

The Representations of the Oscillator Group*

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Abstract. Using the Mackey theory of induced representations all the unitary continuous irreducible representations of the 4-dimensional Lie group G generated by the canonical variables and a positive definite quadratic 'hamiltonian' are found. These are shown to be in a one to one correspondence with the orbits under G in the dual space \mathcal{G}' to the Lie algebra \mathcal{G} of G , and the representations are obtained from the orbits by inducing from one-dimensional representations provided complex subalgebras are admitted. Thus a construction analogous to that of KIRILLOV and BERNAT gives all the representations of this group.

Introduction

The general theory of induced representations as developed by MACKEY [1] allows one to classify and explicitly construct all the unitary irreducible continuous representations of any semi-direct product of groups whose projective representations are known, provided the semi-direct product is 'regular'. The criterion of regularity can be examined explicitly in given cases, and holds for many groups of interest, such as the Poincaré group.

By applying Mackey's theory inductively to nil-potent Lie groups, KIRILLOV [2] has given a very neat method for finding all the (unitary continuous irreducible) representations of any nil-potent group. This method can be applied to any Lie group, whose structure may not be as simple in terms of semi-direct products as the nil-potent case. The question then arises, does the Kirillov construction give all the representations of a more general group?

For solvable Lie groups, with the extra property of being *exponential*, BERNAT [3] has proved that the Kirillov construction does indeed give all the representations (from now on, representations will mean unitary continuous). For compact semi-simple groups, it gives all the representations, and for non-compact semi-simple groups it seems to give many of them. The question arises, for which groups does it give all the representations?

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In this paper we study the group with 4 generators H, P, Q, E , with the commutation relations

$$\begin{aligned}
 [H, P] &= -Q & [H, Q] &= P \\
 [P, Q] &= E, & \text{with the rest zero.}
 \end{aligned}$$

To show this is a Lie algebra, we need to verify the Jacobi identity between all triplets of elements. But this is true, since it holds in the model

$$2H = \frac{-\partial^2}{\partial x^2} + x^2, \quad P = \frac{\partial}{\partial x}, \quad Q = x, \quad E = 1$$

acting on functions of x . This is just the Harmonic oscillator problem, and so we propose to call this group the *harmonic oscillator group*, just as the group generated by P, Q, E is known as the *Heisenberg group*¹, which is thus a subgroup of the oscillator group. Matrices satisfying these relations are

$$\begin{aligned}
 H &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & P &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 Q &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} & E &= \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
 \end{aligned}$$

The typical group element is

$$\exp \alpha E \exp x P \exp y Q \exp t H = (\alpha, x, y, t),$$

which can then be computed²:

$$(\alpha, x, y, t) = \begin{pmatrix} 1, & -(y \cos t + x \sin t), & (y \sin t - x \cos t), & 2\alpha \\ 0, & \cos t, & -\sin t, & x \\ 0, & \sin t, & \cos t, & y \\ 0, & 0, & 0, & 1 \end{pmatrix}.$$

The Heisenberg group is *nil-potent*, that is

Definition. A Lie group is said to be nil-potent if there exists a $k < \infty$ such that $[G[G \dots [G, G] \dots]] = 0$ where the set $[G_1, G_2]$ is the set of elements z of G s.t. $z = [x, y]$ with $x \in G_1, y \in G_2$ for any two subsets G_1, G_2 of G . That is, $[G_1, G_2]$ is the set of commutators of G_1 , with G_2 . We may define the bracket $[G[G \dots [G, G] \dots]]$ to be G^k , where k is the number of G 's occurring. The oscillator group is not nil-potent, since $[H[G, G]] = [G, G] =$ Heisenberg group, and so $G^k =$ Heisenberg group for all k . The oscillator group, however, is *solvable*.

¹ Sometimes called the Weyl group.

² The author thanks N. VILENKIN for this remark.

Definition. Let $\underline{G}^2 = \underline{G}_2 = [\underline{G}, \underline{G}]$, $\underline{G}_3 = [\underline{G}_2, \underline{G}_2]$, $\underline{G}_4 = [\underline{G}_3, \underline{G}_3]$, etc. Then \underline{G} is said to be *solvable* if there exists a k such that $\underline{G}_k = \{0\}$. Then if \underline{G} is the oscillator group, $[\underline{G}, \underline{G}] = \text{Heisenberg group}$, $\underline{G}_3 = \{E\}$ and $\underline{G}_4 = \{0\}$, and so the oscillator group is solvable.

For any Lie group G , with Lie algebra \underline{G} (finite dimensional), there is a canonical map from \underline{G} into G given by the *exponential map*. To construct this, we first define the adjoint representation of \underline{G} . Now \underline{G} is a finite dimensional vector space, of dimension n say, with basis vectors x_1, \dots, x_n . We may represent $x \in \underline{G}$ by the operator adx defined on \underline{G} by $(ad x)x_k = [x, x_k] \in \underline{G}$. In this way any element of \underline{G} is represented by an $n \times n$ matrix. The set of $n \times n$ matrices of the form $\exp(\alpha x)$ for some $x \in \underline{G}$, generate a group G_0 of matrices isomorphic with G (or its connected component if G is not connected; from now on we assume G connected). The isomorphism $G \leftrightarrow G_0$ constitutes the *adjoint representation* of G . Using this isomorphism we get a map $\exp: \underline{G} \rightarrow G$, called the *exponential map*. In general this map is into G but not onto G , i.e. the image of \exp is not the whole of G , though it generates G . This may seem surprising, but is less so when expressed as follows: the image of \exp is the set of points in G that are elements of a one-parameter subgroup of G . For certain Lie groups the exponential map is *onto*; these may be called *exponential groups*. DIXMIER has given a necessary and sufficient condition for the exponential map for a solvable group to be onto. (See 3). We shall see that, according to this criterion, the oscillator group is not an exponential group.

If G is a Lie group and \underline{G} its Lie algebra then the set of linear functionals on \underline{G} form a vector space \underline{G}' , the dual to \underline{G} . There is a natural way to make \underline{G}' into a G space, namely by duality. If $g \in G$, denote by $ad g$ the $n \times n$ matrix representing g in the adjoint representation. Then if $\omega \in \underline{G}'$, $x \in \underline{G}$, the action of g on ω is

$$\omega_g(x) = \omega(ad g x)$$

or in terms of the generators

$$\langle ad x', y \rangle = - \langle x', ad y \rangle .$$

Here, $x' \in \underline{G}'$, $y \in \underline{G}$. One may define the elements h', p', q', e' of \underline{G}' (for the oscillator group) by their scalar products in a natural way, e.g. $\langle h', h \rangle = 1$, $\langle h', p \rangle = 0$ etc. The action of G on \underline{G}' will be called the *co-adjoint* action, and can be represented by matrices that are hermitian conjugates of the adjoint representation. The adjoint representation is not in general unitary, and so the generators are not in general hermitian matrices.

There are three other subjects which need reviewing, before the main result of this paper can be proved. They are,

1. the method of induced representations
2. the Mackey theory of regular semi-direct products
3. the Kirillov method.

These will be reviewed in later sections. Given a group G and a unitary representation L of a subgroup $H \subset G$, then the representation of G induced by L is written U^L ; it will always be unitary but not always irreducible. In fact, the bigger the group H the more likely it is that U^L is irreducible.

A semi-direct product $H \circledast N$ of a group H and a group N , which is an H -space is the set of pairs (h, n) , with the multiplication law $(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1 h_1(n_2))$ where $h(n)$ is the action of H on N . If N and H are topological groups, then there is a natural way to put a topology on $N \times H$; if N and H are Lie groups, then $G = H \circledast N$ is a Lie group, and the Lie algebra \mathcal{G} of $H \circledast N$ is the semi-direct sum of the Lie algebras \underline{N} and \underline{H} of N and H . That is, if N and H have dimensions j and k , then \mathcal{G} has dimension $j + k$, spanned by the generators of the groups N and H . The bracket between an element of \underline{N} and \underline{H} lies in \underline{N} since H maps N into itself. Examples of semi-direct products are (i) the Poincaré group $\mathcal{L} \circledast T_4$ where the Lorentz group \mathcal{L} acts on the translation group T_4 in the usual way: $a^\mu \rightarrow (\Lambda a)^\mu = \Lambda^{\mu\nu} a^\nu$ (ii) the oscillator group, where H consists of the 'time displacements' e^{iHt} and N is the Heisenberg group.

If the semi-direct product is 'regular' in the sense of MACKEY then any representation is obtained by inducing up from a representation of a subgroup. We shall see that the oscillator group is a regular semi-direct product, and so all its irreducible representations can be determined. This is done in section 3. The Kirillov method is explained in section 4, and the representations of the oscillator group given by the method are listed. They are shown to exhaust the set of all representations, as found by the Mackey theory.

2. Induced representations

Suppose G is a separable locally compact group; then there exists an invariant measure on G , the Haar measure; this means, we take the measure space to be G and the Borel σ -algebra generated by the open subsets, and if μ is Haar measure, $\mu(Ex) = \mu(E)$ for all E (Borel) and x . For any separable locally compact group we may therefore define the *regular representation* as follows: the representation space is the space of μ -square integrable functions on the group, with the obvious scalar product, and the representation $g \rightarrow U(g)$ is

$$(U(g)\Psi)(f) = \Psi(fg).$$

Thus for a finite group with n elements and the discrete topology the regular representation is n -dimensional since functions on the group are defined by the n 'components' $\Psi(g_1), \dots, \Psi(g_n)$. It is known that for finite groups the regular representation contains every irreducible representation with a multiplicity equal to the dimension of the irreducible representation. Thus we should expect the regular representation in general to be reducible and its reduction to be a source of irreducible representations of the group. However it is not true in general that the regular representation contains *all* the representations of a group. However for certain groups the following generalization of the regular representation does give all representations.

Let H be a subgroup of G and suppose L is a representation of H in a Hilbert space H_L . Consider the maps $G \xrightarrow{f} H_L$ satisfying

$$f(hg) = L(h) f(g)$$

and such that f is measurable. Then f is in fact a function only of the coset space G/H . MACKEY has shown that G/H can be made into a measure space in a natural way, i.e. we can integrate functions on G/H , thus getting a scalar product space \mathcal{H} . The action of the group is

$$(U(g_1)f)(g_2) = f(g_2g_1) \quad \text{for } g_1, g_2 \in G.$$

This action depends only on the coset to which g_2 belongs, and so maps \mathcal{H} into itself. It is easy to see that the map $g \rightarrow U(g)$ is a representation, and can be made unitary if the definition is weighted suitably. The map $U(g)$ is called the representation *induced by* L and is written U^L . In general, U^L is reducible; e.g. if L is reducible, then so is U^L . U^L will also be reducible if the subgroup H is too small, e.g. if H is a trivial then U^L is the regular representation. If the representation L is one-dimensional then U^L is a sub-representation of the regular representation. The reduction is achieved by looking at the subset of $\mathcal{L}^2(G, \mu)$ satisfying $f(hg) = \chi(h) f(g)$, $\chi(h) = L(h)$, being the character of the representation, a condition which usually means that functions in the subspace depend on fewer variables than the dimension of G . For the representation U^χ so obtained to be irreducible, H must be maximal in some sense; in particular we shall see that H must have the largest dimension possible. For this procedure to exhaust the representation of G , it must be a group for which the regular representation contains all representations in the generalized sense.

3. Regular semi-direct products

The most famous application of Mackey theory to semi-direct products is that of the Poincaré group. In that case one has the semi-direct product $\mathcal{L} \circledast T_4$, where T_4 is the 4-dimensional translation group, an

abelian group. In this case the irreducible unitary representations of T_4 are in one-to-one correspondence with the elements of the group \hat{T}_4 , the Pontrajin dual to T_4 . In this case \hat{T}_4 is just *momentum space*. Now, the Lorentz group \mathcal{L} acts on T_4 and splits it into *orbits*, the well known mass-hyperboloids, light-cones etc. Different points on the same orbit correspond to different representations of T_4 but are "equivalent" via a Lorentz transform. MACKKEY says that the orbits are smooth if there exists a Borel subset of \hat{T}_4 (or a countable union of Borel subsets) cutting each orbit exactly once. It is easy to see that this holds for the Poincaré group. In this case MACKKEY says that the semi-direct product is 'regular', and all representations can be found by the general theory. When one has the semi-direct product $H \circledast N$, where N is not abelian, then an analogous theory can be set up. In place of the dual space of characters, \hat{T}_4 , MACKKEY defines \hat{N} to be the space of equivalence classes of irreducible unitary continuous representations of N . It is then possible to put a Borel structure on \hat{N} in a natural and unique way. The group H will act in \hat{N} as follows: let $n \rightarrow U(n)$ be a unitary representation, i.e. an element of \hat{N} . Then the representation $n \rightarrow U(g(n))$ is said to be equivalent using H , $g \in H$ to $n \rightarrow U(n)$. We put two (possibly inequivalent) representations of N in the same orbit if they are equivalent using H . Thus \hat{N} is again divided up into orbits. One may again define the semi-direct product to be 'regular' if there exists a countable number of Borel subsets of \hat{N} whose union meets each orbit exactly once. If this is the case then the following construction, (which works for any semi-direct product but in general does not give *all* representations) does indeed give all representations.

Let χ be a point in \hat{N} . Then the stability group of χ (= little group of χ) H_χ is the subgroup of H such that $\chi_h \cong \chi$ for all $h \in H$ where the action $\chi \rightarrow \chi_h$ of H on \hat{N} , is the dual action of H on N that is, if χ is the representation $n \rightarrow \chi(n)$ of N , then χ_h is the representation $n \rightarrow \chi(h(n))$. If χ_1 and χ_2 are two points on the same orbit then $H\chi_1$ is isomorphic to $H\chi_2$, and so we may associate one abstract group with each orbit, the little group for the orbit. One may then obtain a representation of the group $G\chi_1 = H\chi_1 \circledast N$ as follows.

Let L be a representation of H_χ in a Hilbert space \mathcal{H}_1 and suppose $\chi \in \hat{N}$ acts in a Hilbert space \mathcal{H}_2 . Since χ_h is unitary equivalent to χ for $h \in H_\chi$ we may identify the carrier space of χ_h with \mathcal{H}_2 , for any $h \in H_\chi$. Then there exists a unique^{2a} operator $W(h) : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ such that $\chi_h(n) = W(h) \chi(n) W^{-1}(h)$ for all $h \in H_\chi$ and $n \in N$. The map $(h, n) \rightarrow \chi(n) W(h)$ then defines a representation^{2b} of G_χ in \mathcal{H}_2 . A more general

^{2a} Up to a factor.

^{2b} In general, a projective representation.

representation is then of the form $(h, n) \rightarrow L(h) \otimes \chi(n) W(h)$ acting in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. The representation of $H \circledast N$ induced by this is thus labelled by χ and L , and will be written $\rho(L, \chi)$. If the semi-direct product is regular, and H has no non-trivial multipliers, then all the representations of G are of this form.

If χ_1 and χ_2 are on the same orbit then the resultant representations obtained by irreducing are equivalent if the same representation of $G_{\chi_i} \cong G_{\chi_j}$ is used. Thus the representations of a regular semi-direct product are labelled by the orbits in \hat{N} , and the representations of G_χ .

If we wish to find all the representations we must allow the representations of G_χ , namely L , to be projective, and we must allow for possible projective representations of N , and for certain multipliers between the representations. However, it follows from MACKEY's theorem that if H has no non-trivial multipliers then all true representations of $H \circledast N$ are obtained from true representations of H and N . In particular if H is a one-dimensional abelian Lie group, it has no projective representations which are not projectively equivalent to true representations, and so we can ignore the question of multipliers for the harmonic oscillator group.

4. Kirillov theory for nil-potent groups

Nil-potent groups can be completely analysed in terms of a sequence of semi-direct products and these can be completely analysed by the preceding analysis. By repeated application one can prove that all representations can be induced from one-dimensional representations. These are classified as follows. Let G be a nil-potent group and \underline{G} its (real) Lie algebra; \underline{G} is an n -dimensional vector space which carries the adjoint representation of \underline{G} : if $x \in \underline{G}$, we define $\text{Ad } x$ as the operator $\text{Ad}(x)y = [x, y] \in \underline{G}$; and let \underline{G}' be the real dual to \underline{G} , carrying the coadjoint representation defined by duality³: $\langle \text{Ad } x', y \rangle = -\langle x', \text{Ad } y \rangle$. We can define the orbits in \underline{G}' as follows: taking a point x' in \underline{G}' , the set $\{\text{ad } g x'\}$, $g \in G$ is called the orbit of x' .

The point x' defines a functional on \underline{G} . A sub-Lie algebra $\underline{H} \subset \underline{G}$, is said to be *subordinate to x'* if the restriction of x' to $[\underline{H}, \underline{H}]$ vanishes. If \underline{H} is subordinate to x' , then x' defines a character on H , the group of the Lie algebra \underline{H} ; if $h = \exp(\alpha x)$, $x \in \underline{H}$, $h \in H$, we define $x'(h) = \exp(i \langle x', x \rangle)$. Because x' vanishes on $[\underline{H}, \underline{H}]$, the function $x'(h)$ satisfies the character relation $x'(h_1 h_2) = x'(h_1) x'(h_2)$, i.e. defines a one-dimensional representation of H . KIRILLOV proves that the representation $\rho(x', H)$ of G induced by the representation x' of H is irreducible if and only if \underline{H} is a *maximal* sub-algebra subordinate to x' . We say \underline{H} is maximal if it has a dimension greater than or equal to the dimension of any other algebra subordinate

³ In later sections we omit the minus sign.

to x' . If \underline{H}_1 and \underline{H}_2 are two maximal algebras subordinate to x' , then the representations $\varrho(x', \underline{H}_1)$ and $\varrho(x', \underline{H}_2)$ are unitary equivalent. One can then easily deduce that if f_1 and f_2 are two points on the same orbit, and \underline{H}_1 and \underline{H}_2 are respectively maximal dimensions sub-algebras subordinate to f_1 and f_2 , then $\varrho(f_1, \underline{H}_1) \cong \varrho(f_2, \underline{H}_2)$. Moreover, every continuous irreducible representation of a nil-potent group can be obtained in this way.

Now this is a very neat way to characterize the representations of nil-potent groups, and the construction of \underline{G}' , maximal \underline{H} and $\varrho(x', \underline{H})$ etc. can be done for any Lie group, and the questions that can be asked are

i) Does the construction give all representations ?

ii) If \underline{H} is maximal, is $\varrho(x', \underline{H})$ (with \underline{H} subordinate to x') necessarily irreducible ?

As remarked, (i) can hold only for groups whose regular representation contains all representations, and so we might expect trouble with the non-type I groups, i.e. Lie groups with representations which generate a non-type I von Neumann algebra. Among such groups may be found certain solvable ones.

P. BERNAT has shown, however, that for solvable groups such that the exponential map is onto, the Kirillov construction does indeed give all the representations. But he gives an example for which (ii) is false and so, in fact, there exists an orbit containing x' , and maximal sub-algebras \underline{H}_1 and \underline{H}_2 subordinate to x' , such that $\varrho(x', \underline{H}_1)$ is not equivalent to $\varrho(x', \underline{H}_2)$.

Our results for the harmonic oscillator group, obtained below, show that the Kirillov method using the real Lie algebras \underline{G} and \underline{G}' , does not give all the representations. Thus one may deduce that this group, while solvable, is not exponential. A more direct argument is given below. However, if one used complex Lie algebras \underline{G} and \underline{G}' then all the irreducible representations are in fact given by orbits.

For the harmonic oscillator group, with a positive definite Hamiltonian $P^2 + Q^2$, one can prove that the exponential map is not onto, and so the problem is outside the scope of Bernat's theorem. To see this, note that DIXMIER has given a criterion which determines whether the exponential map is onto or not, in terms of linear forms called the roots of the group. This analysis goes as follows: let $\underline{G} = \underline{G}_1 \supset \underline{G}_2 \supset \underline{G}_3 \cdots \supset \underline{G}_k = \{0\}$ be a series of ideals of \underline{G} , with \underline{G}_{k+1} a proper maximal sub-ideal of \underline{G} (maximal in \underline{G}_k). A theorem of Lie for solvable groups then assures us that the dimension of $\underline{G}_j/\underline{G}_{j+1}$ is less than or equal to 2. Moreover, if

a) $\text{Dim } \underline{G}_j/\underline{G}_{j+1} = 1$, then if we write $\underline{G}_j = \underline{G}_{j+1} + \tilde{\underline{G}}_j$ there exists a non-zero element a_j of $\tilde{\underline{G}}_j$, an element ψ_j of \underline{G}' and a linear map

$u_j : \underline{G} \rightarrow \underline{G}_{j+1}$ such that for all $x \in \underline{G}$.

$$[x, a_j] = \psi_j(x) a_j + u_j(x) .$$

b) If $\text{Dim } \underline{G}_j / \underline{G}_{j+1} = 2$ then if $\underline{G}_j = \underline{G}_{j+1} + \tilde{\underline{G}}_j$ there exists a base (a_j, a'_j) of $\tilde{\underline{G}}_j$, two elements ψ_j, ψ'_j of \underline{G}' , ψ'_j being non-zero, and two linear maps u_j, u'_j from \underline{G} into \underline{G}_{j+1} such that for all $x \in \underline{G}$

$$\begin{aligned} [x, a_j] &= \psi_j(x) a_j - \psi'_j(x) a'_j + u_j(x) \\ [x, a'_j] &= \psi'_j(x) a_j + \psi_j(x) a'_j + u'_j(x) . \end{aligned}$$

Putting $\psi'_j = 0$ if $\text{dim } \underline{G}_j / \underline{G}_{j+1} = 1$, the roots of \underline{G} are the linear forms $\psi_j \pm i \psi'_j$. Then we have the following result: \underline{G} is exponential if and only if ψ'_j is proportional to ψ_j , that is, if and only if ψ_j is non-zero and there exists a real number α_j such that $\psi'_j = \alpha_j \psi_j$. For the harmonic oscillator group the series of ideals is

$$\underline{G} = (H, P, Q, E), \quad \underline{G}_1 = (P, Q, E), \quad \underline{G}_2 = (E) .$$

$\underline{G}_1 / \underline{G}_2$ is spanned by the vectors P and Q and we are concerned with the linear forms $\psi_1(H)$ and $\psi'_1(H)$. We see

$$\begin{aligned} [H, P] &= -Q = \psi_1(H) P - \psi'_1(H) Q \\ [H, Q] &= P = \psi'_1(H) P + \psi_1(H) Q . \end{aligned}$$

Therefore, identifying, we have

$$\psi_1(H) = 0, \quad \psi'_1(H) = 1 .$$

Since $\psi_1(H) = 0$, the harmonic oscillator group is not an exponential group⁴.

5. The Mackey theory for the oscillator group

The group is the semi-direct product $H \circledast N$ where H is the one dimensional group generated by the time translations and N is the Heisenberg group. We are interested in \hat{N} , the space of equivalence classes of unitary representations of N . Since N is nil-potent, its unitary representations are given by the Kirillov method.

Let us denote the 3-dimensional vector space spanned by P, Q, E by \underline{N} and let us denote by p, q, e vectors spanning an isomorphic space. The adjoint representation of N is thus given by the following action

$$\begin{aligned} Pp &= 0 & Pq &= e, & Pe &= 0 \\ Qp &= -e, & Qq &= 0, & Qe &= 0 . \\ Ep &= Eq &= Ee &= 0 . \end{aligned}$$

⁴ BERNAT has shown [3] that a similar 4-dimensional group (but with indefinite hamiltonian) is a basic ingredient of all exponential solvable groups.

The contragredient action in \underline{N}' is therefore determined up to sign by

$$\begin{aligned}
 \langle Pp' | p \rangle &= 0; & \langle Pp' | q \rangle &= 0; & \langle Pp' | e \rangle &= 0 \therefore Pp' = 0 \\
 \langle Pp' | p \rangle &= 0; & \langle Pq' | q \rangle &= 0; & \langle Pq' | e \rangle &= 0 \therefore Pq' = 0 \\
 \langle Pe' | p \rangle &= 0; & \langle Pe' | q \rangle &= 1; & \langle Pe' | e \rangle &= 0 \therefore Pe' = q' \\
 \langle Qp' | p \rangle &= 0; & \langle Qp' | q \rangle &= 0; & \langle Qp' | e \rangle &= 0 \therefore Qp' = 0 \\
 \langle Qq' | p \rangle &= 0; & \langle Qq' | q \rangle &= 0; & \langle Qq' | e \rangle &= 0 \therefore Qq' = 0 \\
 \langle Qe' | p \rangle &= -1; & \langle Qe' | q \rangle &= 0; & \langle Qe' | e \rangle &= 0 \therefore Qe' = -p'.
 \end{aligned}$$

Clearly $E = 0$ on N' .

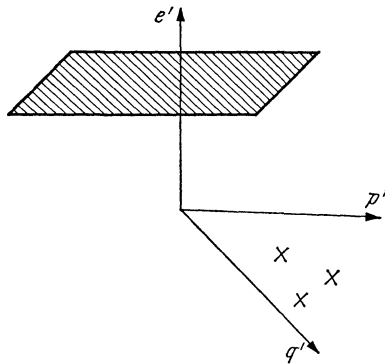


Fig. 1. Orbits in \underline{N}' under N

Thus the point (o, p', q') is unmoved by the group and each such point leads to an irreducible representation. Since e' does not appear on the right hand side, the value of the coefficient of e' does not change under the action of the group. But if this coefficient is non-zero, any value of (p, q) can be reached by acting with the group, and so the orbit is the whole plane (e', p, q) . Thus the irreducible unitary continuous representations of N are labelled by $(e' = o, p, q)$ or $e' \neq o$, and the space \hat{N} may be pictured as a plane with a line perpendicular to it.

We must now determine the orbits in \hat{N} of a point in \hat{N} under the action of the other group in the semi-direct product, namely the one parameter group of time-displacements generated by the hamiltonian. These orbits in \hat{N} must not be confused with the orbits in \underline{N}' just found; in fact orbits in \underline{N}' under N are just the elements of \hat{N} .

In order to find the action of H on a point in \hat{N} we first construct the representations i.e. the points in \hat{N} , by the Kirillov method. If $e' \neq o$ the orbit in \underline{N}' is the plane through the point (e', o, o) , and since any point on the orbit may be chosen, we choose (e', o, o) . The maximal subalgebra of \underline{N} subordinate to this linear form has dimension 2, and may be

taken to be e.g. (P, E) or (Q, E) (i.e. the algebras spanned by these vectors). Consider the subgroup (E, P) (the choice (E, Q) leads to an equivalent representation, by Kirillov's theorem). The subgroup generated by (E, P) is a two-dimensional abelian group with typical element

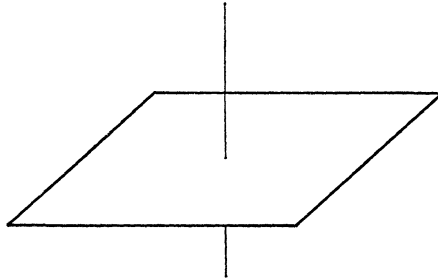


Fig. 2. The space \hat{N}

$(\alpha, x) = \exp(E\alpha) \exp(xP)$ and since (E, P) is subordinate to (e', o, o) the following defines a character: $-\chi(\alpha, x) = e^{i\alpha e'}$. From this representation we can induce a representation of N , as follows. We consider functions $f(\alpha, x, y)$ on the group N (whose typical element is $\exp(E\alpha) \exp(Px) \exp(Qy) = (\alpha, x, y)$). These must satisfy the "left covariant on cosets" condition, namely

$$f((\alpha_1, x_1) \cdot (\alpha, x, y)) = \chi(\alpha_1, x_1) f(\alpha, x, y)$$

i.e.

$$f((\alpha + \alpha_1, x + x_1, y)) = e^{i\alpha e'} f(\alpha, x, y).$$

It follows that the state space is determined by the value of f at $\alpha = x = 0$, and we may identify it with $\mathcal{L}^2(\mathbb{R}) : \psi(y) = f(o, o, y)$. The action of the group N is

$$\begin{aligned} (\alpha_1 x_1 y_1) f(y) &= f(y \cdot (\alpha_1 x_1 y_1)) \\ &= f(\exp Q y \exp E \alpha_1 \exp P x_1 \exp Q y_1) \\ &= f(\exp E (\alpha_1 - y x_1) \exp P x_1 \exp Q (y + y_1)) \\ &= e^{i e' (\alpha_1 - y x_1)} f(y + y_1). \end{aligned}$$

Thus the action of $e^{Q y_1}$ is $\psi(y) \rightarrow \psi(y + y_1)$, i.e. Q is represented by $\frac{\partial}{\partial y}$. The action of $e^{P x_1}$ is $\psi(y) \rightarrow e^{-i e' y x_1} \psi(y)$, so P is multiplication by $-i e' y$. Thus the (P, Q) are represented by the usual Schrödinger operators, and E generates a phase transformation.

The action of the one parameter group of time-translations is given by the rule $x \rightarrow t x t^{-1}$ i.e. if $\mathcal{D}^{e'}$ is the above representation of N then the time displaced one is

$$\mathcal{D}_t^{e'}(n) = \mathcal{D}^{e'}(t(n))$$

where $n \in N$ and $n \rightarrow t(n) \in N$ is the action of a time-displacement.

To calculate this we need to know

$$e^{Ht} e^{xP} e^{-Ht} \quad \text{and} \quad e^{Ht} e^{yQ} e^{-Ht} .$$

By the usual rules, $e^{Ht} e^{xP} e^{-Ht} = e^{xP(t)}$ and $e^{Ht} e^{yQ} e^{-Ht} = e^{yQ(t)}$, where

$$P(t) = e^{Ht} P e^{-Ht}; \quad Q(t) = e^{Ht} Q e^{-Ht} .$$

In fact this leads to

$$\left. \frac{dP(t)}{dt} \right|_0 = [H, P] = -Q; \quad \left. \frac{dQ(t)}{dt} \right|_0 = [H, Q] = P ,$$

with the solution $P(t) = P \cos t - Q \sin t$; $Q(t) = Q \cos t + P \sin t$. It follows that

$$\begin{aligned} e^{tH} e^{\alpha E} e^{xP} e^{yQ} e^{-tH} &= e^{\alpha E} \exp [x(P \cos t - Q \sin t)] \exp [y(Q \cos t + P \sin t)] \\ &= e^{\alpha E} e^{xP \cos t} e^{-xQ \sin t} e^{-\frac{1}{2}(x^2 E \cos t \sin t)} e^{yP \sin t} e^{yQ \cos t} e^{\frac{1}{2}y E \cos t \sin t} \end{aligned}$$

using the Baker-Hausdorff formula

$$\begin{aligned} &= \exp \left(E \left(\alpha - \frac{1}{2} x^2 \cos t \sin t + \frac{1}{2} y^2 \cos t \sin t - xy \sin^2 t \right) \right) \\ &\quad e^{P(x \cos t + y \sin t)} e^{Q(-x \sin t + y \cos t)} . \end{aligned}$$

Therefore the representation $\mathcal{D}_t^{e'}$, which is irreducible, is the following

$$\begin{aligned} (\mathcal{D}_t^{e'}(\alpha_1, x_1, y_1) \psi)(y) &= \left(\mathcal{D}^{e'} \left(\alpha_1 - \frac{1}{2} x_1^2 \cos t \sin t + \frac{1}{2} y_1^2 \cos t \sin t - \right. \right. \\ &\quad \left. \left. - x_1 y_1 \sin^2 t, x_1 \cos t + y_1 \sin t, -x_1 \sin t + y_1 \cos t \right) \psi \right)(y) \\ &= \exp i e' \left[\alpha_1 - \frac{x_1^2}{2} \cos t \sin t + \frac{1}{2} y_1^2 \sin t \cos t - x_1 y_1 \sin^2 t - y(x_1 \cos t + \right. \\ &\quad \left. + y_1 \sin t) \right] \psi(y + y_1 \cos t - x_1 \sin t) . \end{aligned}$$

If we restrict this representation to the subgroup $\exp \alpha E$, we see that it is merely the phase $\mathcal{D}_t(\alpha) = e^{ie'a}$. But this representation is unitary equivalent to the one $\mathcal{D}^{e'}$, since this phase, if non-zero, determines the representation. It follows that if $e' \neq 0$, the stability group of the orbit (or any point on the orbit) is the whole group $\exp(tH)$. Thus we get a representation of $H \circledast N$ immediately; in fact $\mathcal{D}_t^{e'}$ is unitary equivalent to $\mathcal{D}^{e'}$, and since $\mathcal{D}^{e'}$ is irreducible, the operator giving equivalence is unique up to a phase, say $W(t)$, i.e.

$$\mathcal{D}_t^{e'}(\alpha, x, y) = W(t) \mathcal{D}^{e'}(\alpha, x, y) W^{-1}(t) .$$

The operator $e^{it} \left(-x^2 + \frac{\partial^2}{\partial x^2} + it h' \right) = \exp it(P^2 + Q^2 + h')$ has this property, and so by the uniqueness of $W(t)$, this is $W(t)$. But then the resultant representation of G

$$(\alpha, x, y, t) \rightarrow \mathcal{D}^{e'}(\alpha, x, y) W(t)$$

is just the usual harmonic oscillator solution, with e' related to \hbar and $h' = \text{zero-point energy}^5$.

This completes the construction of the representations of G associated with the points (e', o, o) in the space \underline{N}' . These representations all have the property that, restricted to the Heisenberg sub-group N , they are irreducible, and therefore must agree with the 'usual' solution to the Harmonic oscillator; the energy is bounded below by $h' + \frac{1}{4}$, and e' merely gives the value of \hbar .

The remaining representations of N are characterised by the point (o, p', q') in \underline{N}' . It is clear that this point is moved by the group H unless $p' = q' = 0$. For, the representation of N determined by (o, p', q') can be explicitly determined as follows. The maximal sub-algebra of \underline{N} subordinate to (o, p', q') is \underline{N} itself, and (o, p', q') therefore determines a character of \underline{N} ,

$$\chi(\exp E\alpha) = 1, \quad \chi(\exp Px) = e^{i p' x}; \quad \chi(\exp Qy) = e^{i q' y}.$$

The action of $\exp(Ht)$ on this representation is

$$\begin{aligned} \chi_t(\alpha, x, y) &= \chi(e^{+Ht} \exp E\alpha \exp Px \exp Qy \exp -Ht). \\ &= \chi\left(\exp E\left(\alpha + \frac{1}{2}x^2 \cos t \sin t - \frac{1}{2}y^2 \cos t \sin t - xy \sin^2 t\right)\right. \\ &\quad \left.\exp P(x \cos t - y \sin t) \exp Q(x \sin t + y \cos t)\right) \\ &= e^{i p'(x \cos t - y \sin t)} e^{i q'(x \sin t + y \cos t)} \\ &= e^{i x(p' \cos t + q' \sin t)} e^{i y(q' \cos t - p' \sin t)}. \end{aligned}$$

Therefore χ_t is associated with the representation

$$(o, p'_t, q'_t) = (o, p' \cos t + q' \sin t, q' \cos t - p' \sin t) = (o, p', q')$$

only if $t = 0$. Thus the little group is trivial. Thus according to the general theory the resultant representations of G are merely those obtained by inducing from this representation of N ; we note that restricted to N it is reducible; and it has not got positive energy.

We see immediately that the semi-direct product is *regular*. For, the Mackey Borel structure for \hat{N} is undoubtedly that of the product of the plane (omitting the origin) and the line⁶. Each point on the line is an orbit in \hat{N} under H , and the orbits in the plane are the ellipses $p'(t) = p' \cos t + q' \sin t$, $q'(t) = q' \cos t - p' \sin t$. Thus the following Borel set meets each orbit only once: the line $(\rho e', o, o)$, $-\infty < \rho < \infty$ and $(o, \rho p', o')$ $p' \neq o, o < \rho < \infty$.

⁵ In the general construction of § 3, a representation of $H(\mathfrak{s})N$ is determined by a representation of N and a representation of H . But since H is one-dimensional its representations are characters $e^{ih't}$, which we have incorporated in $W(t)$.

⁶ For the Heisenberg group N this has been proved.

Then by Mackey's theorem, we have found all the continuous unitary irreducible representations.

The representations of the form (o, p', q') for N do not occur in quantum mechanics. This is because the restriction to N is reducible, i.e. at a sharp time, $P(t), Q(t)$ form a reducible set of operators. In fact, even in a finite strip of time, a time slab, these operators are reducible. (In any representation in which (P, Q) are irreducible, the solution must agree with the usual dynamics.) But the main physical trouble with these representations is that the energy is not bounded below.

6. Kirillov theory of the oscillator group

The group is four dimensional with elements H, P, Q, E : we consider an isomorphic vector space with basis h, p, q, e . The action of \underline{G} is

$$\begin{aligned} Hh &= o & Hp &= -q & Hq &= p & He &= o \\ Ph &= q & Pp &= o & Pq &= e & Pe &= o \\ Qh &= -p & Qp &= -e & Qq &= o & Qe &= o \\ Eh &= o & Ep &= o & Eq &= o & Ee &= o \end{aligned}$$

One can therefore calculate the contragredient action of the group G on the dual space \underline{G}' :

$$\begin{aligned} \langle Hh' | h \rangle &= o & \langle Hh' | p \rangle &= o & \langle Hh' | q \rangle &= o & \langle Hh' | e \rangle &= o \\ \langle Hp' | p \rangle &= o & \langle Hp' | p \rangle &= o & \langle Hp' | q \rangle &= 1 & \langle Hp' | e \rangle &= o \\ \langle Hq' | h \rangle &= o & \langle Hq' | p \rangle &= -1 & \langle Hq' | q \rangle &= o & \langle Hq' | e \rangle &= o. \end{aligned}$$

Thus

$$Hh' = o; \quad Hp' = q; \quad Hq' = -p'; \quad He' = o.$$

Also

$$\begin{aligned} \langle Pp' | h \rangle &= o & \langle Pp' | p \rangle &= o & \langle Pp' | q \rangle &= o & \langle Pp' | e \rangle &= o \\ \langle Ph' | h \rangle &= o & \langle Ph' | p \rangle &= o & \langle Ph' | q \rangle &= o & \langle Ph' | e \rangle &= o \\ \langle Pq' | h \rangle &= 1 & \langle Pq' | p \rangle &= o & \langle Pq' | q \rangle &= o & \langle Pq' | e \rangle &= o \\ \langle Pe' | h \rangle &= o & \langle Pe' | p \rangle &= o & \langle Pe' | q \rangle &= 1 & \langle Pe' | e \rangle &= o \end{aligned}$$

Thus

$$Ph' = o, \quad Pp' = o, \quad Pq' = h', \quad Pe' = q'.$$

Similarly

$$Qh' = o, \quad Qp' = -h', \quad Qq' = o, \quad Qe' = -p'$$

and $E = o$ as an operator in \underline{G}' .

We are now in a position to determine the orbits in \underline{G}' .

Since $E = o$, the action of $\exp \alpha E$ does not move any point in \underline{G}' .

A general point in \underline{G}' will be denoted by $\alpha h' + \beta p' + \gamma q' + \delta e'$. Since e' does not appear on the right hand side of the equations, the value of δ is constant on any orbit. Once we have chosen δ , we may draw the orbits in three dimensions. If $\delta = 0$, the orbits can be computed quite simply.

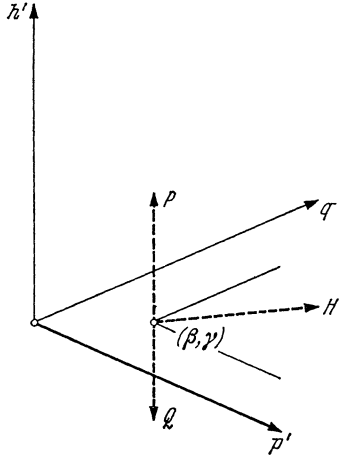


Fig. 3. Action of Lie algebra on a point (β, γ) in the (p', q') -plane, if $\delta = 0$. Action of P is up. Action of Q is down. Action of H is horizontal

The action of H moves a point in the (p', q') -plane. We get, in a small time dt ,

$$\begin{aligned} (\beta, \gamma) &\rightarrow (\beta, \gamma) + \beta H p' dt + \gamma H q' dt \\ &= (\beta, \gamma) + (-\gamma dt, \beta dt) \\ d\beta &= -\gamma dt \\ d\gamma &= \beta dt. \end{aligned}$$

The path is therefore determined by the equations

$$\frac{d\beta}{dt} = -\gamma(t) \quad \frac{d\gamma}{dt} = \beta(t)$$

whose solutions is an ellipse in all cases except $\beta = \gamma = 0$. The action of P and Q does not alter the value of (p', q') ; both change h' if p' or $q' \neq 0$. Thus the orbits, for $\delta = 0$, are elliptic cylinders, except for $p' = q' = 0$. If $p' = q' = 0$ there is one representation for each h' . The algebra subordinate to a point on an orbit containing $(o(p', q')o)$ is the Heisenberg algebra N , and gives a character on N . The resultant induced representation of G is therefore the same as that given by the Mackey theory, using the points in \underline{N}' such that $e' = 0$, p' and q' not both zero. If $p' = q' = 0$ the representation is the identity if $h' = 0$, and if $h' \neq 0$ we get the representations obtained from the point $e' = p' = q' = 0$ in \underline{N}' .

If now $\delta \neq 0$, we can discuss the orbits in the (p', q') plane, i.e. the orbits sliced by the plane $\alpha = \text{const.}$ Now, the value of α does not influence the shape of the orbits in the plane $\alpha = \text{const.}$ Therefore all the pictures of orbits, sliced by $\alpha = \text{const.}$, look the same. To find the shape of the orbits in the plane $\alpha = \text{const.}$, we must first find the subset of G that leaves the coefficient of h' the same, i.e. leaves us in the plane $\alpha = \text{const.}$ This will clearly contain H , since

$$H(\alpha h' + \beta p' + \gamma q' + \delta e') = \beta q' - \gamma p'.$$

The operators P and Q separately do change the fourth component, since $Qp' = -h'$ and $Pq' = h'$. But the point $\alpha h' + \beta p' + \gamma q' + \delta e'$ is left in the same plane α by $\beta P + \gamma Q$. This operator moves the point $\alpha h' + \beta p' + \gamma q' + \delta e'$ to the same point $+\delta(\beta q' - \gamma p')$, i.e. in the same direction as the operator H . Therefore the orbit in this plane is one-dimensional instead of two dimensional, and so are ellipses as before. If $\beta = \gamma = 0$, $\delta \neq 0$, then both P and Q infinitesimally leave us in the plane $\alpha = \text{const.}$, i.e. we have an infinitesimal 2-dimensional surface in the orbit. Therefore the orbit touches the plane $\alpha = \text{const.}$ at $\beta = \gamma = 0$, and we have the following picture: each orbit is a parabolic shape with elliptic section (drawn with $\delta > 0$). To see that the orbit is bounded below

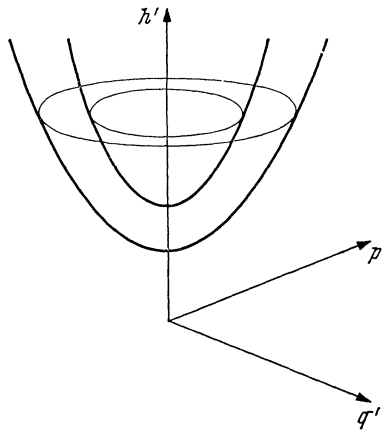


Fig. 4. Orbits if $\delta > 0$

in h' , note that if the coefficient γ of q' is > 0 , the action of P : $Pq' = h'$, $Pe' = q'$ increases both the q' -value and the h' -value, while if γ is < 0 the action of P decreases h' while q' is increased. Thus $\gamma = 0$ is a minimum. Similarly, if $\beta > 0$ the action $Qp' = -h'$, $Qe' = -p'$ shows that Q decreases h' and p' , and if $\beta < 0$, Q increases h' while decreasing p' . Thus $\beta = 0$ is a minimum also in this direction. Thus if $\delta \neq 0$ an orbit is

determined by the intersection (unique) with $\beta = \gamma = 0$, i.e. by the element (α, o, o, δ) .

If we now proceed with the Kirillov construction we find that we get a reducible representation of G , unless complex subalgebras are admitted. For, the dimension of a maximal algebra subordinate to (α, o, o, δ) is two if it is required to be real; for example, it could be (P, E) or (Q, E) ; but the representation obtained from the resulting character of such a sub-group must be reducible. For it can easily be seen not to be unitary equivalent to the only remaining class of irreducible representations (those with positive energy).

However, if we allow complex coefficients, there is a three dimensional sub-algebra subordinate to the functional (α, o, o, δ) . For if we define $A = P - iQ$, then the algebra generated by (H, A, E) is subordinate to (α, o, o, δ) , since $[H, A] = -Q - iP = -i(P - iQ)$; we make this choice if $\alpha < 0$. (Similarly one could use H, A^* and E if $\alpha > 0$). Then we have $[A, A^*] = 2iE$ where $A^* = P + iQ$. Thus we may consider the group to consist of elements $(\varphi, z, \bar{z}, t) = e^{\varphi E} e^{zA^*} e^{\bar{z}A} e^{tH}$ or more conveniently (φ, t, z, \bar{z}) with multiplication

$$\begin{aligned} (o, o, z', o) (\varphi, t, z, \bar{z}) &= e^{A^*z'} e^{E\varphi} e^{Ht} e^{A^*z} e^{A\bar{z}} \\ &= e^{E\varphi} e^{Ht} e^{A^* \theta z'} e^{A^*z} e^{A\bar{z}} \end{aligned}$$

where $\theta = e^{-it}$.

Therefore

$$(o, o, z', o) \cdot (\varphi, t, z, \bar{z}) = (\varphi, t, z + z' e^{-it}, \bar{z}).$$

Also

$$\begin{aligned} (o, o, o, \bar{z}') \cdot (\varphi, t, z, \bar{z}) &= e^{A\bar{z}'} e^{E\varphi} e^{Ht} e^{A^*z} e^{A\bar{z}} \\ &= e^{E\varphi} e^{Ht} e^{A\bar{z}'} e^{i\theta} e^{A^*z} e^{A\bar{z}} \\ &= e^{E\varphi} e^{Ht} e^{A^*z} e^{A(\bar{z} + \bar{z}' e^{i\theta})} e^{2iE\bar{z}'z e^{i\theta}} \\ &= (\varphi + 2i\bar{z}'z e^{i\theta}, t, z, \bar{z} + \bar{z}' e^{i\theta}). \end{aligned}$$

The post-multiplication is

$$\begin{aligned} (\varphi, t, z, \bar{z}) \cdot (o, t', o, o) &= e^{E\varphi} e^{Ht} e^{A^*z} e^{A\bar{z}} e^{Ht'} \\ &= \exp \varphi E \exp(H(t + t')) \exp(A^*z e^{-it'}) \exp(A\bar{z} e^{it'}). \end{aligned}$$

Therefore

$$(\varphi, t, z, \bar{z}) (o, t', o, o) = (\varphi, t + t', z e^{-it'}, \bar{z} e^{it'}).$$

We now induce up from the subgroup of (H, A, E) . The functional (α, o, o, δ) defines the following character on the group of this algebra:

$$\chi(\varphi, t, z, \bar{z}) = e^{i\varphi\alpha} e^{i\delta t}.$$

Thus the Hilbert space of U^z , the corresponding induced representation, consists of functions on the group, i.e. functions Ψ of φ, t, z, \bar{z} , satisfying

$$(i) \quad \Psi((\varphi', o, o, o) \cdot (\varphi, t, z, \bar{z})) = e^{i\varphi'\alpha} \Psi(\varphi, t, z, \bar{z})$$

i.e.

$$\Psi(\varphi + \varphi', t, z, \bar{z}) = e^{i\varphi'\alpha} \Psi(\varphi, t, z, \bar{z})$$

i.e.

$$\Psi(\varphi, t, z, \bar{z}) = e^{i\varphi\alpha} \Psi(o, t, z, \bar{z}).$$

Moreover

$$(ii) \quad \Psi((o, t', o, o) \cdot (\varphi, t, z, \bar{z})) = e^{it'\delta} \Psi(\varphi, t, z, \bar{z})$$

i.e.

$$\Psi(\varphi, t, z, \bar{z}) = e^{i\alpha\varphi} e^{i\delta t} \Psi(o, o, z, \bar{z}).$$

Moreover

$$(iii) \quad \Psi((o, o, z', o) (\varphi, t, z, \bar{z})) = \Psi(\varphi, t, z, \bar{z}); \quad \text{put } \varphi = t = o;$$

then we get

$$e^{2i^2\alpha\bar{z}'z} \Psi(o, o, z, \bar{z} + \bar{z}') = \Psi(o, o, z, \bar{z}).$$

Conditions (i), (ii) and (iii) are the 'left covariance' conditions.

The Haar measure on the group reduces to the invariant Liouville volume element $dz d\bar{z}$. From (iii)

$$\begin{aligned} \frac{\partial}{\partial \bar{z}'} \{ \Psi(z, \bar{z} + \bar{z}') e^{-2\alpha\bar{z}'z} \} &= \frac{\partial}{\partial \bar{z}'} \Psi(z, \bar{z}) = 0 \\ \therefore \frac{\partial \Psi}{\partial \bar{z}} e^{-2\alpha\bar{z}'z} + \Psi(z, \bar{z} + \bar{z}') (-2\alpha z) e^{-2\alpha\bar{z}'z} &= 0. \end{aligned}$$

Putting $\bar{z}' = 0$ we get the equation $\frac{\partial \Psi}{\partial \bar{z}} = 2\alpha z \Psi$, whose solution

$$\Psi(z, \bar{z}) = f(z) e^{2\alpha\bar{z}z}$$

where $f(z)$ is a holomorphic function of z . It is clear that this form is sufficient to satisfy all the conditions. The choice of α determines the scale of \hbar and the choice of δ adds a constant to the Hamiltonian⁷. The action of A is $\Psi(z, \bar{z})^{\exp A \bar{z}'} \rightarrow \Psi(z, \bar{z} + \bar{z}')$, which multiplies $f(z)$ by $e^{2\alpha z \bar{z}'}$ i.e. A is multiplication by $2\alpha z$. The action of $e^{z' A^*}$ takes $\Psi(z, \bar{z})$ to $e^{-2\alpha z' \bar{z}} \Psi(z + z', \bar{z})$, so that $f(z) = e^{-2\bar{z}z\alpha} \Psi(z, \bar{z})$ is transformed by post-multiplication into

$$e^{-2\alpha(z+z')\bar{z}} \Psi(z + z', \bar{z}) = f(z + z').$$

Therefore A^* is represented by $\frac{\partial}{\partial z}$. Thus we have arrived at the Bargmann-Segal representation of the harmonic oscillator by holomorphic functions, [4], [5] well known to coincide with the 'usual' representation, and therefore with the remaining irreducibles (depending on α and δ) given by the Mackey theory.

⁷ In fact, negative α correspond to positive \hbar ; if $\alpha > 0$, we must interchange the roles of A and A^* in order to obtain a convergent exponential.

We note that the action of time-displacement will be

$$\begin{aligned} \Psi(\varphi, t, z, \bar{z}) &\xrightarrow{t'} \Psi((\varphi, t, z, \bar{z}) (o, t', o, o)) \\ &= \Psi(\varphi, t + t', ze^{-it'}, \bar{z}e^{it'}) \\ &= e^{i\delta(t+t')} e^{i\alpha\varphi} e^{2z\bar{z}z} f(ze^{-it'}). \end{aligned}$$

Therefore the eigenstates are found to be functions for which $f(ze^{-it'}) = e^{-int'} f(z)$ i.e. the homogeneous polynomials of degree n , and the eigenvalues are $n + \delta$. The fact that these eigenvalues are positive (i.e. bounded below) is ensured by the fact that the corresponding Kirillov orbit, shown in the figure, is bounded below by δ . Thus the orbits give a simple criterion for determining the spectrum of the associated operators in that representation.⁸

The fact that the left-covariance conditions sometimes lead to holomorphy in certain variables was discovered in other special cases by GELFAND and GRAEV [6]. The procedure of the present paper has been termed 'holomorphic' induction by DIXMIER, and forms part of a general result of KOSTANT who showed the existence of holomorphic sections for any Lie group; and gives the measure on the cosetspace. For the general solvable Lie group there is as yet no proof that all the representations are given by orbits, and this would seem unlikely in view of the existence of solvable groups not of type I . It can be shown by explicit computation that all the representations of $0(2, 1)$ correspond to orbits (S. DUNNE, private communication), except that the supplementary series is not obtained directly, but requires analytic continuation from the principal series.

Conclusion

We have found that for a certain solvable Lie group, the oscillator group, the exponential map is not onto, and so the theorem of BERNAT giving all the representations is not necessarily true; that is, the Kirillov prescription need not give all the representations; in fact, restricting ourselves to real Lie algebras, the Kirillov theory fails. However, if we allow complex Lie algebras, one does arrive at all the representations, as is proved by comparing with the Mackey theory. The latter gives all the representations since the group in question is a regular semi-direct product. Applying the Kirillov technique leads directly to the holomorphic function realization of the harmonic oscillator.

⁸ For nilpotent groups KIRILLOV has proved that the spectrum of an operator X in the Lie algebra, in a representation corresponding to an orbit, is contained in the projection of the orbit onto X' , the dual to X . This result also holds for O_3 and the oscillator group, as is seen explicitly.

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