^{\mathsf{T}\,-1}$.

4. Continuity and Differentiability of the Family of Intertwining Operators

In II we proved that the displacement operator $D(f) \in L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$ is differentiable as a function of $f \in \mathscr{S}^*$. The object of this section is to prove an analogous result for the intertwining operator $\Omega(\mathbf{U})$.

We start by evaluating the bilinear form $\langle\!\langle \Phi, \Omega\,\Psi \rangle\!\rangle$ for Φ, Ψ belonging to the summand spaces of $\tilde{\mathfrak{S}}$.

Lemma 4. *Let* $\mathbf{U} \in \Sigma_0$, *and let* $\Phi = a^*(\varphi)^n\,\Psi_0$, $\Psi = a^*(\psi)^m\,\Psi_0$. *Then*

$$\langle\!\langle \Phi, \Omega(\mathbf{U})\,\Psi \rangle\!\rangle =$$

$$= \sum_{\substack{2p+r=m \\ 2q+r=n}} \frac{m!\,n!}{2^{p+q}\,p!\,q!\,r!}\, \langle \gamma(\mathbf{U}^{-1})\,\bar{\psi}, \psi \rangle^p\, \langle \varphi, U^{-1}\psi \rangle^r\, \langle \varphi, \gamma(\mathbf{U})\,\bar{\varphi} \rangle^q .$$

Proof. With the notation of Theorem 3, we have

$$\langle\!\langle \Phi, \Omega(\mathbf{U})\,\Psi \rangle\!\rangle = \langle\!\langle a_{\mathrm{U}}(\bar{\psi})^m\, a^*(\varphi)^n\,\Psi_0, \Gamma(\gamma) \rangle\!\rangle .$$

Since

$$a_{\mathrm{U}}(\bar{\psi})^m = \sum_{2s+t+u=m} \frac{m!}{2^s\,s!\,t!\,u!} \langle \psi, U\,V^{\mathsf{T}}\,\bar{\psi} \rangle^s\, a^*(V^{\mathsf{T}}\,\bar{\psi})^t\, a(U^{\mathsf{T}}\,\bar{\psi})^u$$

we get

$$a_{\mathrm{U}}(\bar{\psi})^m\, a^*(\varphi)^n\,\Psi_0 = \sum_{\substack{2s+t+u=m \\ u+v=n}} \frac{m!\,n!}{2^s\,s!\,t!\,u!\,v!} \times$$

$$\times \langle \psi, U\,V^{\mathsf{T}}\,\bar{\psi} \rangle^s\, \langle \psi, U\,\varphi \rangle^v\, a^*(V^{\mathsf{T}}\,\bar{\psi})^t\, a^*(\varphi)^v\,\Psi_0 .$$

Consider a term $\langle\!\langle a^*(V^{\mathsf T}\bar\psi)^t a^*(\varphi)^v \Psi_0, \Gamma(\gamma)\rangle\!\rangle$. For $t+v$ odd it is zero, and for $t+v=2w$ we get

$$\begin{aligned}\langle\!\langle a^*(V^{\mathsf T}\bar\psi)^t a^*(\varphi)^v \Psi_0, \Gamma(\gamma)\rangle\!\rangle &= \\ &= \langle ((2w)!)^{\frac12} \operatorname{sym}((V^{\mathsf T}\bar\psi)^{t\otimes} \otimes \varphi^{v\otimes}),\ 2^{-w}\binom{2w}{w}^{\frac12} \gamma_{\mathsf K}^{w\otimes}\rangle \\ &= 2^{-w}(w!)^{-1} \sum_{\substack{2x+y=t\\ 2q+y=v}} \frac{w!t!v!\,2^y}{x!y!q!} \langle V^{\mathsf T}\bar\psi \otimes V^{\mathsf T}\bar\psi, \gamma_{\mathsf K}\rangle^x \times \\ &\quad\times \langle V^{\mathsf T}\bar\psi\otimes\varphi, \gamma_{\mathsf K}\rangle^y \langle\varphi\otimes\varphi,\gamma_{\mathsf K}\rangle^q .\end{aligned}$$

Thus

$$\begin{aligned}\langle\!\langle \Phi, \Omega(\mathbf U)\Psi\rangle\!\rangle &= \\ &= \sum_{\substack{2s+u+2x+y=m\\ 2u+y+2q=n}} \frac{m!n!}{2^{s+x+q}\, s!u!x!y!q!} \langle U V^{\mathsf T}\bar\psi, \psi\rangle^s \langle\varphi, U^*\psi\rangle^u \times \\ &\quad\times \langle V^{\mathsf T}\bar\psi\otimes V^{\mathsf T}\bar\psi, \gamma_{\mathsf K}\rangle^x \langle V^{\mathsf T}\bar\psi\otimes\varphi,\gamma_{\mathsf K}\rangle^y \langle\varphi\otimes\varphi,\gamma_{\mathsf K}\rangle^q .\end{aligned}$$

To complete the proof, observe first that in view of Lemma 3,

$$\begin{aligned}&\langle U V^{\mathsf T}\bar\psi, \psi\rangle + \langle V^{\mathsf T}\bar\psi\otimes V^{\mathsf T}\bar\psi, \gamma_{\mathsf K}\rangle \\ &= \langle (U V^{\mathsf T} + V\gamma^* V^{\mathsf T})\bar\psi, \psi\rangle \\ &= \langle V\,\bar U^{-1}\bar\psi, \psi\rangle = \langle \gamma(\mathbf U^{-1})\bar\psi, \psi\rangle ,\end{aligned}$$

and similarly,

$$\langle\varphi, U^*\psi\rangle + \langle V^{\mathsf T}\bar\psi\otimes\varphi, \gamma_{\mathsf K}\rangle = \langle\varphi, U^{-1}\psi\rangle .$$

Hence the result follows by summing first over all pairs (s, x) with $s+x=p$ and over all pairs (u, y) with $u+y=r$.

Corollary 1. *If* $\mathbf U\in\Sigma_0$, *then* $\langle\!\langle\Phi, \Omega(\mathbf U)\Psi\rangle\!\rangle = \langle\!\langle\Omega(\mathbf U^{-1})\Phi, \Psi\rangle\!\rangle$ *for all* $\Phi, \Psi\in\tilde{\mathfrak S}$.

We now define

$$:a^{*m\otimes}\otimes a^{n\otimes}(k_{m+n})\,A:\ \in L(\tilde{\mathfrak S}, \tilde{\mathfrak S}^*)$$

for $k_{m+n}\in\mathscr S^{m*}\otimes\mathscr S^{n*}$ and $A\in L(\tilde{\mathfrak S},\tilde{\mathfrak S}^*)$ by putting

$$:a^{*m\otimes}\otimes a^{n\otimes}(f_m\otimes g_n)\,A: = a^{*m\otimes}(f_m)\,A\,a^{n\otimes}(g_n) \tag{4.1}$$

for $f_m\in\mathscr S^{m*}$, $g_n\in\mathscr S^{n*}$ and extending linearly. It can be shown that the mapping

$$k_{m+n}\rightharpoonup\ :a^{*m\otimes}\otimes a^{n\otimes}(k_{m+n})\,A:$$

is continuous when $\mathscr S^{m*}\otimes\mathscr S^{n*}$ is provided with the topology of $\mathscr S^{(m+n)*}$, and since $L(\tilde{\mathfrak S},\tilde{\mathfrak S}^*)$ is complete, this mapping has a unique continuous extension to $\mathscr S^{(m+n)*}$. We denote this extension by the same symbol.

The operator: $(a^{*m\otimes}\otimes a^{n\otimes}(k_{m+n}))^r A:$ is defined by induction as:

$$:a^{*m\otimes}\otimes a^{n\otimes}(k_{m+n})\Big(:(a^{*m\otimes}\otimes a^{n\otimes}(k_{m+n}))^{r-1} A:\Big): .$$

Lemma 5. *Let $\alpha, \beta, \gamma \in \mathscr{S}^{2*}$. Then there exists a unique linear operator $\Lambda(\alpha, \beta, \gamma) \in L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$ such that*

$$\langle\langle \Phi, \Lambda(\alpha, \beta, \gamma) \Psi \rangle\rangle = \sum_{\substack{2p+r=m\\2q+r=n}} \frac{m!\,n!}{p!\,q!\,r!} \langle \bar{\psi} \otimes \bar{\psi}, \alpha\rangle^p \langle \varphi \otimes \bar{\psi}, \beta\rangle^r \langle \varphi \otimes \varphi, \gamma\rangle^q \tag{4.2}$$

for all $\Phi = a^*(\varphi)^n \Psi_0$, $\Psi = a^*(\psi)^m \Psi_0$. The mapping Λ is differentiable from $\mathscr{S}^{2*} \oplus \mathscr{S}^{2*} \oplus \mathscr{S}^{2*}$ into $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$ with the differential

$$d\Lambda = :(a \otimes a(d\alpha) + a^* \otimes a(d\beta) + a^* \otimes a^*(d\gamma))\,\Lambda(\alpha, \beta, \gamma): \tag{4.3}$$

Proof. The uniqueness of $\Lambda(\alpha, \beta, \gamma)$ follows from the fact that linear combinations of elements of the form $a^*(\varphi)^n \Psi_0$, $\varphi \in \mathscr{S}$, $n = 0, 1, 2, \ldots$, are dense in $\tilde{\mathfrak{S}}$ (cf. II, Lemma 1). On the other hand, consider the operator

$$\Lambda = \exp(a^* \otimes a^*(\gamma)) : \exp(a^* \otimes a(\beta))\, P_0 : \exp(a \otimes a(\alpha)) \tag{4.4}$$

where

$$\exp(a \otimes a(\alpha)) = \sum_{p=0}^{\infty} (p!)^{-1}(a \otimes a(\alpha))^p \in L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}),$$

$$\exp(a^* \otimes a^*(\gamma)) = \sum_{q=0}^{\infty} (q!)^{-1}(a^* \otimes a^*(\gamma))^q \in L(\tilde{\mathfrak{S}}^*, \tilde{\mathfrak{S}}^*),$$

$$:\exp(a^* \otimes a(\beta))\, P_0 : = \sum_{r=0}^{\infty} (r!)^{-1} : (a^* \otimes a(\beta))^r\, P_0 : \in L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*).$$

Here P_0 denotes the projection

$$P_0 \Psi = \langle\langle \Psi_0, \Psi \rangle\rangle \Psi_0 .$$

As noted in I (cf. Theorem 5.17) the convergence of the above series follows from the special character of the topologies of $\tilde{\mathfrak{S}}$ and $\mathfrak{S}^*$.

Straightforward computation shows that the operator Λ satisfies (4.2).

Let Φ, Ψ be of the form considered above, let $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1 \in \mathscr{S}^{2*}$, and define

$$R = \Lambda(\alpha + \alpha_1, \beta + \beta_1, \gamma + \gamma_1) - \Lambda(\alpha, \beta, \gamma) - \\ - :(a \otimes a(\alpha_1) + a^* \otimes a(\beta_1) + a^* \otimes a^*(\gamma_1))\,\Lambda(\alpha, \beta, \gamma):$$

Then, obviously, $\langle\langle \Phi, R\, \Psi \rangle\rangle$ is a finite linear combination of terms of the form

$$\langle \bar{\psi} \otimes \bar{\psi}, \alpha\rangle^s \langle \varphi \otimes \bar{\psi}, \beta\rangle^t \langle \varphi \otimes \varphi, \gamma\rangle^u \times \\ \times \langle \bar{\psi} \otimes \bar{\psi}, \alpha_1\rangle^v \langle \varphi \otimes \bar{\psi}, \beta_1\rangle^x \langle \varphi \otimes \varphi, \gamma_1\rangle^y$$

with $v + x + y \geqq 2$, and the differentiability of Λ follows by arguments similar to those in II, Section 4. At the same time we have proved the formulae (4.3) and (4.4).

As an immediate corollary to this result and Lemma 3 we have:

Theorem 4. *Denote by* σ_0 *the subset of* $\sigma = L(\mathscr{S}, \mathscr{S}^*)^3$ *consisting of all elements of the form* $(\bar{\gamma}(\mathbf{U}^{-1}), U^{-1}, \gamma(\mathbf{U}))$ *where*

$$\mathbf{U} = \begin{pmatrix} U & V \\ \overline{V} & \overline{U} \end{pmatrix} \in \Sigma_0 \,.$$

Consider $\Omega(\mathbf{U})$ *as a function defined on* σ_0. *This function has a differentiable extension from* σ *into* $L(\tilde{\mathfrak{S}}, \mathfrak{S}^*)$. *At elements of* σ_0 *the differential of this extension (which we also denote by* Ω*) is given by*

$$i\, d\Omega = \, : dK\, \Omega : \,,$$

where

$$dK = \frac{i}{2}\, a \otimes a\big(d(\bar{\gamma}(\mathbf{U}^{-1})_{\mathsf{K}})\big) + i\, a^* \otimes a\big(d(U^{-1})_{\mathsf{K}}\big) + \\ + \frac{i}{2}\, a^* \otimes a^*\big(d(\gamma(\mathbf{U})_{\mathsf{K}})\big)\,.$$

5. Intertwining Operators in $\mathfrak{H}$

For the applications it is of particular importance to have a characterization of those elements $\mathbf{U} \in \Sigma$, for which there exists an intertwining operator Ω, which can be extended to a unitary operator in the Hilbert space $\mathfrak{H}$ obtained by completing $\tilde{\mathfrak{S}}$ in the norm $|||\cdot|||$.

For this discussion it is convenient to introduce some further notation.

We denote by $\mathscr{H}^n_+$ the completion of $\mathscr{S}^n_+$ in the norm $\|\cdot\|$, and by $\tilde{\mathfrak{H}}$ the direct sum $\tilde{\mathfrak{H}} = \sum_{n=0}^{\infty} \mathscr{H}^n_+$ of these provided with the direct sum topology. Thus, $\tilde{\mathfrak{H}}$ is not a Hilbert space. The dual space $\tilde{\mathfrak{H}}^*$ of $\tilde{\mathfrak{H}}$ can be identified with the product $\prod_{n=0}^{\infty} \mathscr{H}^n_+$, which as a set is identical with the subspace of $\tilde{\mathfrak{S}}^*$ consisting of all elements $\mathbf{T} = \{T_n\}$, for which $T_n \in \mathscr{H}^n_+$ for all n.

It is clear that

$$\tilde{\mathfrak{S}} \subset \tilde{\mathfrak{H}} \subset \mathfrak{H} \subset \tilde{\mathfrak{H}}^* \subset \tilde{\mathfrak{S}}^*$$

and that both $\tilde{\mathfrak{H}}$ and $\tilde{\mathfrak{H}}^*$ are invariant under $a(\overline{\varphi})$ and $a^*(\varphi)$ for $\varphi \in \mathscr{S}$. Also, the mappings a and a^* have unique continuous extensions from $\mathscr{H}$ into $L(\tilde{\mathfrak{H}}, \tilde{\mathfrak{H}}) \cap L(\tilde{\mathfrak{H}}^*, \tilde{\mathfrak{H}}^*)$, these extensions are given by the natural formulae [cf. (2.3) in II], they satisfy the canonical coommutation relations, and $a(\overline{\varphi})$ [resp. $a^*(\varphi)$] in $L(\tilde{\mathfrak{H}}, \tilde{\mathfrak{H}})$ has the dual $a^*(\varphi)$ [resp. $a(\overline{\varphi})$] in $L(\tilde{\mathfrak{H}}^*, \tilde{\mathfrak{H}}^*)$ for all $\varphi \in \mathscr{H}$.

Assume now that the symplectic transformation $\mathbf{U} \in \Sigma$ has an associated intertwining operator Ω, which maps $\tilde{\mathfrak{S}}$ into $\tilde{\mathfrak{H}}^*$ (this is in particular the case if Ω is the restriction to $\tilde{\mathfrak{S}}$ of a transformation in $\mathfrak{H}$).

Exactly as in Section 3 we see that if $U^{\mathsf{T}}(\mathscr{S})$ is dense in $\mathscr{H}$, then the family of intertwining operators with this property is at most one-dimensional.

We shall only discuss the case $\mathbf{U} \in \Sigma_1$; the general case seems to involve considerable extra work of a rather technical nature, but we believe that essentially the same results are valid (except the one-dimensionality).

The purpose of the present section is to prove the following results.

Theorem 5. *Assume that* $\mathbf{U} \in \Sigma_1$. *A necessary and sufficient condition in order that the equations*

$$a_{\mathbf{U}}(\varphi)\,\Psi = 0 \quad \forall\, \varphi \in \mathscr{S} \tag{5.1}$$

have a non-zero solution $\Psi \in \tilde{\mathfrak{H}}^*$ *is that* $\gamma(\mathbf{U})$ *be of Hilbert-Schmidt class, and then the space of solutions in* $\tilde{\mathfrak{H}}^*$ *to* (5.1) *is one-dimensional.*

A necessary and sufficient condition in order that (5.1) *have a non-zero solution* $\Psi \in \mathfrak{H}$ *ist that* $\gamma(\mathbf{U})$ *be of Hilbert-Schmidt class and* $\|\gamma(\mathbf{U})\|_{op} < 1$, *or equivalently, that* V *be of Hilbert-Schmidt class and* U *be bounded as operators in* $\mathscr{H}$.

If $\Psi \in \mathfrak{H}$ *is a normalized solution to* (5.1), *then the operator* Ω *defined by* (3.4) *is the restriction to* $\tilde{\mathfrak{S}}$ *of a unitary operator in* $\mathfrak{H}$.

Proof. The first two statements of the theorem are immediate consequences of the arguments in Section 2 and 3 except for the equivalence of the conditions

$$\|\gamma\|_{op} < 1 \quad \text{and} \quad \|U\|_{op} < \infty\,.$$

To prove this equivalence, assume first that $\|\gamma\|_{op} = c < 1$. The first relation in (3.13) then gives

$$\|U^{*-1}\varphi\|^2 = \|\varphi\|^2 - \|\bar{\gamma}\,\varphi\|^2 \geqq (1-c^2)\|\varphi\|^2$$

for $\varphi \in \mathscr{H}$, and in particular for $\varphi \in U^*\,\psi$, where $\psi \in \mathscr{S}$, whence

$$\|U^*\,\psi\|^2 \leqq (1-c^2)^{-1}\,\|\psi\|^2$$

for $\psi \in \mathscr{S}$.

It follows that the restriction of U^* to $\mathscr{S}$ has a unique extension to a bounded operator from $\mathscr{H}$ into $\mathscr{H}$, and since U^* is continuous from $\mathscr{S}^*$ into $\mathscr{S}^*$, this extension to $\mathscr{H}$ coincides on $\mathscr{H}$ with U^*.

Since the operators U on $\mathscr{S}$ and U^* on $\mathscr{S}^*$ are each other's dual, it follows that

$$|\langle U\,\varphi, f\rangle| = |\langle \varphi, U^*\,f\rangle| \leqq \|\varphi\|\,\|U^*\|_{op}\,\|f\|$$

for $\varphi \in \mathscr{S}$ and $f \in \mathscr{H}$, and hence that also U is a bounded operator of $\mathscr{H}$ into $\mathscr{H}$.

Conversely, if $\|U\|_{op} < \infty$, insertion of the inequality

$$\|U^{*-1}\,\varphi\| \geqq \|U^*\|_{op}^{-1}\,\|\varphi\| = \|U\|_{op}^{-1}\,\|\varphi\|$$

in the first relation (3.13) proves that $\|\gamma\|_{op} < 1$.

Observe also that it follows from (3.13) that $\|\gamma_1\|_{op} = \|\gamma\|_{op}$.

Assume now that Ψ is a normalized solution to (5.1), then a simple formal argument yields

$$\langle\!\langle a_{\mathrm{U}}^*(\varphi)^m \Psi, a_{\mathrm{U}}^*(\psi)^n \Psi\rangle\!\rangle = m!\,\langle\varphi, \psi\rangle^m \delta_{mn},$$

whence the theorem would follow. However, the operators $a_{\mathrm{U}}^*(\varphi)$ and $a_{\mathrm{U}}(\bar\varphi)$ are not known to be dual on $\Omega(\mathrm{U})\,\tilde{\mathfrak{S}}'$, in fact, it is not yet proved that $\Omega(\mathrm{U})\,\tilde{\mathfrak{S}}' \subset \mathfrak{H}$. The following three lemmas are concerned with problems related to the unboundedness of $a_{\mathrm{U}}^*(\varphi)$ and $a_{\mathrm{U}}(\bar\varphi)$ and contain the remaining part of the proof of Theorem 5.

Lemma 6. *Let $\mathrm{U} \in \Sigma_1$ be such that U is bounded in $\mathscr{H}$ and V is of Hilbert-Schmidt class. Let $\Gamma = \{\Gamma_j\} \in \mathfrak{H}$ be the Gaussian element associated with $\gamma = \gamma(\mathrm{U})$, and put $c = \|\gamma\|_{op}$. Define the projection p_k in $\tilde{\mathfrak{H}}^*$ by*

$$p_k(\{T_n\}) = \{\delta_{kn}\, T_n\}\,.$$

We then have

$$|||p_{2k+\varepsilon}\, a_{\mathrm{U}}^*(\varphi)^n\, \Gamma|||^2 \leqq \frac{(2n)!}{n!}\left(\frac{\|\varphi\|^2}{c}\right)^n \sum_{j=0}^{k} \binom{n-1+k-j}{n-1} c^{2(k-j)+\varepsilon}\, \|\Gamma_{2j}\|^2 \tag{5.2}$$

for all $\varphi \in \mathscr{S}$, all positive n and all $k \geqq 0$. Here $\varepsilon = \varepsilon(n)$ is equal to 0 or 1 according to whether n is even or odd.

Proof. Define A_k^n and B_k^n by

$$A_k^n(\varphi) = |||p_{2k+\varepsilon(n)}\, a_{\mathrm{U}}^*(\varphi)^n\, \Gamma|||^2\,,$$
$$B_k^n(\varphi;\psi) = \langle\!\langle p_{2k+1+\varepsilon(n)}\, a_{\mathrm{U}}^*(\varphi)^{n-1}\,\Gamma,\ a^*(\psi)\, a_{\mathrm{U}}^*(\varphi)^n\,\Gamma\rangle\!\rangle\,.$$

Repeated application of the equations

$$a_{\mathrm{U}}^*(\varphi) = a_{\mathrm{U}}(\bar\gamma_1\,\varphi) + a^*(U^{-1}\varphi)\,,$$
$$a(\varphi) = a_{\mathrm{U}}(U^{\mathsf{T}-1}\varphi) + a^*(\gamma\,\varphi)\,,$$
$$a(\varphi)\,p_k = p_{k-1}\,a(\varphi)\,,$$
$$p_k\,a^*(\varphi) = a^*(\varphi)\,p_{k-1}\,,$$

and

$$a_{\mathrm{U}}(\varphi)\,\Gamma = 0$$

yields the inequality

$$\begin{aligned}
A_k^n(\varphi) &= \langle\!\langle p_{2k+\varepsilon}(a_{\mathrm{U}}(\bar\gamma_1\,\varphi) + a^*(U^{-1}\varphi))\, a_{\mathrm{U}}^*(\varphi)^{n-1}\,\Gamma,\ a_{\mathrm{U}}^*(\varphi)^n\,\Gamma\rangle\!\rangle \\
&= (n-1)\,\langle\bar\gamma_1\,\varphi, \bar\varphi\rangle \langle\!\langle p_{2k+\varepsilon}\, a_{\mathrm{U}}^*(\varphi)^{n-2}\,\Gamma,\ (a_{\mathrm{U}}(\bar\gamma_1\,\varphi) + \\
&\qquad\qquad + a^*(U^{-1}\varphi))\, a_{\mathrm{U}}^*(\varphi)^{n-1}\,\Gamma\rangle\!\rangle + \\
&\quad + \langle\!\langle p_{2k-1+\varepsilon}\, a_{\mathrm{U}}^*(\varphi)^{n-1}\,\Gamma,\ (a_{\mathrm{U}}(U^{\mathsf{T}-1}\,\overline{U}^{-1}\bar\varphi) + a^*(\gamma\,\overline{U}^{-1}\bar\varphi))\, a_{\mathrm{U}}^*(\varphi)^n\,\Gamma\rangle\!\rangle \\
&= (n-1)^2\,|\langle\bar\gamma_1\,\varphi, \bar\varphi\rangle|^2\, A_k^{n-2}(\varphi) + \\
&\quad + (n-1)\,\langle\bar\gamma_1\,\varphi, \bar\varphi\rangle\, B_{k-\varepsilon(n-1)}^{n-1}(\varphi; U^{-1}\varphi) + \\
&\quad + n\,\|U^{-1}\varphi\|^2\, A_{k-\varepsilon(n-1)}^{n-1}(\varphi) + \\
&\quad + B_{k-1}^n(\varphi; \gamma\,\overline{U}^{-1}\bar\varphi) \leqq \\
&\leqq (n-1)^2\, c^2\,\|\varphi\|^4\, A_k^{n-2}(\varphi) + \\
&\quad + (n-1)\, c\,\|\varphi\|^2\, |B_{k-\varepsilon(n-1)}^{n-1}(\varphi; U^{-1}\varphi)| + \\
&\quad + n\,\|\varphi\|^2\, A_{k-\varepsilon(n-1)}^{n-1}(\varphi) + |B_{k-1}^n(\varphi; \gamma\,\overline{U}^{-1}\bar\varphi)|\,.
\end{aligned}$$

In a similar way one proves that

$$\begin{aligned}|B_k^n(\varphi;\psi)| \leqq\ & (n-1)^2 c\,\|\psi\|\,\|\varphi\|^3 A_k^{n-2}(\varphi) + \\ & + (n-1)\,\|\psi\|\,\|\varphi\|\,|B_{k-\varepsilon(n-1)}^{n-1}(\varphi, U^{-1}\varphi)| + \\ & + n\,c\,\|\psi\|\,\|\varphi\|\,A_{k-\varepsilon(n-1)}^{n-1}(\varphi) + \\ & + |B_{k-1}^n(\varphi;\gamma\bar\gamma\psi)|\,.\end{aligned}$$

From these inequalities the estimate (5.2) and the estimate

$$|B_k^n(\varphi;\psi)| \leqq \frac{(2n)!}{n!}\frac{\|\psi\|}{\|\varphi\|}\left(\frac{\|\varphi\|^2}{c}\right)^n \sum_{j=0}^{k}\binom{n-1+k-j}{n-1} c^{2(k-j)+\varepsilon+1}\,\|\Gamma_{2j}\|^2$$

follow by induction.

Lemma 7. *Let $\mathbf{U} \in \Sigma_1$ be such that U is bounded in $\mathscr{H}$ and V is of Hilbert-Schmidt class. Then $\Omega(\mathbf{U})$ is a multiple of an isometry of $\mathfrak{S}$ into $\mathfrak{H}$. The unique extension of $\Omega(\mathbf{U})$ to a continuous operator from $\mathfrak{H}$ into $\mathfrak{H}$ is also denoted $\Omega(\mathbf{U})$.*

Proof. Let $\varphi \in \mathscr{S}$, then it follows from Lemma 6 that

$$\begin{aligned}|||\Omega(\mathbf{U})\,a^{*n\otimes}(\varphi^{n\otimes})\,\Psi_0|||^2 &= |||a_{\mathrm{U}}^*(\varphi)^n\,\Gamma|||^2 \leqq \\ &\leqq \frac{(2n)!}{n!}\|\varphi\|^{2n} c^{\varepsilon-n}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{n-1+k-j}{n-1} c^{2(k-j)}\|\Gamma_{2j}\|^2 \\ &= \frac{(2n)!}{n!}\|\varphi\|^{2n} c^{\varepsilon-n}\sum_{n=0}^{\infty}\binom{-n}{m}(-c^2)^m\,|||\Gamma(\mathbf{U})|||^2 \\ &= \frac{(2n)!}{n!}\|\varphi\|^{2n} c^{\varepsilon-n}\,|||\Gamma(\mathbf{U})|||^2(1-c^2)^{-n} < \infty\,,\end{aligned}$$

and hence that $\Omega(\mathbf{U})\,a^{*n\otimes}(\varphi^{n\otimes})\,\Psi_0 \in \mathfrak{H}$. It then follows from II, Lemma 1 that $\Omega(\mathbf{U})$ maps $\mathfrak{S}'$ into $\mathfrak{H}$, and in order to prove the assertion concerning the isometry on $\mathfrak{S}$ it is sufficient to prove it on $\mathfrak{S}'$. Since a_{U} and a_{U}^* satisfy the canonical commutation relations, the result follows by the argument used in the proof of I, Theorem 3.14 provided we show that $a_{\mathrm{U}}(\bar\varphi)$ and $a_{\mathrm{U}}^*(\varphi)$ are adjoint on $\Omega(\mathbf{U})\,\mathfrak{S}'$.

In order to prove this, define

$$P_k = \sum_{j=0}^{k} p_j$$

and consider the difference

$$\begin{aligned}\langle\!\langle P_k\,a_{\mathrm{U}}^*(\omega)\,a_{\mathrm{U}}^*(\varphi)^n\,\Gamma,\ a_{\mathrm{U}}^*(\psi)^m\,\Gamma\rangle\!\rangle - \langle\!\langle P_{k-1}\,a_{\mathrm{U}}^*(\varphi)^n\,\Gamma,\ a_{\mathrm{U}}(\bar\omega)\,a_{\mathrm{U}}^*(\psi)^m\,\Gamma\rangle\!\rangle \\ = \langle\!\langle (p_{k+1}+p_k)\,a_{\mathrm{U}}^*(\varphi)^n\,\Gamma,\ a^*(V^{\mathsf{T}}\bar\omega)\,a_{\mathrm{U}}^*(\psi)^m\,\Gamma\rangle\!\rangle\,. \qquad (5.3)\end{aligned}$$

From the first part of the proof follows that the left-side of (5.3) converges for $k \to \infty$ towards the limit

$$\langle\!\langle a_{\mathrm{U}}^*(\omega)\,a_{\mathrm{U}}^*(\varphi)^n\,\Gamma,\ a_{\mathrm{U}}^*(\psi)^m\,\Gamma\rangle\!\rangle - \langle\!\langle a_{\mathrm{U}}^*(\varphi)^n\,\Gamma,\ a_{\mathrm{U}}(\bar\omega)\,a_{\mathrm{U}}^*(\psi)^m\,\Gamma\rangle\!\rangle$$

so that the result follows if we prove that the right-side tends to 0.

Now, if $n - m$ is even, it is identically 0, and if $n - m$ is odd, we have, with $\varepsilon = \varepsilon(m)$:

$$\begin{aligned}
c_k(\omega) &= |\langle\langle p_{2k+\varepsilon+1}\, a^*_{\mathbf{U}}(\varphi)^n\, \Gamma, a^*(\omega)\, a^*_{\mathbf{U}}(\psi)^m\, \Gamma\rangle\rangle| \\
&= |\langle\langle p_{2k+\varepsilon}(a_{\mathbf{U}}(U^{\mathsf{T}-1}\,\bar\omega) + a^*(\gamma\,\bar\omega))\, a^*_{\mathbf{U}}(\varphi)^n\, \Gamma, a^*_{\mathbf{U}}(\psi)^m\, \Gamma\rangle\rangle| \leqq \\
&\leqq |n\,\langle U^{*-1}\,\omega, \varphi\rangle \langle\langle p_{2k+\varepsilon}\, a^*_{\mathbf{U}}(\varphi)^{n-1}\, \Gamma, a^*_{\mathbf{U}}(\psi)^m\, \Gamma\rangle\rangle| + \\
&+ |\langle\langle p_{2(k-1)+\varepsilon+1}\, a^*_{\mathbf{U}}(\varphi)^n\, \Gamma, (a_{\mathbf{U}}(U^{\mathsf{T}-1}\,\bar\gamma\,\omega) + a^*(\gamma\,\bar\gamma\,\omega))\, a^*_{\mathbf{U}}(\psi)^m\, \Gamma\rangle\rangle| \leqq \\
&\leqq n\,\|\omega\|\,\|\varphi\|\,|||p_{2k+\varepsilon}\, a^*_{\mathbf{U}}(\varphi)^{n-1}\, \Gamma|||\,|||p_{2k+\varepsilon}\, a^*_{\mathbf{U}}(\psi)^m\, \Gamma||| + \\
&+ m\,c\,\|\omega\|\,\|\psi\|\,|||p_{2k-\varepsilon(n)}\, a^*_{\mathbf{U}}(\varphi)^n\, \Gamma|||\,|||p_{2k-\varepsilon(m-1)}\, a^*_{\mathbf{U}}(\psi)^{m-1}\, \Gamma||| + \\
&+ |\langle p_{2(k-1)+\varepsilon+1}\, a^*_{\mathbf{U}}(\varphi)^n\, \Gamma, a^*(\gamma\,\bar\gamma\,\omega)\, a^*_{\mathbf{U}}(\psi)^m\, \Gamma\rangle\rangle| \leqq \\
&\leqq A_p\,\|\omega\|\,\|\varphi\|^n\,\|\psi\|^m \sum_{j=0}^{k} \binom{p-1+k-j}{p-1} c^{2(k-j)}\,\|\Gamma_{2j}\|^2 + c_{k-1}(\gamma\,\bar\gamma\,\omega)\,,
\end{aligned}$$

where $p = \max(m, n)$, and A_p depends only upon p and c. In the last step (5.2) has been used. It follows by induction that

$$\begin{aligned}
&|\langle\langle p_{2k+\varepsilon+1}\, a^*_{\mathbf{U}}(\varphi)^n\, \Gamma, a^*(\omega)\, a^*_{\mathbf{U}}(\psi)^m\, \Gamma\rangle\rangle| \leqq \\
&\leqq A_p\,\|\omega\|\,\|\varphi\|^n\,\|\psi\|^m \sum_{j=0}^{k} \binom{p+k-j}{k-j} c^{2(k-j)}\,\|\Gamma_{2j}\|^2\,.
\end{aligned} \tag{5.4}$$

Since the k'th term in a convergent series tends to 0, the result follows by the argument in the beginning of this proof.

Lemma 8. *Let $\mathbf{U} \in \Sigma_1$ be such that U is bounded and V is of Hilbert-Schmidt class in $\mathfrak{X}$. Let $\Omega_1(\mathbf{U})$ denote the isometry of $\mathfrak{H}$ into $\mathfrak{H}$ obtained by normalizing $\Omega(\mathbf{U})$. Then*

$$\Omega_1(\mathbf{U})\,\Omega_1(\mathbf{U}^{-1}) = \Omega_1(\mathbf{U}^{-1})\,\Omega_1(\mathbf{U}) = 1\,,$$

and hence $\Omega_1(\mathbf{U})$ is unitary.

Proof. It follows from Corollary 1 (obviously this holds for $\mathbf{U} \in \Sigma_1$ as well) that $\Omega(\mathbf{U}^{-1})$ and $\Omega(\mathbf{U})$ are each other's duals as operators from $\tilde{\mathfrak{S}}$ into $\tilde{\mathfrak{S}}^*$, and the equation (3.15) then shows that

$$\begin{aligned}
\Omega(\mathbf{U}^{-1})\, a^*_{\mathbf{U}}(\varphi) &= a^*(\varphi)\,\Omega(\mathbf{U}^{-1}) \\
\Omega(\mathbf{U}^{-1})\, a_{\mathbf{U}}(\varphi) &= a(\varphi)\,\Omega(\mathbf{U}^{-1})
\end{aligned} \tag{5.5}$$

in $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$ for all $\varphi \in \mathscr{S}$. Since both sides of (5.5) are continuous from $\tilde{\mathfrak{H}}$ into $\tilde{\mathfrak{H}}^*$, the relation (5.5) holds in $L(\tilde{\mathfrak{H}}, \tilde{\mathfrak{H}}^*)$ as well. Consequently we have

$$\begin{aligned}
&a^*(\varphi)\,\Omega(\mathbf{U}^{-1})\,P_k\,\Omega(\mathbf{U})\,\Psi = \Omega(\mathbf{U}^{-1})\,a^*_{\mathbf{U}}(\varphi)\,P_k\,\Omega(\mathbf{U})\,\Psi = \\
&= \Omega(\mathbf{U}^{-1})\,P_{k-1}\,\Omega(\mathbf{U})\,a^*(\varphi)\,\Psi + \Omega(\mathbf{U}^{-1})\,(p_{k+1} + p_k)\,a^*(U^*\,\varphi)\,\Omega(\mathbf{U})\,\Psi
\end{aligned}$$

for all $\varphi \in \mathscr{S}$, all $\Psi \in \tilde{\mathfrak{S}}'$, and all $k = 1, 2, \ldots$.

Since $\Omega(\mathbf{U})\,\Psi \in \mathfrak{H}$ and $\Omega(\mathbf{U})\,a^*(\varphi)\,\Psi \in \mathfrak{H}$, we have in $\tilde{\mathfrak{H}}^*$:

$$a^*(\varphi)\,\Omega(\mathbf{U}^{-1})\,\Omega(\mathbf{U})\,\Psi = \lim_{k\to\infty} a^*(\varphi)\,\Omega(\mathbf{U}^{-1})\,P_k\,\Omega(\mathbf{U})\,\Psi\,,$$

and

$$\Omega(\mathbf{U}^{-1})\,\Omega(\mathbf{U})\,a^*(\varphi)\,\Psi = \lim_{k\to\infty} \Omega(\mathbf{U}^{-1})\,P_{k-1}\,\Omega(\mathbf{U})\,a^*(\varphi)\,\Psi\,.$$

It follows from (5.5) that

$$\Omega(\mathbf{U}^{-1})\,(p_{k+1}+p_k)\,a^*(U^*\varphi)\,\Omega(\mathbf{U})\,\Psi$$
$$=\Omega(\mathbf{U}^{-1})\,(p_{k+1}+p_k)\,\Omega(\mathbf{U})\,a^*_{\mathbf{U}^{-1}}(U^*\varphi)\,\Psi\,,$$

and since $\Omega(\boldsymbol{U})\,a^*_{\mathbf{U}^{-1}}(U^*\varphi)\,\Psi \in \mathfrak{H}$, this term tends to 0 for $k\to\infty$, and consequently

$$a^*(\varphi)\,\Omega(\mathbf{U}^{-1})\,\Omega(\mathbf{U}) = \Omega(\mathbf{U}^{-1})\,\Omega(\mathbf{U})\,a^*(\varphi)$$

in $L(\tilde{\mathfrak{H}}, \tilde{\mathfrak{H}}^*)$. Similarly one proves that

$$a(\varphi)\,\Omega(\mathbf{U}^{-1})\,\Omega(\mathbf{U}) = \Omega(\mathbf{U}^{-1})\,\Omega(\mathbf{U})\,a(\varphi)$$

for all $\varphi \in \mathscr{S}$. Hence $\Omega(\mathbf{U}^{-1})\,\Omega(\mathbf{U})$ is an intertwining operator associated with the identity in Σ, and thus $\Omega(\mathbf{U}^{-1})\,\Omega(\mathbf{U}) = c\,I$ for some $c \in \mathbf{C}$.

Since $\Omega(\mathbf{U}^{-1}) = \Omega(\mathbf{U})^*$, c must be positive, and the result follows.

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