Commun. math. Phys. 5, 131—156 (1967)

Analytic Continuation of Group Representations. IV

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Received November 25, 1966

Abstract. The main problem, deforming a subalgebra of a Lie algebra, is treated algebraically, requiring an extensive development of methods of defining multiplications on Lie algebra cohomology cochains. Some applications to differential geometry are also presented.

I. Introduction

As we have already seen [2], one of our main problems can be described in the following way: Suppose $\mathbf{G}$ and $\mathbf{L}$ are Lie algebras, with Φ a homomorphism of $\mathbf{G}$ to $\mathbf{L}$. It is possible to "deform" these structures in the sense of defining:

a. A Lie algebra structure $[X, Y]_\lambda$ on $\mathbf{G}$ varying with the parameter λ, reducing to the given Lie algebra when $\lambda = 0$.

b. A one-parameter family of linear maps $\Phi_\lambda : \mathbf{G} \to \mathbf{L}$, each of which is a homomorphism of the $[\ ,\ ]_\lambda$ Lie algebra structure on $\mathbf{G}$.

The INONU-WIGNER idea of "contraction" of Lie algebras and representations and the Gell-Mann method of "expanding" representations both suggest that this is the fundamental problem.

In this paper, we shall develop the full algebraic formalism necessary to discuss this deformation problem. As can be seen from Ref. [4], this necessitates studying the "multiplicative" structure on the chochains associated with Lie algebra cohomology. We have delayed presenting this theory because of its complexity, but in this paper we can present a relatively simple independent exposition, and show how it is applied to the interesting deformation problems in a straightforward way. There is considerable overlap in results with work done by A. NIJENHUIS and R. RICHARDSON [5, 6, 9, 10]. However, the methods presented here are perhaps better adapted to the explicit calculations that are necessary to apply the theory to interesting problems of group representations and differential geometry.

It is extremely interesting to notice that our problem (deforming Lie algebras and their representations) and that of K. KODAIRA and D. C. SPENCER [3, 11] on deformation of differential geometric structures

* Work supported by the U.S. Atomic Energy Commission.

are basically the same. We will present some work designed to show this connection, without getting involved with the full details of this.

I would like to thank P. GRIFFITHS and R. RICHARDSON with whom I have had many conversations about the material presented here on deformations.

II. The Multiplicative Structure on Cochains

Let $\mathfrak{G}$ be a Lie algebra. Suppose V_1, V_2 and V_3 are vector spaces, and that Φ_1, Φ_2, Φ_3 are representations of $\mathfrak{G}$ by linear transformations of V_1, V_2, V_3, respectively. Suppose also that $\alpha: V_1 \times V_2 \to V_3$ is a bilinear map, commuting with the action of $\mathfrak{G}$, i. e.,

$$\Phi_3(X)\,(\alpha(v_1, v_2)) = \alpha(\Phi_1(X)\,v_1, v_2) + \alpha(v_1, \Phi_2(X)\,v_2) \quad \text{for} \quad X \in \mathfrak{G}, v_1 \in V_1, v_2 \in V_2 .$$

Let $C^r(\Phi_i)$, $i = 1, 2, 3$, $r = 0, 1, \ldots,$ be the r-cochains of $\mathfrak{G}$ with coefficients in these three representations. (At this point, we will need the notations and concepts of Lie algebra cohomology theory as presented in [2] Part II.) Our aim is to show that α induces a bilinear map, which we also denote by α, of $C^r(\Phi_1) \times C^s(\Phi_2) \to C^{r+s}(\Phi_3)$, for each pair (r, s) of non-negative integers.

Now, for $r = s = 0$, $C^r(\Phi_1) = V_1$, $C^s(\Phi_2) = V_2$, $C^{r+s}(\Phi_3) = V_3$. We require in this case that α be the same as the map we are given. We will now proceed by induction on $(r + s)$, assuming that α is defined on $C^{r'} \times C^{s'}$, for $r' + s' < r + s$, and show that it can be well-defined on $C^r \times C^s$. For this purpose, we "postulate" the following law connecting the multiplication and the operation of contraction by an element of $\mathfrak{G}$.

$$X \,\lrcorner\, \alpha(\omega_1, \omega_2) = \alpha(X \,\lrcorner\, \omega_1, \omega_2) + (-1)^r\, \alpha(\omega_1, X \,\lrcorner\, \omega_2) \quad \text{for} \quad X \in \mathfrak{G}, \omega_1 \in C^r(\Phi_1), \omega_2 \in C^s(\Phi_2) . \tag{2.1}$$

Following a pattern established earlier, this rule enables us to *define* $\alpha(\omega_1, \omega_2)$ by induction on $r + s$.

$$\begin{aligned} \alpha(\omega_1, \omega_2)\,(X_1, \ldots, X_{r+s}) &= X_1 \,\lrcorner\, \alpha(\omega_1, \omega_2)\,(X_2, \ldots, X_{r+s}) \\ &= \alpha(X_1 \,\lrcorner\, \omega_1, \omega_2)\,(X_2, \ldots, X_{r+s}) + \\ &+ (-1)^r\, \alpha(\omega_1, X_1 \,\lrcorner\, \omega_2)\,(X_2, \ldots, X_{r+s}) \quad \text{for} \quad X_1, \ldots, X_{r+s} \in \mathfrak{G} . \end{aligned}$$

We must show that $\alpha(\omega_1, \omega_2)$ actually is a cochain, i. e., depends skew-symmetrically on $X_1, \ldots, X_{r+s}$. The above formula (and the induction hypothesis) makes it obvious that it depends skew-symmetrically on $X_3, \ldots, X_{r+s}$. We must consider interchange of X_1 and X_2.

$$\begin{aligned} \alpha(\omega_1, \omega_2)\,(X_1, \ldots, X_{r+s}) &= X_2 \,\lrcorner\, \alpha(X_1 \,\lrcorner\, \omega_1, \omega_2) + \\ &+ (-1)^r\, X_2 \,\lrcorner\, \alpha(\omega_1, X_1 \,\lrcorner\, \omega_2)\,(X_3, \ldots, X_{r+s}) \\ &= \alpha(X_2 \,\lrcorner\, X_1 \,\lrcorner\, \omega_1, \omega_2) + (-1)^{r-1}\, \alpha(X_1 \,\lrcorner\, \omega_1, X_2 \,\lrcorner\, \omega_2) + \\ &+ (-1)^r\, \alpha(X_2 \,\lrcorner\, \omega_1, X_1 \,\lrcorner\, \omega_2) + (-1)^{2r}\, \alpha(\omega_1, X_2 \,\lrcorner\, X_1 \,\lrcorner\, \omega_2) \\ &\quad (X_3, \ldots, X_{r+s}) . \end{aligned}$$

This makes it evident that the dependence on X_1 and X_2 is also skew-symmetric.

Having *defined* $\alpha(\omega_1, \omega_2)$ so that Eq. (2.1) is satisfied, we must now investigate by the same sort of inductive reasoning how the other algebraic operations we have defined on cochains are compatible with this product. First, consider the Lie derivative:

$$X\,\alpha(\omega_1, \omega_2) = \alpha(X(\omega_1), \omega_2) + \alpha(\omega_1, X(\omega_2)) \quad \text{for} \\ \omega_1 \in C^r(\Phi_1),\ \omega_2 \in C^s(\Phi_2),\ X \in \mathrm{G}\,. \tag{2.2}$$

Proof. Let $Y \in \mathrm{G}$. We must show that $Y \rfloor$ applied to both sides of Eq. (2.2) gives the same result, if it is assumed that Eq. (2.2) is true for cochain of lower degree. (Notice that for $r = s = 0$, it just expresses the fact that α commutes with the action of G.)

$$\begin{aligned} Y \rfloor X\,\alpha(\omega_1, \omega_2) &= X(Y \rfloor \alpha(\omega_1, \omega_2)) - [X, Y] \rfloor \alpha(\omega_1, \omega_2) \\ &= X(\alpha(Y \rfloor \omega_1, \omega_2) + \\ &+ (-1)^r \alpha(\omega_1, Y \rfloor \omega_2)) - \alpha([X, Y] \rfloor \omega_1, \omega_2) - (-1)^r \alpha(\omega_1, [X, Y] \rfloor \omega_2) \\ &= \alpha([X, Y] \rfloor \omega_1, \omega_2) + \alpha(Y \rfloor X(\omega_1), \omega_2) + \alpha(Y \rfloor \omega_1, X(\omega_2)) + \\ &+ (-1)^r \alpha(X(\omega_1), Y \rfloor \omega_2) + (-1)^r \alpha(\omega_1, [X, Y] \rfloor \omega_2) + \\ &+ (-1)^r \alpha(\omega_1, Y \rfloor X(\omega_2)) - \alpha([X, Y] \rfloor \omega_1, \omega_2) - (-1)^r \alpha(\omega_1, \\ &[X, Y] \rfloor \omega_2) \\ &= Y \rfloor [\alpha(X(\omega_1), \omega_2) + \alpha(\omega_1, X(\omega_2))] \end{aligned}$$

q.e.d.

Now, turn to the following formula:

$$d\,\alpha(\omega_1, \omega_2) = \alpha(d\omega_1, \omega_2) + (-1)^r \alpha(\omega_1, d\omega_2) \quad \text{for} \\ \omega_1 \in C^r(\Phi_1),\ \omega_2 \in C^r(\Phi_2)\,. \tag{2.3}$$

Proof. For $X \in \mathrm{G}$.

$$X \rfloor d\,\alpha(\omega_1, \omega_2) = X(\alpha(\omega_1, \omega_2)) - d(X \rfloor \alpha(\omega_1, \omega_2)) = ,$$

using that Eq. (2.3) is true for forms of total degree less than $r + s$,

$$\begin{aligned} &\alpha(X(\omega_1), \omega_2) + \alpha(\omega_1, X(\omega_2)) - d(\alpha(X \rfloor \omega_1, \omega_2) + (-1)^r \alpha(\omega_1, X \rfloor \omega_2)) \\ &= \alpha(X(\omega_1), \omega_2) + \alpha(\omega_1, X(\omega_2)) - \alpha(d(X \rfloor \omega_1), \omega_2) - \\ &- (-1)^{r-1} \alpha(X \rfloor \omega_1, d\omega_2) - (-1)^r\big(\alpha(d\omega_1, X \rfloor \omega_2) - \alpha(\omega_1, d(X \rfloor \omega_2))\big) \\ &= \alpha(X \rfloor d\omega_1, \omega_2) + \alpha(\omega_1, X \rfloor d\omega_2) + (-1)^r \alpha(X \rfloor \omega_1, d\omega_2) - \\ &- (-1)^r \alpha(d\omega_1, X \rfloor \omega_2) = X \rfloor \big(\alpha(d\omega_1, \omega_2) + (-1)^r \alpha(\omega_1, d\omega_2)\big)\,. \end{aligned}$$

This proves Eq. 2.3.

Equation 2.3 indicates that the map α on cochains induces a bilinear map (that we also denote by α) on cohomology classes, $\alpha: H^r(\Phi_1) \times H^s(\Phi_2) \to H^{r+s}(\Phi_3)$. Suppose that ω_1 and ω_2 are cocycles belonging to given cohomology classes $\bar{\omega}_1$ and $\bar{\omega}_2$. Put $\alpha(\bar{\omega}_1, \bar{\omega}_2) = \overline{\alpha(\omega_1, \omega_2)}$. We must verify several facts to make this definition legitimate:

$d(\alpha(\omega_1, \omega_2)) = 0$, i. e. ,$\alpha(\omega_1, \omega_2)$ is a cycle. [This follows from Eq. (2.3.)]

If ω_1 and ω_2 are replaced by ω_1' and ω_2' in the same cohomology class, then

$$\overline{\alpha(\omega_1, \omega_2)} = \overline{\alpha(\omega_1', \omega_2')} \tag{2.4}$$

For the proof, notice that:

$\alpha(d\theta_1, \omega_2) = d\alpha(\theta_1, \omega_2) - (-1)^{r-1}\alpha(\theta_1 d\omega_2) = d\alpha(\theta_1, \omega_2)$, i.e., $\overline{\alpha(d\theta_1, \omega_2)} = 0$

Thus,

$\alpha(\omega_1, \omega_2) = \alpha(\omega_1 - \omega_1', \omega_2 - \omega_2') + \alpha(\omega_1', \omega_2) + \alpha(\omega_1, \omega_2) - \alpha(\omega_1', \omega_2',)$

hence 2.4.

Finally, let us suppose that $V_1 = V_2 = V$ that $\Phi_1 = \Phi_2 = \Phi$, and that $\alpha(v_2, v_2) = c\,\alpha(v_1, v_1)$ for $v_1, v_2 \in V$, where c is constant independent of v_1 and v_2.

Then we have:

$$\alpha(\omega_1, \omega_2) = (-1)^{rs} c\,\alpha(\omega_2, \omega_1) \quad \text{for} \quad \omega_1 \in c^r(\Phi),\ \omega_2 \in c^s(\Phi)\,. \tag{2.5}$$

Proof. Again, by induction on $(r + s)$.

$$\begin{aligned} X \rfloor \alpha(\omega_1, \omega_2) &= \alpha(X \rfloor \omega_1, \omega_2) + (-1)^r \alpha(\omega_1, X \rfloor \omega_2) \\ &= (-1)^{(r-1)s} c\,\alpha(\omega_2, X\,\Gamma\,\omega_1) + \\ + (-1)^{r+r(s-1)} c\,\alpha(X \rfloor \omega_2, \omega_1) &= (-1)^{rs} c\big((-1)^s \alpha(\omega_2, X \rfloor \omega_1) + \\ &+ \alpha(X \rfloor \omega_2, \omega_1)\big) = (-1)^{rs} c\,X \rfloor \alpha(\omega_2, \omega_1) \end{aligned}$$

q.e.d.

The general problem inherent in Eq. (2.5) is that of determining how algebraic relations among the representations of G used to define cochains induce algebraic relations among the cochains themselves. Let us turn to another example of this that is of interest in the application of Lie algebra cohomology to deformation problems, namely that related to "associative" laws.

Suppose we are given five vector spaces $V_1, \ldots, V_5$ with representations $\Phi_1, \ldots, \Phi_5$ of G on each.

Consider bilinear maps

$$\alpha : V_1 \times V_2 \to V_3\,; \quad \beta : V_3 \times V_4 \to V_5\,.$$

Form $\beta(\alpha(v_1, v_2), v_4)$, $(v_1, \ldots,$ are typical elements of $V_1, \ldots, V_5)$. Similarly, form $\beta'(v_1, \alpha'(v_2, v_4))$ and $\beta''(v_2, \alpha''(v_1, v_4))$, where the following bilinear maps have the following domains and ranges:

$$\alpha' : V_2 \times V_4 \to V_3\,; \quad \beta' : V_1 \times V_3 \to V_5$$
$$\alpha'' : V_1 \times V_4 \to V_3\,; \quad \beta'' : V_2 \times V_3 \to V_5\,.$$

Suppose now that all these bilinear maps commute with the action of G, and that the following relations hold:

$$\beta(\alpha(v_1, v_2), v_4) = a\,\beta'(v_1, \alpha'(v_2, v_4)) + b\,\beta''(v_2, \alpha''(v_1, v_4)) \qquad (2.6)$$

where a and b are scalar constants.

Our problem is to find if there is a relation similar to Eq. (2.6) among cochains. Suppose then that

$$\omega_1 \in C^r(\Phi_1),\ \omega_2 \in C^s(\Phi_2),\ \omega_4 \in C^t(\Phi_4)\,.$$

Based on our acquired experience with this sort of thing, let us try to prove the following law as the extension of Eq. (2.6) to cochains:

$$\beta(\alpha(\omega_1,\omega_2), \omega_4) = a\,\beta'(\omega_1, \alpha'(\omega_2, \omega_4)) + (-1)^{rs} b\,\beta''(\omega_2, \alpha''(\omega_1, \omega_4)) \qquad (2.7)$$

As before, we apply $X \rfloor$ to both sides of Eq. 2.7, where $X \in \mathrm{G}$, with Eq. (2.6) starting off the induction.

$$\begin{aligned}
X \rfloor \beta(\alpha(\omega_1, \omega_2), \omega_4) &= \beta(X \rfloor \alpha(\omega_1, \omega_2), \omega_4) + \\
&\quad + (-1)^{r+s}\,\beta(\alpha(\omega_1, \omega_2), X \rfloor \omega_4)) \\
&= \beta(\alpha(X \rfloor \omega_1, \omega_2), \omega_4) + (-1)^r\,\beta(\alpha(\omega_1, \times \\
&\quad \times X \rfloor \omega_2), \omega_4) + \\
&\quad + (-1)^{r+s}\,\beta(\alpha(\omega_1, \omega_2), X \rfloor \omega_4) \\
&=, \text{ using Eq. (2.7)}\,,
\end{aligned}$$

$$\begin{gathered}
a\,\beta'(X \rfloor \omega_1, \alpha'(\omega_2, \omega_4)) + (-1)^{(r-1)s} b\,\beta''(\omega_2, \alpha''(X \rfloor \omega_1, \omega_4)) + \\
+ (-1)^r [a\,\beta'(\omega_1, \alpha'(X \rfloor \omega_2, \omega_4)) + (-1)^{r(s-1)} b\,\beta''(X \rfloor \omega_2, \times \\
\times \alpha''(\omega_1, \omega_4))] + \\
+ (-1)^{r+s} [a\,\beta'(\omega_1, \alpha'(\omega_2, X \rfloor \omega_4)) + (-1)^{rs} b\,\beta''(\omega_2, \alpha''(\omega_1, X] \rfloor \omega_4))].
\end{gathered}$$

Now

$$\begin{aligned}
X \rfloor \beta'(\omega_1, \alpha'(\omega_2, \omega_4)) &= \beta'(X \rfloor \omega_1, \alpha'(\omega_2, \omega_4)) + \\
&\quad + (-1)^r\,\beta'(\omega_1, X \rfloor \alpha'(\omega_2, \omega_4) \\
&= \beta'(X \rfloor \omega_1, \alpha'(\omega_2, \omega_4)) + (-1)^r\,[\beta'(\omega_1, \alpha'(X \rfloor \omega_2, \omega_4)) + \\
&\quad + (-1)^s\,\beta'(\omega_1, \alpha'(\omega_2, X \rfloor \omega_4))]
\end{aligned}$$

$$\begin{aligned}
X \rfloor \beta''&(\omega_2, \alpha''(\omega_1, \omega_4)) \\
&= \beta''(X \rfloor \omega_1, \alpha''(\omega_2, \omega_4)) + (-1)^r\,\beta''(\omega_1, \alpha''(X \rfloor \omega_2, \omega_4)) \\
&= \beta''(X \rfloor \omega_2, \alpha''(\omega_1, \omega_4)) + (-1)^s [\beta''(\omega_2, \alpha''(X \rfloor \omega_1, \omega_4)) + \\
&\quad + (-1)^r\,\beta''(\omega_2, \alpha''(\omega_1, X \rfloor \omega_4))]\,.
\end{aligned}$$

These identities prove Eq. (2.7).

III. Deformation of Lie Algebra Homomorphisms

We will now consider what is perhaps the simplest situation where one can see how the "multiplicative" structure of the cochains is related to deformation problems. Suppose that $\mathbf{G}$ and $\mathbf{L}$ are Lie algebras, and that $\lambda \to \Phi_\lambda$ is a one parameter family of Lie algebra homomorphisms from $\mathbf{G}$ to $\mathbf{L}$. Suppose that:

$$\Phi_\lambda(X) = \sum_{j=0}^{\infty} \omega_j(X)\, \lambda^j$$

Let Φ' be the following representation of $\mathbf{G}$ by linear transformations on $\mathbf{L}$:

$$\Phi'(X)\,(A) = [\Phi_0(X), A] \quad \text{for} \quad X \in \mathbf{G}, A \in \mathbf{L}\,.$$

Then, each $\omega_j(X)$, for $j \geqq 1$, defines a 1-cochain of $\mathbf{G}$ with coefficients defined by the representation Φ'. Let us examine the conditions determined by requiring that each Φ_λ be a Lie algebra homomorphism.

$$\Phi_\lambda([X, Y]) = [\Phi_\lambda(X), \Phi_\lambda(Y)]\,, \quad \text{for} \quad X, Y \in \mathbf{G}\,, \quad \text{or}$$

$$\sum_j \omega_j([X, Y])\, \lambda^j = \sum_{i,j} [\omega_i(X), \omega_j(Y)]\, \lambda^{i+j}\,, \quad \text{or}$$

$$\omega_j([X, Y]) = \sum_{k=0}^{j} [\omega_{k-j}(X), \omega_j(Y)] = [\Phi_0(X), \omega_k(Y)] + \tag{3.1}$$

$$+ [\omega_k(X), \Phi_0(Y)] + \sum_{k=1}^{j-1} [\omega_{k-j}(X), \omega_j(Y)]\,.$$

Now

$$\begin{aligned} d\omega_j(X, Y) &= \Phi'(X)\,(\omega_j(Y)) - \Phi'(Y)\,(\omega_j(X)) - \omega_j([X, Y]) \\ &= [\Phi_0(X), \omega_j(Y)] - [\Phi_0(Y), \omega_j(X)] - \omega_j([X, Y])\,. \end{aligned}$$

Hence, Eq. (3.1) can be rewritten as:

$$d\omega_j(X, Y) + \sum_{k=1}^{j-1} [\omega_{k-j}(X), \omega_j(Y)] = 0\,. \tag{3.2}$$

Now let α be the map: $\mathbf{L} \times \mathbf{L} \to \mathbf{L}$ defined by: $\alpha(A, B) = [A, B]$. Then,

$$\begin{aligned} \alpha(\omega_{k-j}, \omega_j)\,(X, Y) &= X \rfloor \alpha(\omega_{k-j}, \omega_j)\,(Y) \\ &= \alpha(\omega_{k-j}(X), \omega_j) - \alpha(\omega_{k-j}, \omega_j(X))\,(Y) \\ &= \alpha(\omega_{k-j}(X), \omega_j(Y)) - \alpha(\omega_{k-j}(Y), \omega_j(X)) \\ &= [\omega_{k-j}(X), \omega_j(Y)] - [\omega_{k-j}(Y), \omega_j(X)]\,. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j=1}^{k-1} \alpha(\omega_{k-j}, \omega_j)\,(X, Y) &= ([\omega_{k-j}(X), \omega_j(Y)] + [\omega_j(X), \omega_{k-j}(Y)]) \\ &= 2 \sum [\omega_{k-j}(X), \omega_j(Y)]\,. \end{aligned}$$

Thus, Eq. (3.2) can be rewritten as:

$$d\omega_j + \frac{1}{2} \sum_{k=1}^{j-1} \alpha(\omega_{j-k}, \omega_k) = 0\,. \tag{3.3}$$

It all the data depends analytically on λ, then Eq. (3.3) gives the condition (for $j = 1, 2, \ldots$) that $\lambda \to \Phi_\lambda$ be, for each λ, an homomorphism.

Notice that Eq. 3.3, for $j = 1$, gives the condition we already know, namely $d\omega_1 = 0$. As we have seen [2], the cohomology class determined by ω_1 in $H^1(\Phi)$ is the first "obstruction" to showing that each member of family $\lambda \to \Phi_\lambda$ of homomorphisms is equivalent under the group of inner automorphisms of $\mathbf{L}$ to Φ_0 itself. In fact, the known theorem [5] is that, if $\mathbf{G}$ and $\mathbf{L}$ are finite dimensional, if $H^1(\Phi') = 0$, then Φ_λ is equivalent to Φ_0 for λ sufficiently small. The proof involves as its basic tool the implicit function theorem. It is sometimes possible to extend it to certain infinite dimensional situations by a judicious use of the implicit function theorem in Banach spaces. However, the situation of main interest to us, where $\mathbf{L}$ is the space of skew-Hermitian operators on a Hilbert space, does not seem readily amenable to such techniques. We will then present the primitive, but more explicit technique for carrying out such calculations.

Let L be the group of inner automorphisms of $\mathbf{L}$. L is the group of transformations of $\mathbf{L}$ generated by those of the form:

$$\begin{aligned}\operatorname{Exp}(\operatorname{Ad} A) : B \to \operatorname{Exp}(\operatorname{Ad} A)\,(B) \\ = \sum_{j=0}^{\infty} \frac{(\operatorname{Ad} A)^j}{j'}(B) = B + [A, B] + \frac{1}{2}[A, [A, B]] + \cdots\end{aligned}$$

where A and B are elements of $\mathbf{L}$.

We now have:

Theorem 3.1. *Suppose that $H^1(\Phi') = 0$. For any sequence $A_1, A_2, \ldots$ of elements of $\mathbf{L}$, consider the following formulas:*

$$\Phi_\lambda^j(X) = \operatorname{Exp}(A\,d\,\lambda^j A_j) \ldots \operatorname{Exp}(A\,d\,\lambda\, A_1)\,\Phi_\lambda(X),\ \Phi_\lambda^j(X) = \sum_{k=0}^{\infty} \omega_k^j(X)\,\lambda^j\,.$$

Our assertion is that this sequence can be chosen so that

$$\omega_k^j(X) = 0 \quad \textit{for} \quad 0 < k \leqq j\,, \quad \textit{all} \quad Z \in \mathbf{G}\,. \tag{3.4}$$

Proof. Choose A_1 so that $dA_1 = \omega_1$. Then, for $X \in \mathbf{G}$,

$$\begin{aligned}\operatorname{Exp}(\operatorname{Ad} \lambda\, A_1)\,(\Phi_0 + \omega_1\,\lambda + \cdots)\,(X) &= (1 + \lambda \operatorname{Ad} A_1 + \cdots)\,(\Phi_0 + \\ &\quad + \omega_1\,\lambda + \cdots)\,(X) \\ &= \Phi_0(X) + \lambda([A_1, \Phi_0(X)] + \omega_1(X)) + \cdots \\ &= \Phi_0(X) + \lambda([A_1, \Phi_0(X)] + [\Phi_0(X), A_1]) + \cdots \\ &= \Phi_0(X) + (\text{terms involving } \lambda^2, \lambda^3, \ldots)\,.\end{aligned}$$

Proceed by induction on j, assume that: $\Phi_\lambda^j(X) = \Phi_0(X) + {}$ $+ \omega_j(X)\,\lambda^j + \cdots$ By 3.3, $d\omega_j = 0$. Choose A_j so that

$$dA_j = \omega_j\,,$$

again using the fact that $H^2(\Phi') = 0$.

Put $\Phi_\lambda^{j+1}(X) = \operatorname{Ad} \operatorname{Exp}(\lambda^j A_j)(\Phi_j(X))$

$$= (1 + \lambda^j \operatorname{Ad} A_j + \cdots)(\Phi_0(X) + \omega_j(X)\lambda^j + \cdots 1$$
$$= \Phi_0(X) + \lambda^j([A_j, \Phi_0(X)] + dA_j(X)) + \cdots$$
$$= \Phi_0(X) + (\text{terms involving } \lambda^{j+1}, \ldots) \qquad \text{q.e.d.}$$

Theorem 3.1 is purely algebraic, of course. However, put

$$g_j = \operatorname{Exp}(Ad\, \lambda^j A_j) \ldots \operatorname{Exp}(Ad\, \lambda A_1), \text{ an element of } L.$$

If g_j converges as $j \to \infty$, to, say $g(\lambda)$, note that:

$$\operatorname{Ad} g(\lambda)\, \Phi_\lambda(X) = \Phi_0(X),$$

i. e., each of the homomorphisms Φ_λ is equivalent under L to Φ_0, i. e., the deformation $\lambda \to \Phi_\lambda$ of homomorphism as "trivial". The next step in the program should be to consider conditions for the convergence. In this paper, we will pass them by.

Next, we consider a situation where the algebraic structure on the cochains plays a more important role. Suppose $H^1(\Phi')$ is not zero. Given a cohomology class in $H^2(\Phi')$, we inquire whether there actually is a deformation $\lambda \to \Phi_\lambda = \sum_j \omega_j \lambda^j$ of Φ_0 with Φ_1 in that cohomology class. There is a standard answer that this is so if $H^2(\Phi') = 0$. We shall now proceed to consider the formal aspects of this.

As we have seen, it suffices (modulo the convergence problems for the series $\Phi_\lambda = \sum_j \omega_j \lambda^j$, which we will again pass by), to show that a sequence $\omega_1, \omega_2, \ldots$ of chochains satisfying 3.3 exists, starting off with ω_1 given. However, this can easily be done by induction: Assume $\omega_1, \ldots, \omega_j$ exists satisfying 3.3. We shall show that ω_{j+1} exists.

Put:

$$\theta_{j+1} = -\frac{1}{2} \sum_{k=1}^{j} \alpha(\omega_{j+1-k}, \omega_k).$$

We must show that $d\theta_{j+1} = 0$. For then, our assumption that $H^2(\Phi') = 0$ would guarantee that ω_{j+1} could be chosen as the cochain such that $d\omega_{j+1} = \theta_{j+1}$. Now

$$d\theta_{j+1} = -\frac{1}{2} \sum_{k=1}^{j} \alpha(d\omega_{j+1-k}, \omega_k) - \alpha(\omega_{j+1-k}, d\omega_k)$$
$$= \sum_{k=1}^{j} \alpha(d\omega_k, \omega_{j+1-k})$$
$$= -\frac{1}{2} \sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\alpha(\omega_{k-l}, \omega_l), \omega_{j+1-k})$$
$$= \frac{1}{2} \sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\omega_{k-l}, \alpha(\omega_l, \omega_{j+1-k}) + \alpha(\omega_l, \alpha(\omega_{k-l}, \omega_{j+1-k})).$$

Consider

$$\sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\omega_l, \alpha(\omega_{k-l}, \omega_{j+1-k})) . \tag{3.5}$$

Make the following change of variables in the summation:

$$k - l \to l\,, \quad j + 1 - k \to k - l\,.$$

Then, also $l \to j + 1 - k$. The limits of summation remain the same. The sum 3.5 is then:

$$\begin{aligned} &\sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\omega_{j+1-k}, \alpha(\omega_l, \omega_{k-l})) \\ &\qquad = - \sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\alpha(\omega_{k-l}, \omega_l), \omega_{j+1-k}) \\ &\qquad = 2 d\theta_{j+1} . \end{aligned}$$

This gives the identity:

$$d\theta_{j+1} = -\frac{1}{2} \sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\omega_{k-l}, \alpha(\omega_l, \omega_{j+1-k})) - d\theta_{j+1}\,,$$

or

$$d\theta_{j+1} = -\frac{1}{4} \sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\omega_{k-l}, \alpha(\omega_l, \omega_{j+1-k})) . \tag{3.6}$$

However, we can also make a different change of variables in the summation in 3.5: $j + 1 - k \to l$; $k - l \to j + 1 - k$. Then, also $l \to k - l$. The limits again remain the same. 3.5 then becomes

$$\begin{aligned} &\sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\omega_{k-l}, \alpha(\omega_{j+1=k}, \omega_l)) \\ &= - \sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\omega_{k-l}, \alpha(\omega_l, \omega_{j+1-k})) . \end{aligned}$$

Hence,

$$d\theta_{j+1} = - \sum_{k=1}^{j} \sum_{l=1}^{k-1} \alpha(\omega_{k-l}, \alpha(\omega_l, \omega_{j+1-k})) .$$

This is incompatible with 3.6 unless

$$d\theta_{j+1} = 0$$

Summing up, we have then formed:

Theorem 3.2. *Suppose that* $H^2(\Phi') = 0$, *and that* ω_1 *is a* 1*-cocycle.* Then *there exists* 1*-cochains* $\omega_2, \omega_3, \ldots$ *such that the formal power series*

$$\Phi_\lambda(X) = \Phi_0(X) + \sum_{j=1}^{\infty} \omega_j(X)\, \lambda^j$$

satisfies the equations which, if the series converges, implies that each Φ_λ *is a homomorphism from* $\mathfrak{G}$ *to* $\mathbf{L}$.

IV. Algebraic Aspects of the Deformation of Lie Algebras

As we have just seen, the formal properties of deformation of homomorphism to all orders involves a multiplicative algebraic structure on the cochains. We shall now consider the similar, but more complicated, structure involved in deformations of Lie algebras alone. (This has been recently considered by NIJENHUIS and RICHARDSON [6] and S. PIPER [8] but since our work here is from a slightly different point of view, we shall briefly indicate how it can be applied to this case.)

Suppose G is a vector space, with $[X, Y]_\lambda$ a Lie algebra type of product, defined for all values of the parameter λ. Consider its Taylor expansion:

$$[X, Y]_\lambda = \sum_j \omega_j(X, Y)\, \lambda^j .$$

Let Φ be the adjoint representation of the initial Lie algebra $[X, Y]_0 = [X, Y]$, i. e.,

$$\Phi(X)\,(Y) = [X, Y] .$$

Then, each ω_j is a 2-cochain of G with coefficients in the representation Φ. The algebraic conditions imposed on these cochains are determined by the Jacobi identity, which should be true for each value of λ:

$$[X, [Y, Z]_\lambda]_\lambda = [[X, Y]_\lambda, Z]_\lambda + [Y, [X, Z]_\lambda]_\lambda , \quad \text{or}$$

$$\sum_{j,k} \omega_j(X, \omega_k(Y, Z))\, \lambda^{j+k} = \sum_{j,k} \omega_j(\omega_k(X, Y), Z) + \omega_j(Y, \omega_k(X, Z))\, \lambda^{j+k} ,$$

or

$$\sum_{j=0}^{k} \omega_j(X, \omega_{k-j}(Y, Z)) - \omega_j(\omega_{k-j}(X, Y), Z) - \omega_j(Y, \omega_{k-j}(X, Z)) = 0 .$$

The terms for $J = 0$ and $j = k$ just form the six terms of $d\omega_k(X, Y, Z)$, hence we have:

$$d\omega_k(X, Y, Z) + \sum_{j=1}^{k-1} \omega_j(X, \omega_{k-j}(Y, Z)) - \omega_j(\omega_{k-j}(X, Y), Z) - \\ - \omega_j(Y, \omega_{k-j}(X, Z)) = 0 . \qquad (4.1)$$

Now, after the pattern found for the case of deformation of homomorphismu considered earlier, one would expect to find the second term of 4.1 to be a multiplicative operation on cochains. However, at first sight, is has a different form than any we have considered before. We will now show that it does in fact fit into the same pattern as the unified theory of such multiplications given in Section II.

Construct a representation Φ' as the "adjoint representation" of Φ. Explicitly, let V be the vector space of all linear mappings $A : G \to G$.

For $X \in G$,

$$\Phi'(X)\,(A) = [\mathrm{Ad}\, X, A] = \mathrm{Ad}\, X\, A - A\, \mathrm{Ad}\, X . \qquad (4.2)$$

Let α be the bilinear mapping: $V \times \mathfrak{G} \to \mathfrak{G}$ defined by:

$$\alpha(A, X) = A(X) .$$

Then, it is readily seen that α commutes with the action of $\mathfrak{G}$ via Φ and Φ'.

Define a mapping $\beta : C^r(\Phi) \to C^{r-1}(\Phi')$ as follows:

If

$$\omega : (X_1, \ldots, X_r) \to \omega(X_1, \ldots, X_r)$$

is an r-cochain, then:

$$\beta(\omega)(X_1, \ldots, X_{r-1})(X) = \omega(X_1, \ldots, X_{r-1}, X) ,$$

$$\text{i. e. } \beta(\omega)(X_1, \ldots, X_{r-1})$$

is a linear transformation $\mathfrak{G} \to \mathfrak{G}$.

Lemma 4.1. $d\beta(\omega) = \beta(d\omega)$.

Proof. By induction on r. By definition, if $r = 0$, $\beta(\omega) = 0$, $d\beta\,\omega = 0$. Assume it is true for cochains of degree less than r, and let $\omega \in C(\Phi)$.

$$\begin{aligned} Y \lrcorner \beta(\omega)(X_1, \ldots, X_{r-2})(X) &= \omega(Y, X_1, \ldots, X_{r-2}, X) \\ &= (Y \lrcorner \omega)(X_1, \ldots, X_{r-2}, X) \\ &= \beta(Y \lrcorner \omega)(X_1, \ldots, X_{r-2})(X) , \end{aligned}$$

i. e.,

$$Y \lrcorner \beta(\omega) = \beta(Y \lrcorner \omega) .$$

$$\begin{aligned} Y(\beta(\omega))(X_1, \ldots, X_{r-1})(X) &= \Phi'(Y)(\beta(\omega)(X_1, \ldots, X_{r-1}))(X) - \\ &- \beta(\omega)([Y, X_1], \ldots, X_{r-1})(X) - \cdots \\ &= [\Phi(Y), \omega(X_1, \ldots, X_{r-1}, X)] - \omega(X_1, \ldots, X_{r-1}, [Y, X]) - \\ &- \omega([Y, X_1], \ldots, X_{r-1}, X) - \cdots \\ &= Y(\omega)(X_1, \ldots, X_{r-1}, X) = \beta(Y(\omega))(X_1, \ldots, X_{r-1})(X) , \end{aligned}$$

i. e.,

$$Y(\beta(\omega)) = \beta(Y(\omega)) . \tag{4.3}$$

Finally then,

$$\begin{aligned} Y \lrcorner (d\beta(\omega) - \beta(d\omega)) &= Y(\beta(\omega)) - d(Y \lrcorner \beta(\omega)) - \beta\, Y \lrcorner \beta(d\omega) \\ &= \beta(Y(\omega)) - d\beta(Y \lrcorner (\omega)) - \beta(Y \lrcorner d\omega) . \end{aligned}$$

By induction hypothesis,

$$d\beta(Y \lrcorner \omega) = \beta\, d(Y \lrcorner \omega) , \quad \text{hence} \quad Y \lrcorner (d\beta(\omega) - \beta(d\omega)) = 0 .$$

Since this holds for all $Y \in \mathfrak{G}$, Lemma 4.1 is proved.

Suppose now that ω and ω' are elements of $C^2(\Phi)$. Let us compute: $\alpha(\beta(\omega'), \omega)$, which is an element of $C^3(\Phi)$.

$\alpha(\beta(\omega'), \omega)(X, Y, Z) = \big(\alpha(\beta(X \rfloor \omega'), \omega) - \alpha(\beta(\omega'), X \rfloor \omega)\big)(Y, Z)$

$= \big(\alpha(\beta(X \rfloor \omega'), Y \rfloor \omega)\big) - \alpha(Y \rfloor \beta(\omega'), X \rfloor \omega) + \alpha(\beta(\omega'), \omega(X, Y))(Z)$

$= \alpha(\beta(X \rfloor \omega'), \omega(Y, Z)) - \alpha(Y \rfloor \beta(\omega'), \omega(X, Z)) +$

$+ \alpha(Z \rfloor \beta(\omega'), \omega(X, Y))$

$= \beta(X \rfloor \omega')(\omega(Y, Z)) - \beta(Y \rfloor \omega')(\omega(X, Z)) + \beta(Z \rfloor \omega')(\omega(X, Y))$

$= \omega'(X, \omega(Y, Z)) - \omega'(Y, \omega(X, Z)) + \omega'(Z, \omega(X, Y))$.

Then 4.1 can be written as:

$$d\omega_k + \sum_{j=1}^{k-1} \alpha(\beta(\omega_j), \omega_{k-j}) = 0 \quad k = 1, 2, \ldots . \tag{4.4}$$

With this formula in hand, and the rules we have derived for computing $d\alpha(\beta(\omega_j), \omega_{k-j})$, it is now a routine matter to carry through the KODAIRA-SPENCER "deformation program", as sketched, for example, in [4]; it may be considered as an exercise for the reader.

V. Simultaneous Deformations of Lie Algebras and their Homomorphisms

As we have seen [2], the INONU-WIGNER idea of "contraction" of the Lie algebra *together* with its representations suggests the study of the following deformation problem: Let $\mathfrak{G}$ be a vector space, and let $\mathbf{L}$ be a Lie algebra. Consider one-parameter family $(X, Y) \to [X, Y]_\lambda$ of Lie algebra structures on $\mathfrak{G}$, together with a one-parameter family $\Phi_\lambda : \mathfrak{G} \to \mathbf{L}$ of linear transformation, each of which is homomorphism of the λ-th Lie algebra on $\mathfrak{G}$, i. e.,

$$\Phi_\lambda([X, Y]_\lambda) = [\Phi_\lambda(X), \Phi_\lambda(Y)] . \tag{5.1}$$

As before, let us expand in a Taylor's series:

$$[X, Y]_\lambda = \sum_j \omega_j(X, Y)\, \lambda^j ,$$

$$\Phi_\lambda(X) = \sum_k \theta_k(X)\, \lambda^k .$$

The cochains ω_j satisfy 4.4.

5.1 now leads to the equations:

$$\sum_{k,j} \theta_k(\omega_j(X, Y))\, \lambda^{j+k} - [\theta_j(X), \theta_k(X)]\, \lambda^{j+k} = 0 ,$$

or

$$\sum_{j=0}^{k} \theta_j(\omega_{k-j}(X, Y)) - [\theta_j(X), \theta_{k-j}(Y)] = 0 . \tag{5.2}$$

Suppose we relabel

$$[X, Y] = [X, Y]_0 = \omega_0(X, Y)\,,$$
$$\Phi(X) = \Phi_0(X) = \theta_0(X)\,.$$

Following the pattern already established, split off from 5.2 the terms corresponding to $j = 0$ and $j = k$.

$$\Phi(\omega_k(X, Y)) - [\Phi(X), \theta_k(Y)] + \theta_k([X, Y]) - [\theta_k(X), \Phi(Y)] + \\ + \sum_{j=1}^{k-1} \theta_j(\omega_{k-j}(X, Y)) - [\theta_j(X), \theta_{k-j}(Y)] = 0\,. \tag{5.3}$$

Our next task is to interpret the first four terms in 5.3 via Lie algebra cohomology. First, θ_k is an element of $C^1(\Phi')$, where Φ' is the representation defined by:

$$\Phi'(X)\,(A) = [\Phi(X), A] \quad \text{for} \quad A \in \mathbf{L}\,. \tag{5.4}$$

Then,

$$d\theta_k(X, Y) = \Phi'(X)\,(\theta_k(Y)) - \Phi'(Y)\,(\theta_k(X)) - \theta_k([X, Y])$$
$$= [\Phi(X), \theta_k(Y)] - [\Phi(Y), \theta_k(X)] - \theta_k([X, Y])\,.$$

ω_k is a 2-cochain with coefficients in the adjoint representation of $\mathfrak{G}$. Φ is a linear map: $\mathfrak{G} \to \mathbf{A}$ which commutes with the action of $\mathfrak{G}$ via the adjoint representation on $\mathfrak{G}$ and via Φ' on $\mathbf{A}$, i. e.,

$$\Phi'(X)\,(\Phi(Y)) = [\Phi(X), \Phi(Y)]$$
$$= \Phi([X, Y])$$
$$= \Phi(\mathrm{Ad}\, X\,(Y))\,.$$

Thus, $\Phi(\omega_k)$ is well-defined as an element of $C^2(\Phi')$ by the formula:

$$\Phi(\omega_k)\,(X, Y) = \Phi(\omega_k(X, Y))\,.$$

The reader may readily verify that it follows from this remark that the mapping

$$\Phi : C^k(\mathrm{Ad}\,\mathfrak{G}) \to C^k(\Phi')$$

commutes with the Φ-operator, Lie derivative, and inner product. Thus, 5.3 can be written as:

$$(\Phi(\omega_k) - d\theta_k)\,(X, Y) + \sum_{j=1}^{k-1} \theta_j(\omega_{k-j}(X, Y)) - \\ - [\theta_j(X), \theta_{k-j}(Y)] = 0\,. \tag{5.5}$$

We must now interpret the remaining terms by constructing the appropriate multiplicative structure on the cochains. The fourth term in 5.5 is easy: Let α' be the bilinear map: $\mathbf{L} \times \mathbf{L} \to \mathbf{L}$ given by

$$\alpha'(A, B) = [A, B]\,.$$

Then,

$$\alpha'(\theta_j, \theta_{k-j})\,(X, Y) = \big(\alpha(\theta_j(X), \theta_{k-j}) - \alpha'\big(\theta_j, \theta_{k-j}(X)\big)\big)\,(Y)\,.$$

The interpretation of $\theta_j(\omega_{k-j}(X, Y))$ is more complicated. Following the pattern used in the last section, let V be the space of linear mapping of $\mathbf{G} \to \mathbf{L}$. A typical element of V will be denoted by E. Construct a representation Φ'' on V as follows:

$$\Phi''(X)(E)(Y) = \Phi'(X)(E(Y)) - E(A\,dX\,(Y)) = \Phi(X), E(Y) - E([X,Y]),$$
$$\text{for} \quad X, Y \in \mathbf{G}, E \in V .$$

Define a linear mapping $\beta' : C^r(\Phi') \to C^{r-1}(\Phi'')$ as follows:

$$\beta'(\theta)\,(X_1, \ldots, X_{r-1})\,(X) = \theta(X_1, \ldots, X_{r-1}, X) \text{ for } X, X_1, \ldots, X_{r-1} \in \mathbf{G} .$$

One may readily verify, following the pattern established for the similar mapping β defined in Section IV, that β commutes witd d, inner product, and Lie derivative. Now, define a bilinear map $\gamma : V \times \mathbf{G} \to \mathbf{L}$ by the formula:

$$\gamma(E, X) = E(X) .$$

Again, it is readily verified that it commutes with the action of $\mathbf{G}$, as expressed by Φ'', $A\,d\,\mathbf{G}$, and Φ'. Then,

$$\gamma(\beta'(\theta_j), \omega_{k-j})$$

is a 2-cochain in $C^2(\Phi')$, since $\beta'(\theta_j)$ is a 2-cochain,

$$\gamma(\beta'(\theta_j), \omega_{k-j})\,(X, Y) = \gamma(\beta'(\theta_j), \omega_{k-j}(X, Y)) = \theta_j(\omega_{k-j}(X, Y)) .$$

Finally, 5.5, the basic "deformation equation", can be rewritten as:

$$\Phi(\omega_k) - d\theta_k + \sum_{j=1}^{k-1} \gamma(\beta'(\theta_j), \omega_{k-j}) + \frac{1}{2}\alpha'(\theta_j, \theta_{k-j}) = 0 . \qquad (5.6)$$

Again, the algebraic part of the deformation program can now be considered to be in standard form, since the rules for applying it to all the terms in 5.6 are known.

VI. Study of the First Order Terms of the Deformation Equation

We continue with the problem studied in the last section, and, in particular, with the main deformation Equation 5.5 or 5.6. Let us write it out for $k = 1$

$$\Phi(\omega_1(X, Y)) = d\theta_1(X, Y) . \qquad (6.1)$$

Both sides of this equation are cochains of $\mathbf{G}$ with respect to the representation Φ' of $\mathbf{G}$ given

$$\Phi'(X)\,(A) = [\Phi(X), A] \quad \text{for} \quad X \in \mathbf{G}, A \in \mathbf{L} .$$

Notice that Φ' is a reducible (but not necessarily completely reducible!) representation, since

$$\Phi'(\mathbf{G})\,(\Phi(\mathbf{G})) \subset \Phi(\mathbf{G}) .$$

Also, the cochain on the left hand side of 6.1 takes its values in this invariant subspace $\Phi(\mathrm{G})$ of $\mathbf{L}$. This suggests that we "divide out" by this subspace, i. e., construct $V = \mathbf{L}/\Phi(\mathrm{G})$, and define Φ'' as the quotient representation of Φ' in V. Then, if $X \to \theta_1'(X)$ is the cochain that assigns the image of $\theta_1(X)$ in V, we see from 6.1 that:

$$d\theta_1' = 0\,,$$

i. e., θ_1' is a 1-cocycle, hence determines a cohomology class in $H^1(\Phi'')$ or the "first obstruction" to the deformation problem. This first cohomology class is typical of the KODAIRA-SPENCER theory.

Let us discuss in an informal way what happens if $H^1(\Phi'') = 0$. Then, we can find an element A_1 of $\mathbf{L}$ such that

$$[\Phi(X), A] - \theta_1(X) \in \Phi(\mathrm{G}) \quad \text{for} \quad X \in \mathrm{G}\,.$$

Let us suppose further that Φ is one-one. Then, there exists anelement Y_X such that:

$$[\Phi(X), A] - \theta_1(X) = \Phi(Y_X) \quad \text{for} \quad X \in \mathrm{G}\,.$$

Now,

$$\begin{aligned}
\Phi(Y_{[X_1, X_2]}) &= [\Phi([X_1, X_2])\, A] - \theta_1([X_1, X_2]) \\
&= [[\Phi(X_1), A], \Phi(X_2)] + [\Phi(X_1), [\Phi(X_2), A]] + \\
&\quad + d\theta_1(X_1, X_2) - [\Phi(X_1), \theta_1(X_2)] + [\Phi(X_2), \theta_1(X_1)] \\
&= [[\Phi(X_1), A], \Phi(X_2)] + [\Phi(X_1), [\Phi(X_2), A]] + \Phi(\omega_1(X_1, X_2)) - \\
&\quad - [\Phi(X_1), [\Phi(X_2), A] - \Phi(Y_{X_2})] + [\Phi(X_2), [\Phi(X_1), A] - \Phi(Y_{X_1})] \\
&= \Phi(\omega_1(X_1, X_2)) + \Phi([X_1, Y_{X_2}] - [X_2, Y_{X_1}])
\end{aligned}$$

hence,

$$Y_{[X_1, Y_2]} = \omega_1(X_1, X_2) + [X_1, Y_{X_2}] - [X_2, Y_{X_1}]\,. \tag{6.2}$$

But, notice now that relation 6.2 is just that which asserts that d applied to the 1-cochain $X \to Y_X$ of G with coefficients in the adjoint representation of G is just ω_1, i. e., ω_1 is a cohomology. We know the interpretation of this: The one parameter family $\lambda \to \Phi_\lambda$ of homomorphisms $\mathrm{G} \to \mathbf{L}$ can be changed to $\lambda \to \delta_\lambda = \Phi_\lambda\, T_\lambda$, where T_λ is a one-parameter family of invertible linear maps: $\mathrm{G} \to \mathrm{G}$, so that the cocycle of type ω_1 attached to this new family $\lambda \to \delta_\lambda$ is *zero*. The new cochain of type θ_1 is now a cocycle itself, hence determines a cohomology class in $H^1(\Phi')$. Again, since the pattern of further development should be clear, we wull not carry the analysis further at this point.

VII. Deformation of Subalgebras of Lie Algebras

Suppose $\mathbf{L}$ is a Lie algebra, and G is a given subalgebra of L. We want to study possible deformations $\lambda \to \mathrm{G}_\lambda$ which assign to each value of λ a subalgebra of λ, reducing to the given one at $\lambda = 0$. The situation may be more general than the one considered previously, in the sense that the dimension of these subalgebras may not be the same, hence one cannot set up a common isomorphism between their underlying vector spaces. Yet, to apply our "Taylor's series" methods, it is necessary to parametrize the problem in some convenient way. We can do this by using the "dual" method of parametrizing subspaces, namely assuming that there is a one-parameter family $\lambda \to P_\lambda$ of linear projection operators $\mathbf{L} \to \mathbf{L}$ such that:

$$\begin{aligned} &\text{a) } P_\lambda^2 = P_\lambda, \\ &\text{b) } P_\lambda(\mathbf{L}) = \mathfrak{G}_\lambda, \text{ hence, since } \mathfrak{G}_\lambda \text{ is a subalgebra,} \\ &\text{c) } P_\lambda[P_\lambda X, P_\lambda Y] = [P_\lambda X, P_\lambda Y] \text{ for } X, Y \in \mathbf{L}. \end{aligned} \tag{7.1}$$

Having described the problem in this way, we are free to use the standard methods, i. e., expand P_λ in a Taylor's series, interpret the individual terms as cochains, then interpret the conditions on the cochains resulting from 7.1 in terms of multiplicative structures on the cochains.

$$P_\lambda = \sum_{j=0}^{\infty} P_j \lambda^j ,$$

where each P_j is a linear map: $\mathbf{L} \to \mathbf{L}$.

(7.1) a) gives the condition:

$$\sum_{j,k} P_j P_k \lambda^{j+k} = \sum_j P_j \lambda^j , \quad \text{or}$$

$$\sum_{j=0}^{k} P_j P_{k-j} = P_k, k = 0, 1, \ldots, \quad \text{or}$$

$$P P_k + P_k P - P_k = \sum_{j=1}^{k-1} P_j P_{k-j} . \tag{7.2}$$

Consider 7.1 b):

$$\sum_{j,k,l} P_j[P_k X, P_l Y] \lambda^{j+k+l} = \sum_{j,k} [P_j X, P_k Y] \lambda^{j+k} \quad \text{or}$$

$$\sum_{0 \leqq j+k \leqq l} P_j[P_j(X), P_{l-j-k}(Y)] = \sum_{j=0}^{l} [P_j X, P_{l-j}(Y)], \quad l = 1, 2, \ldots, \quad \text{or}$$

$$P[P(X), P_l(Y)] + \sum_{j+k=l} P_j[P_k X, P(Y)] = [P X, P_l Y] + [P_l X, P Y] + \\ + \sum_{j=1}^{l-1} [P_j X, P_{l-j} Y] ,$$

or

$$P[P X, P_k Y] + P[P_k X, P Y] + P_k[P X, P Y] - [P X, P_k Y] - \\ - [P_k X, P Y] \\ = \sum_{j=1}^{k-1} [P_j X, P_{k-j} Y] - P_j[P_{k-j} X, P Y] . \tag{7.3}$$

The first step is to consider the left hand of 7.3 separately as a function of X and Y, which we define as $\theta_k(X, Y)$. Notice then that:

$$\theta_k(X, Y) = 0 \quad \text{if} \quad P\,X = 0 = P\,Y \tag{7.4}$$

$$\theta_k(X, Y) = P[P_k X, Y] - [P_k X, Y] = (I - P)([Y, P_k X]) \tag{7.5}$$

if $P\,X = 0$, $P\,Y = Y$ (I = identity map of $\mathbf{L} \to \mathbf{L}$).

Now, let $V = (I - P)(\mathbf{L})$, i. e., $V = \{X \in \mathbf{L} : P\,X = 0\}$, identify V with $\mathbf{L}/\mathbf{G}$, and define Φ as the representation of $\mathbf{G}$ in V resulting from passing to the quotient via the adjoint representation of $\mathbf{G}$ in $\mathbf{L}$. Then

$$\Phi(X)(Y) = (I - P)[X, Y] \quad \text{for} \quad Y \in V\,. \tag{7.6}$$

We can now rewrite (7.5) as:

$$\theta_k(X, Y) = \Phi(Y)(P_k(X)) \quad \text{for} \quad X \in V,\ Y \in \mathbf{G}\,. \tag{7.7}$$

Now, work out (7.3) for X, $Y \in \mathbf{G}$. Put:

$$\omega_k(X) = (I - P)\,P_k(X)$$
$$\omega'_k(X) = P\,P_k(X)\,.$$

Interpret ω_k as a 1-cochain of $\mathbf{G}$ with coefficients in V. Then,

$$\theta_k(X, Y) = P[X, P_k Y] + P[P_k X, Y] + P_k[X, Y] - [X, P_k Y] - \\ - [P_k X, Y]\,.$$

$$(I - P)\,\theta_k(X, Y) = \omega_k([X, Y]) - \Phi(X)(\omega_k(Y)) + \Phi(Y)(\omega_k(X)) \\ = -d\omega_k(X, Y) \\ \text{for} \quad X, Y \in \mathbf{G} \tag{7.8}$$

$$P\,\theta_k(X, Y) = P\,P_k([X, Y]) = \omega'_k([X, Y]) \quad \text{for} \quad X, Y \in \mathbf{G}\,. \tag{7.9}$$

(7.8) is the key identity linking the deformation equations (7.3) with cohomology. ($d\omega_k$ is of course the coboundary of the cochain ω_k with respect to the representation Φ of $\mathbf{G}$ in V.)

VIII. Deformations of Complex Structures in Manifolds

There is a close relation between the deformation-of-subalgebra problem and the SPENCER theory of deformations of pseudogroups on manifolds [11]. This way of developing the theory should provide a realistic algebraic model for D. C. SPENCER's monumental work, and provide a unifying framework for many differential-geometric problems. Since the deformation of complex structures has served as a model for most of the work of KODAIRA and SPENCER, it will be instructive to study it from our point of view before proceeding further.

At this point we will have to use the theory of manifolds, for which we refer to HELGASON's book [1] which also contains, in Chapter 8, a short exposition of the notion of complex manifold. (All manifolds, maps, tensor-fields, etc., will be of differentiability class C^∞.) Let M be a manifold. $F(M)$ denotes the ring of real-valued functions on M. $V(M)$ denotes the set of vector fields in M: Each element $X \in V(M)$ is a derivation $f \to X(f)$ of $F(M)$. $V(M)$ is both a Lie algebra over the real numbers (relative to the Jacobi bracket operation $(X, Y \to [X, Y])$ and a module over $F(M)$, i. e., if $f \in F(M)$, $X \in V(M)$, fX is the derivation $f' \to fX(f')$ of $F(M)$. The relation between these two types of algebraic structures on $V(M)$ is given by the following rule:

$$[X, fY] = X(f)\, Y + f[X, Y] \quad \text{for} \quad f \in F(M), X, Y \in V(M)\,.$$

A complex analytic structure on M is defined by an $F(M)$-linear map, typically denoted by J, of $V(M) \to V(M)$ such that:

a) $J^2 = -$ (identity),

b) $[X, Y] + J[JX, Y] = J[X, JY] + [JX, JY]$. (8.1)

Such an operator can be used to define the notion of complex analytic function on M; a complex-valued function $f + i\,g$ is complex analytic if

$$X(f) = JX(g) \quad \text{for all} \quad X \in V(M)\,.$$

(For example, consider the case $M = R^2$, i. e., the space of two real variable x and y. $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are vector fields. J is defined by

$$J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}\,;\, J\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}\,.$$

These equations are just the Cauchy-Riemann equations.)

This differential-geometric version of the notion of complex manifold emphasizes the relation to the underlying "real" manifold structure, and its similarity to other sorts of differential geometric structure, such as Riemannian manifolds, homogeneous spaces, sympletic manifolds, etc. In this picture, however, the integrability conditions (8.1) are in a rather unmanagably complicated form. As is customary in this subject, we introduce complex-valued functions and vector-fields on M in order to simplify it. $F(M, C)$ is the ring of complex-valued functions om M, i. e., $F(M, C) = F(M) + i\,F(M)$. $V(M, C)$ is the set of derivations of $F(M, C)$, which is in fact, just the "complexification" of $V(M)$, i. e.,

$$V(M, C) = V(M) + i\,V(M)\,.$$

Consider a J satisfying (8.1). Extend it to an $F(M, C)$-linear map of $V(M, C) \to V(M, C)$ by the rule:

$$J(V(M) + i\,V(M)) = J(V(M)) + i\,J(V(M))\,.$$

Put:

$$P = \frac{1}{2}(I + i\,J) \quad (I = \text{identity operator})\,.$$

Then:

$$P^2 = \frac{1}{4}(I - J^2 + 2\,i\,J) = \frac{1}{2}(I + i\,J) = P \tag{8.2}$$

i. e., P is a projection operator $V(M, C) \to V(M, C)$.

Notice that:

$$P^* \text{ (the complex conjugate of } P) = \frac{1}{2}(I - i\,J) = I - P \tag{8.3}$$

i. e., P^* is the projection operator on the complementary subspace to $P(V(M, C))$.

The advantage of introducing these notions is that the integrability condition (8.1) b) takes the very convenient form:

$$P(V(M, C)) \text{ is a subalgebra of } V(M, C), \text{ i. e.,}$$

$$P[P\,X, P\,Y] = [P\,X, P\,Y] \quad \text{for} \quad X, Y \in V(M, C)\,. \tag{8.4}$$

Conversely, an operator P satisfying (8.2–4) defines a complex analytic structure on M: Define $J = -i(2P - I)$, and verify reality that (8.1) is satisfied.

Of course, a *deformation* of a complex structure on M would be a one-parameter family $\lambda \to J_\lambda$ of J-operators, each satisfying (8.1), reducing to the given one at $\lambda = 0$. Alternately, we can consider it as a one-parameter family $\lambda \to P_\lambda$ of $F(M, C)$-linear operators: $V(M, C) \to \to V(M, C)$, satisfying (8.2–4) for every λ, reducing to the given P at $\lambda = 0$. Hence, we are in a special case of the general theory sketched in the last chapter:

$$\mathbf{L} = V(M, C),\ \mathfrak{G} = P(V(M, C)),\ V = (I - P)(V(M, C)) = P^*(V(M, C))$$

$$\Phi(X)(Y) = P^*[X, Y) \quad \text{for} \quad X \in \mathfrak{G},\ Y \in V\,.$$

Notice now that *both* $\mathfrak{G}$ and V are stable under multiplication by $F(M, C)$, i. e., both are $F(M, C)$-modules. If

$$P_\lambda = \sum_k P_k\,\lambda^k\,, \quad \text{and} \quad \omega_k = P^*\,P_k\,,$$

with each ω_k interpreted as a 1-cochain: $\mathfrak{G} \to V$, notice that each ω_k is $F(M, C)$-linear. This suggests a study (that we will begin in Section 9) from a purely algebraic point of view of Lie algebra cohomology with an additional module structure imposed. We can immediately check that $d\,\omega_k$ is also $F(M, C)$-linear:

For $f \in F(M, C)$, X, $Y \in \mathfrak{G}$,

$$\begin{aligned} d\,\omega_k(X, f\,Y) &= \Phi(X)\,(\omega_k(f\,Y)) - \Phi(f\,Y)\,(\omega_k(X)) - \omega_k(X, f\,Y) \\ &= P^*[X, P^*\,P_k(f\,Y)] - P^*[f\,Y, P^*\,P_k(X)] - P^*\,P_k[X, Y] \\ = X(f)\,P^*\,P^*\,P_k(Y) &+ P^*\,P_k(X)\,(f)\,P^*(Y) - X(f)\,P^*\,P_k(Y) + \\ &+ f\,d\,\omega_k(X, Y) = f\,d\,\omega_k(X, Y)\,. \end{aligned}$$

This suggests that, in constructing cohomology groups, we restrict ourselves to cochains that are $F(M, C)$-linear. They, in turn, can (in accordance with the general principles of differential geometry [1]) be interpreted as tensor-fields on M. The corresponding cohomology groups are called the Dolbeault cohomology groups for the complex structure on M.

Let us look for a geometric interpretation for the 0-cycles, i. e., – the elements $X \in V$ such that:

$$\Phi(\mathrm{G})(Z) = 0 \,, \quad \text{i. e.,} \quad [\mathrm{G}, Z] \subset \mathrm{G} \,.$$

If Z satisfies this condition, so does $X = \frac{1}{2}(Z + Z^*)$ X is a *real* vector field (i. e., in $V(M)$ itself), and is, in fact, just the "real part" of the complex vector field Z. (Z^* denotes the complex conjugate: If $Z = X + i\,Y$, with $X, Y \in V(M)$, then $Z^* = X - i\,Y$.) Z^* also satisfies:

$$[\mathrm{G}^*, Z^*] = \mathrm{G}^* \,.$$

From this, one sees that $Ad\,Z$ commutes with P, hence also with J, i. e.

$$[X, J\,Y] = J[X, Y] \quad \text{for} \quad Y \in V(M) \,.$$

This says that X is a vector-field generating a one-parameter group that preserves the complex analytic structure, i. e., is what one calls (in the theory of complex analytic manifolds) a *holomorphic* vector field. They form a Lie algebra, that we denote by S. Thus, we have a sequence of vector spaces.

$$0 \to \mathrm{S} \to V(M) \xrightarrow{d} C^1(\Phi) \to \cdots \,.$$

S is the Lie algebra of the "pseudogroup" of all complex analytic transformations of M. This sequence (called the "Spencer resolution" of the pseudogroup) is *exact*, (i. e., the image of each homomorphism = the kernel of the succeding one) if and only if the Dolbeault cohomology groups vanish.

We will leave discussion of this well-known (to mathematicians) example at this point, since we have merely meant it as "geometric" motivation for the general treatment in the next section.

IX. Lie Algebra Cohomology and the Spencer Resolution

Consider a geometric structure on a manifold M which leads to a Lie algebra S of vector fields on M. The *Spencer resolution* construction gives a sequence $E_1, E_2, \ldots$ of vector bundles over M, together with a sequence $D_i : \Gamma(E_i) \to \Gamma(E_{j+1})$ of linear maps, $i = 1, 2, \ldots$ ($\Gamma(E_j)$ is the space of cross section of the i-th bundle) such that:

a) $\Gamma(E_1) = V(M)$.
b) $S =$ kernel D_1.
c) $D_{i+1} D_i = 0$, i. e., image $D_i \subset$ kernel $D_{i=1}$, $i = 1, 1, \ldots$.
Thus, we have a sequence

$$0 \to S \to V(M) \xrightarrow{D_1} \Gamma(E_2) \to \cdots .$$

The main problems of the theory are, first, to construct the resolution, then to prove that under certain "convexity" conditions on M that the sequence is exact, i. e., image $D_i =$ kernel D_{i+1}, $i = 1, 2, \ldots$. We will now present an algebraic construction that might serve as a model for some of the ideas.

Suppose again that $\mathbf{G}$ is a subalgebra of a Lie algebra $\mathbf{L}$, and that $V = \mathbf{G}/\mathbf{L}$. Let Φ be the representation of $\mathbf{G}$ in V obtained by passing to the quotient via the adjoint action of $\mathbf{G}$ in $\mathbf{L}$. Let $C^r(\Phi)$ be the r-cochains of $\mathbf{G}$ with coefficients in V, $r = 0, 1, 2, \ldots$. Let $d : C^r(\Phi) \to C^{r+1}(\Phi)$ be the coboundary operator. Let $Z^r(\Phi)$ be the cocycles $C^r(\Phi)$, i. e., the kernel of d. Then, of course, we have a sequence

$$0 \to Z^0(\Phi) \to C^0(\Phi) \xrightarrow{d} C^1(\Phi) \xrightarrow{d} C^2(\Phi) \to \cdots . \tag{9.1}$$

It is exact if and only if all cohomology groups of dimension $\geqq 1$ are zero. Now, we have:

Theorem 9.1. *Let $N(\mathbf{G})$ be the normalizer of $\mathbf{G}$ in $\mathbf{L}$. Then $Z^0(\Phi)$ is isomorphic to $N(\mathbf{G})/\mathbf{G} = \mathbf{S}$, hence, we have an exact sequence:*

$$0 \to \mathbf{S} \to V \xrightarrow{d} C^1(\Phi) \to \cdots . \tag{9.2}$$

The *proof* should be obvious. If $Z \in \mathbf{L}$, the image of Z in $V = \mathbf{L}/\mathbf{G}$ is in the kernel of d if and only if

$$[X, \mathbf{G}] \subset \mathbf{G} ,$$

i. e., Z is normalizer. Further, the quotient map $N(\mathbf{G}) \to V$ has $\mathbf{G}$ as kernel q.e.d.

Now we may inquire under what condition $C^0(\Phi) = V$ itself can be made into a Lie algebra so that the map $\mathbf{S} \to \mathrm{V}$ of 9. 2 is a Lie algebra homomorphism. We shall give one such condition.

Suppose that $\mathbf{L}$ is, as a vector space, the direct sum $\mathbf{G} \oplus \mathbf{G}'$ of subalgebras, i, e., V can, as a vector space, be identified with $\mathbf{G}'$. We must find the condition that the map $\mathrm{N}(\mathbf{G})/\mathbf{G} \to \mathbf{G}'$ is a Lie algebra homomorphism.

Suppose that $\mathrm{Z}, \mathrm{Z}_1 \in \mathrm{N}(\mathbf{G})$, with $\mathrm{Z} = \mathrm{X} + \mathrm{X}'$, $\mathrm{Z}_1 = \mathrm{X}_1 + \mathrm{X}_1'$; $\mathrm{X}, \mathrm{X}_1 \in \mathbf{G}$, and $\mathrm{X}', \mathrm{X}_1' \in \mathbf{G}'$, Now

$$[\mathrm{Z}, \mathrm{X}_1] = [\mathrm{Z}, \mathrm{Z}_1] - [\mathrm{Z}, \mathrm{X}_1'] \in \mathbf{G},$$

$$[\mathrm{X}, \mathrm{Z}_1] = [\mathrm{Z}, \mathrm{Z}_1] - [\mathrm{X}', \mathrm{Z}_1] \in \mathbf{G} \text{ , hence}$$

$$[Z, X_1'] - [X', Z_1] \in \mathbf{G} . \tag{9.3}$$

Then,

$$2[Z, Z_1] - [Z, X_1'] - [X', Z_1] \in G\,, \quad \text{or} \quad [Z, Z_1] - [Z, X_1'] \in G\,. \tag{9.4}$$

Now, we have:

Theorem 9.2. *Suppose* $N(G) = G \oplus (N(G) \cap N(G'))$. *Then, the map* $S = N(G)/G \to G' = C^0(\Phi)$ *is a homomorphism.*

Proof. Suppose that $Z, Z_1 \in N(G) \cap N(G')$.

By (9.4), projection of $[Z, Z_1]$ on G' = projection of $[Z, X_1']$ on G' =, by (9.4), the projection of $[X', Z_1]$ on G'. Now, $[X', Z_1]$ belongs to G', but also equals $[X', X_1] + [X', X_1']$. Note that:

$$[X', X_1] = [X', Z_1] - [X', X_1'] \in G' = [Z, X_1] - [X, X_1] \in G\,.$$

Since $G \cap G' = (0)$, $[X', X_1] = 0$, i. e.,

$$[(\text{projection of } N(G) \cap N(G') \text{ on } G), \text{projection of } N(G) \cap N(G') \text{ on } G'] = 0\,. \tag{9.5}$$

This implies that projection of $[Z, Z_1]$ on $G' = [X', X_1']$, which shows that, with the identification $C^0(\Phi) = G'$, the map $S \to G'$ is a homomorphism.

X. Lie Algebra Cohomology with an Associated Module Structure

In the last section we have abstracted out one feature of the complex-manifold theory that has general algebraic validity. Now, we will present another general feature.

Suppose that G is a Lie algebra, and also a module over the ring F. We will denote elements of G by X, elements of F by f. Suppose that to each $X \in G$ we are given a derivation $f \to X(f)$ of F, and that:

$$[X, fY] = X(f)\,Y + f[X, Y]\,, \quad \text{for} \quad X, Y \in G, f \in F\,.$$

Suppose that $X \to \Phi(X)$ defines a representation of G by linear transformation on a vector space V, that V is also a module under F, with

$$\begin{aligned} &\Phi(X)(fv) = X(f)\,v + f\,\Phi(X)(v)\,f\,, \\ &\Phi(fX)(v) = f\,\Phi(X)(v)\,, \\ &\text{for} \quad X \in G, v \in V\,. \end{aligned} \tag{10.1}$$

Let $C_F^r(\Phi)$ be the submodule of $C^r(\Phi)$ (the r-cochains of G with coefficients on V) consisting of those r-cochains which are also F-multilinear, i. e., the functions $X_1, \ldots, X_r \to \omega(X_1, \ldots, X_r)$ that satisfy:

$$\omega(fX_1, X_2, \ldots, X_r) = f\,\omega(X_1, \ldots, X_r) \quad \text{for} \quad f \in F, X_1, \ldots, X_r \in G\,.$$

As a consequence of 10.1, one proves easily that:

$$X(f\omega) = X(f)\,\omega + f X(\omega) \quad \text{for} \quad \omega \in C^r(\Phi), f \in F \tag{10.2}$$

($X(\omega)$ denotes the Lie derivative of the cochain ω).

Theorem 10.1. *If $\omega \in C_F^r(\Phi)$, $X \in \mathrm{G}$, then $X(\omega) \in C_F^r(\Phi)$.*

Proof. For $f \in F$,

$$X(\omega)(fX_1, \ldots, X_r) = \Phi(X)(\omega(fX_1, \ldots, X_r)) - \omega([X, fX_1], \ldots, X_r) - \ldots$$
$$- \omega(fX_1, \ldots, [X, X_r]) = X(f)\,\omega(X_1, \ldots, X_r) + f\,\Phi(X)(\omega(X_1, \ldots, X_r))$$
$$- X(f)\,\omega(X_1, \ldots, X_r) - f\omega([X, X_1], \ldots, X_r) -$$
$$- \cdots - f\omega(X_1, \ldots, [X, X_r]) = (fX(\omega)(X_1, \ldots, X_r)) .$$

Theorem 10.2. $d(C_F^r(\Phi)) \subset C_F^{r+1}(\Phi)$, $\quad r = 0, 1, \ldots$.

Proof. We proceed by induction. For $r = 0$: $\omega \in C_F^0(\Phi)$ means $\omega \in V$

$$d\omega(X) = \Phi(X)(\omega) ,$$
$$d\omega(fX) = \Phi(fX)(\omega) = f\,\Phi(X)(\omega) = f\,d\omega(X) .$$

Assume it is true for forms of degree $< r$

$$d\omega(X, F, X_1, \ldots, X_r) = (X \rfloor d\omega)(fX_1, \ldots, X_r)$$
$$= X(\omega)(fX_1, \ldots, X_r) + d(X \rfloor \omega)(fX_1, \ldots, X_r)$$
$$= fX(\omega)(X_1, \ldots, X_r) + f\,d(X \rfloor \omega)(X_1, \ldots, X_r)$$

(by Theorem (10.1) and the induction hypothesis),

$$= f\,d\omega(X, X_1, \ldots, X_r) \qquad \text{q.e.d.}$$

Thus, we can use cochains that are F-multilinear to construct a cohomology group. They are obviously the appropriate group to discuss deformation of homomorphisms, subalgebras, etc., that are F-linear.

Notice the following way of defining an interesting cohomology situation: Suppose G is a Lie algebra that is also an F-module, with $\mathbf{L}$ also acting as derivations in F, as before. Suppose that G is a Lie subalgebra of $\mathbf{L}$ that is also a submodule of G. Then, $V = \mathrm{G}/\mathrm{G}$ is an F-module, and the cohomology theory sketched above can be used.

XI. Deformations of Complex Submanifolds of Complex Manifolds

Now, we proceed to abstract out of KODAIRA's work [3] on deformation of complex submanifolds an interesting algebraic structure. Let M be a manifold, $J: V(M) \to V(M)$ a tensor-field (i. e., an $F(M)$-linear map). Satisfying (8.1). For $p \in M$, let M_p be the tangent space to M at p. Each vector field $X \in V(M)$ determines a tangent vector ar p, i. e., an element of M_p, which is its "value" at p, which we denote by $X(p)$. J has a "value" at p also, which is a linear map (which we also denote by J of $M_p \to M_p$ such that $J^2 = -$ (identity). Then, as definition,

$$J(X)(p) = J(X(p)) .$$

Let N be a submanifold of M. For each point p of N, its tangent space, N_p, is a subspace of M_p. $J(N_p)$ may or may not be equal to N_p. If it is, for each $p \in N$, then obviously N inherits a J-tensor. If this is so, it is readily verified that the "integrability condition" (8.1) is satisfied on N also, i. e., N has a complex manifold structure.

Now, we can consider "deformations", i. e., a one-parameter family $\lambda \to N^\lambda$ of complex submanifolds of M. The "trivial" deformations are those of the form

$$N^\lambda = \Phi_\lambda(N) ,$$

where $\lambda \to \Phi_\lambda$ is a one-parameter family of transformations of M that preserve the complex-analytic structure on M, i. e., that are complex-analytic transformations on M.

Let us formulate this more algebraically. Suppose $F(M, N)$ consists of the functions of M that zero on N. Then, $F(M, N)$ is an ideal in the ring $F(M)$, and $F(N)$, the functions on N, can be identified with the quotient ring $F(M)/F(M, N)$. A vector field $X \in V$ is tangent to N if:

$$X(F(M, N)) \subset F(M, N) .$$

Thus, the action of such an X by derivation on $F(M)$ induces an action by derivation on $F(N) = F(M)/F(M, N)$. This defines a vector field on N, which is just the induced vector field on N. Let $V(M, N)$ denote the set of these vector fields that are tangent to N. Then, the condition that N be a complex submanifold is

$$J(V(M, N)) \subset V(M, N) .$$

Notice that

$$[V, (M, N), V(M, N)] \subset V(M, N) ,$$

i. e., $V(M, N)$ is a subalgebra of $V(M)$.

Also $V(M, N)$ is an $F(M)$-submodule of $V(M)$. Thus, a deformation of submanifolds, $\lambda \to N^\lambda$ (independently of the condition that each submanifold be complex), can be considered as a deformation $\lambda \to V(M, N^\lambda)$ of a subalgebra of $V(M)$.

Now, we can handle deformations of complex submanifolds. As before, "complexify" $F(M)$ and $V(M)$ to $F(M, C)$ and $V(M, C)$. Put $P = 1/2\,(I + i\,J)$: $F(M, N)$ and $V(M, N)$ can be complexified to $F(M, N, C)$ and $V(M, N, C)$. Then, $P(V(M, N, C))$ is a subalgebra of $P(V(M, C))$. A deformation $\lambda \to N^\lambda$ of complex submaifolds leads to a deformation $\lambda \to P(V(M, N^\lambda, C))$ of subalgebras of $P(V(M, C))$.

We have carried the analysis sufficiently far to make it clear that the algebraic formalism sketched above applied to this situation also. However, there is a new feature: We have a Lie algebra $\mathbf{L}\big(= P(V(M, C))\big)$, a subalgebra $\mathfrak{G}\big(= P(V(M, N, C))\big)$, and we count those deformations $\lambda \to \mathfrak{G}^\lambda$ of $\mathfrak{G}$ as "trivial" which are obtained by acting on $\mathfrak{G}$ by a one-parameter family of automorphisms of $\mathbf{L}$ taken from a *given* group of automorphisms. In this case, the Lie algebra of the group of automorphisms is $\mathbf{S}$, the normalizer of $P(V(M,C))$ in $V(M,C)$, modulo $P(V(M,C))$ itself, i. e., the group is the group of automorphisms that is induced by the complex analytic transformations of M on $V(M,C)$. This, is a slightly more general deformation problem that is considered here or in RICHARDSON's paper [10], but the same methods apply to it: We shall return to this in another paper.

XII. Deformations that are Linear in the Deformation Parameter

Let us turn to another problem that is closely related to the "multiplicative" structure on the cochains, namely the problem of deciding when a given deformation is equivalent to one which is linear in the deformation parameter. (For example, the "Gell-Mann formula" type of analytic continuation of Lie algebra representations [2] leads to such types of deformations, in a very natural way.) Since we will only begin this discussion in this paper, we will consider the simplest case, the deformation of Lie algebra homomorphisms.

Let $\mathbf{L}$ and $\mathbf{G}$ be Lie algebra, and let $\Phi : \mathbf{G} \to \mathbf{L}$ be a homomorphism from $\mathbf{G}$ to $\mathbf{L}$. Let V be the underlying vector space to $\mathbf{L}$, and let Φ' be the following representation of $\mathbf{G}$ by linear transformation in $\mathbf{L}$:

$$\Phi'(X)(Y) = [\Phi(X), Y] \quad \text{for} \quad X \in \mathbf{G},\ Y \in \mathbf{L}\,.$$

Let $\omega \in C^2(\Phi')$, i. e., ω is a linear map $\mathbf{G} \to \mathbf{L}$.

Consider

$$\Phi_\lambda(X) = \Phi(X) + \lambda\,\omega(X) \quad \text{for} \quad X \in \mathbf{L}\,.$$

Then,

$$\Phi_\lambda([X, Y]) = \Phi([X, Y]) + \lambda\,\omega([X, Y])\ [\Phi_\lambda(X), \Phi_\lambda(Y)]$$

$$= [\Phi(X) = \lambda\,\omega(X), \Phi(Y) + \lambda\,\omega(Y)]$$

$$= [\Phi(X), \Phi(Y)] + \lambda[\omega(X), \Phi(Y)] + \lambda[\Phi(X), \omega(Y)] + \lambda^2[\omega(X), \omega(Y)]$$

Now, let $\alpha : \mathbf{L} \times \mathbf{L} \to \mathbf{L}$ be the map $\alpha(X, Y) = [X, Y]$. Then α induces a multiplication

$$C^r(\Phi') \times C^R(\Phi') \to C^{r+R}(\Phi')$$

on cochains, as described above.

$$\alpha(\omega, \omega)\,(X, Y) = [\omega(X), \omega(Y)]\,.$$

We see that Φ_λ is a homomorphism for every λ, if and only if:

$$d\omega = 0\,, \tag{12.1}$$

$$\alpha(\omega, \omega) = 0\,. \tag{12.2}$$

Suppose now that a given $\omega \in C^1(\Phi')$ satisfies (12.1), but not (12.2). Then (12.1) says that ω_1 is a cocycle, hence determines a cohomology class that we denote by $\bar{\omega}_1$. Can we change ω_1 within its cohomology class so that 12.2 is *also* satisfied? Now, α induces, as we have seen, a multiplication

$$H^1(\Phi') \times H^1(\Phi') \to H^2(\Phi') .$$

Thus, a necessary condition is that

$$\alpha(\bar{\omega}, \bar{\omega}) = 0 .$$

Now, it is easy to see that it is not always a sufficient condition. However, we can add another condition which makes it also sufficient, and that is satisfied in many examples.

Theorem 12.1. *Suppose* $\omega \in Z^1(\Phi')$ *satisfies*

$$\alpha(\bar{\omega}, \bar{\omega}) = 0 . \tag{12.3}$$

Suppose also that $Z^1(\Phi)$ *can be split up as a direct sum*

$$d C^0(\Phi) \oplus W^1$$

where W^1 *is a subspace of* $Z^1(\Phi)$ *satisfying:*

$$\alpha(W^1, W^1) \cap d C^1(\Phi') = (0) . \tag{12.4}$$

Conclusion: If ω' *is the element of* W^1 *which is the same cohomology class as* ω *then* Φ_λ *defined by:*

$$\Phi_\lambda(X) = \Phi(X) + \lambda\, \omega'(X)$$

is, for each λ, *a homomorphism of* $\mathfrak{G}$ *into* $\mathbf{L}$.

The *proof* is trivial: (12.3) says that $\alpha(\bar{\omega}, \bar{\omega}) = \alpha(\bar{\omega}, \bar{\omega}) = 0$, i. e., $\alpha(\omega', \omega') \in d C^2(\Phi')$, while (12.4) then implies that $\alpha(\omega', \omega') = 0$. Then, (12.1–2) are satisfied with ω replaced by ω', whence the conclusion.

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