Commun. math. Phys. 4, 258—260 (1967)

The Energy Momentum Spectrum of Quantum Fields

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Received October 1, 1966

Abstract. It is proved, assuming Einstein causality, that the energy-momentum spectrum of a quantum field cannot be bounded. More is known under special assumptions [1, 4]. Our main concern is the method and general applicability of the result.

I. Introduction

The Haag-Araki formulation of local quantum field theory associates with open regions $\mathcal{O}$ of Minkowski space-time R^4 von Neumann algebras $\mathcal{R}(\mathcal{O})$ on a Hilbert space $\mathcal{H}$. The self-adjoint operators in $\mathcal{R}(\mathcal{O})$ correspond to the bounded observables of the field localized in the region $\mathcal{O}$ of space-time. The dynamics and relativistic invariance of the field are expressed in terms of a (strongly-continuous) unitary representation U of the Poincaré group G on $\mathcal{H}$ in such a manner that $U(g)\,\mathcal{R}(\mathcal{O})\;U(g)^{-1} = \mathcal{R}(g(\mathcal{O}))$, where $g(\mathcal{O})$ denotes the transform of the region $\mathcal{O}$ by the (inhomogeneous) Lorentz transformation g of space-time. (This is *covariance* of U and $\mathcal{R}$.) Further assumptions are made — among them:

$\{\mathcal{R}(\mathcal{O}) : \mathcal{O}$ open in $R^4\}$ and $\{\mathcal{R}(\mathcal{O}_s) : \{\mathcal{O}_s\}$ an open covering of $R^4\}$ both generate the same C^*-algebra $\mathfrak{A}$ (the *quasi-local algebra of* the system). (1)

$$\mathcal{R}(\mathcal{O}_1) \subseteq \mathcal{R}(\mathcal{O}_2)' \text{ if } \mathcal{O}_1 \text{ and } \mathcal{O}_2 \text{ are space-like separated.} \tag{2}$$

$$\mathcal{R}(\mathcal{O}_0) \subseteq \mathcal{R}(\mathcal{O}) \quad \text{if} \quad \mathcal{O}_0 \subseteq \mathcal{O}\,. \tag{3}$$

According to the theory of unitary representations of locally compact abelian groups (generalization of Stone's theorem) [3: p. 147] the restriction of U from G to the 4-translation group (the additive group of R^4) gives rise to a projection-valued measure E on the dual $\hat{R}^4$ of R^4, this dual being identified with energy-momentum space, such that $U(a) = \int_{\hat{R}^4} \exp(ia\cdot p)\,dE(p)$. Stone's theorem tells us that each of the one-parameter unitary groups $t \to U(ta)$ has an infinitesimal generator P_a which is a (not necessarily bounded) self-adjoint operator on $\mathcal{H}$. If a is space-like P_a is the momentum observable conjugate to translation in the direction a. If a is a vector along the time axis, the generator H is

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* Research supported by ONR and NSF.

identified with the total energy. The projection-valued measure E on $\hat{R}^4$ is simply a synthesis of all the spectral resolutions of the momenta and energy observables. To speak of all the momenta and energy as having finite spectrum is to require that E have support in a bounded region of $\hat{R}^4$ — equivalently, that $\|U(a) - I\| \to 0$ as $a \to 0$.

It is not the unitary operators $U(a)$ which are physically relevant, but rather the automorphisms $\alpha(a)$ defined by $\alpha(a)(A) = U(a) A U(-a)$ which they induce on the quasi-local algebra — so that, from the point of view of physical relevance, we should be concerned with a representation of G by * automorphisms of $\mathfrak{A}$. There is, as yet, no theory associating a "spectrum" with such a representation: though it is clear that a restriction such as boundedness of the "spectrum" should be equivalent to: $\|\alpha(a) - \iota\| \to 0$ as $a \to 0$, where ι is the identity automorphism of $\mathfrak{A}$ and $\|\alpha(a) - \iota\|$ is the bound of $\alpha(a) - \iota$ as an operator on the normed space $\mathfrak{A}$. We say that α is a norm-continuous representation of R^4 by automorphisms of $\mathfrak{A}$, in this case. With this in mind, we make the:

Definition. A *covariance system* is a pair $\{\mathscr{R}, \alpha\}$ where $\mathscr{R}$ is a mapping which assigns a C^*-algebra $\mathscr{R}(\mathscr{O})$ to each bounded region $\mathscr{O}$ of R^4 satisfying (1), (2), and (3), and α is a representation of the additive group of R^4 by * automorphisms of $\mathfrak{A}$ satisfying $\alpha(a)(\mathscr{R}(\mathscr{O})) = \mathscr{R}(\mathscr{O} + a)$. We say that the system has bounded energy-momentum spectrum when α is norm-continuous.

II. The Spectrum

If $\mathfrak{A}$ is a commutative C^*-algebra acting on a Hilbert space $\mathscr{H}$, it is easy to check that $\{\mathscr{R}, \alpha\}$, with $\mathscr{R}(\mathscr{O}) = \mathfrak{A}$ for each bounded open $\mathscr{O}$ and $\alpha(a) = \iota$ for each a in R^4, is a covariance system. We say that such a system is *constant.* Conversely, if $\mathscr{R}(\mathscr{O}) = \mathfrak{A}$ for some bounded open $\mathscr{O}$, translating far enough in a space-like direction relative to $\mathscr{O}$, we see that $\mathfrak{A}$ is isomorphic with a subalgebra commuting with $\mathfrak{A}$. This subalgebra is in the center of $\mathfrak{A}$ and is abelian so that $\mathfrak{A}$ (isomorphic to it) is abelian.

Theorem. *A covariance system with bounded energy-momentum spectrum is constant.*

Proof. Since $t \to \alpha(ta)$ is a norm-continuous, one-parameter group of * automorphisms of $\mathfrak{A}$, there is a derivation δ of $\mathfrak{A}$ such that $\alpha(ta) = \exp t\delta = \iota + t\delta + \frac{t^2\delta^2}{2} + \cdots$ (convergence in the norm topology on bounded operators on $\mathfrak{A}$) [2: Lemma 2]. Let $\mathscr{O}_0$ and $\mathscr{O}$ be the interiors of the spheres with center 0, radii r and $2r$, respectively, in R^4. For each a in R^4 and all sufficiently small t, $\mathscr{O}_0 + ta \subseteq \mathscr{O}$ so that $\alpha(ta)(\mathscr{R}(\mathscr{O}_0)) \subseteq \subseteq \mathscr{R}(\mathscr{O})$, from (3). With B in $\mathscr{R}(\mathscr{O})'$ and A in $\mathscr{R}(\mathscr{O}_0)$, $B\alpha(ta)(A) - - \alpha(ta)(A)B = 0$ for small t. Thus $BA - AB + t(B\delta(A) - \delta(A)B) + + \frac{1}{2}t^2(B\delta^2(A) - \delta^2(A)B) + \cdots = 0$, for small t, so that $B\delta^n(A) -$

$-\delta^n(A)B = 0$ for $n = 0, 1, \ldots$. Hence $B\alpha(ta)(A) - \alpha(ta)(A)B = 0$ for all t and each a in R^4. It follows that B commutes with $\mathcal{R}(\mathcal{O}_0 + a)$ for all a; and, from (1), B commutes with $\mathfrak{A}$. Thus, with $\mathcal{O}_1$ the interior of a sphere space-like separated from $\mathcal{O}$, $\mathcal{R}(\mathcal{O}_1) \subseteq \mathcal{R}(\mathcal{O})' \subseteq \mathfrak{A}'$. Since $\mathcal{R}(\mathcal{O}_1) \subseteq \mathfrak{A}$, $\mathcal{R}(\mathcal{O}_1)$ lies in the center of $\mathfrak{A}$. As each $\alpha(a)$ is an automorphism of $\mathfrak{A}$, $\alpha(a)(\mathcal{R}(\mathcal{O}_1))(= \mathcal{R}(\mathcal{O}_1 + a))$ lies in the center of $\mathfrak{A}$; and, from (1), $\mathfrak{A}$ coincides with its center, that is, $\mathfrak{A}$ is abelian. From [2: Lemma 2] each $\alpha(a)$ arises from a unitary operator in $\mathfrak{A}''$; and, since $\mathfrak{A}''$ is abelian, $\alpha(a) = \iota$ for each a. For each open $\mathcal{O}_2$, $\alpha(a)(\mathcal{R}(\mathcal{O}_2)) = \mathcal{R}(\mathcal{O}_2) = \mathcal{R}(\mathcal{O}_2 + a)$; and from (1), $\mathcal{R}(\mathcal{O}_2) = \mathfrak{A}$. Thus $\{\mathcal{R}, \alpha\}$ is constant.

Since the assumptions of quantum field theory rule out a commutative quasi-local algebra, we have:

Corollary. *No quantum field has a bounded energy-momentum.*

Of course the foregoing applies to covariance systems based on more general groups than R^4 (in particular, on R^n).

References

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