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Fields at a Point

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Abstract. Free fields and Wick products without smearing are studied as operators in a nested Hilbert space. It is shown that Wick products are entire analytic in complex space-time and that products of field operators are holomorphic in the forward tube. The Poincaré group is represented by unitary automorphisms of the Fock space.

I. Introduction

A quantized field at a point cannot be a reasonable operator in the Hilbert space of states [1]. It is known, however, that free fields at a point are mappings from a suitable topological vector space into its dual [2]; one expects therefore that they are operators in some nested Hilbert space[1].

It will be shown here that this is indeed so. The construction is uneventful and involves only Hilbert spaces that are very similar to the "physical" one. The norms in these spaces are *not* Lorentz invariant. Nevertheless, the Poincaré group is represented by unitary automorphisms of the "Fock nested Hilbert space".

Some of the results are stated below:

Theorem. *Let X be the positive hyperboloid of mass $M \geqq 0$, and let μ be the Lorentz invariant measure on X. Denote by $H_I^{(1)}(X;\mu)$ the nested Hilbert space corresponding to $(X;\mu)$ (see Section 2a) and by $T_I(X;\mu)$ the tensor algebra over $H_I^{(1)}(X;\mu)$ (see Section 2b). Then:*

(i) *For any ν space-time points $x_1, \ldots, x_\nu$ $(\nu \geqq 1)$ the Wick product $:A(x_1)\ldots A(x_\nu):$ belongs to $L(T_I; T_I)$. In particular, the free[2] field operator $A(x)$ belongs to $L(T_I; T_I)$ for every x.*

(ii) *The operator family $:A(x_1)\ldots A(x_\nu):$ is the restriction, to real x, of a family $:A(z_1)\ldots A(z_\nu):$ which is entire analytic (in the sense described in Section 2g) in the arguments $z_1, \ldots z_\nu$.*

(iii) *The product $A(z_1)\ldots A(z_\nu)$ (without Wick ordering) is holomorphic in the domain*

$$\operatorname{Im}(z_j - z_{j-1}) \in V_+ \quad (j = 2, \ldots, \nu)$$

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[1] We shall use the terminology and notation of [3] and [4].

[2] For the sake of simplicity, we consider only neutral scalar fields.

in the variables $z_1, \ldots z_\nu$. *Here* V_+ *is the interior of the forward light cone.*

(iv) *The usual representation* $\{a, \Lambda\} \to U_{\underline{0}\underline{0}}(a, \Lambda)$ *of the Poincaré group is the* $\{\underline{0}, \underline{0}\}$*-representative of a representation* $\{a, \Lambda\} \to U(a, \Lambda)$ *of the Poincaré group by unitary automorphisms of* T_I. *Consequently* $U(a, \Lambda)\, A(x)\, U^{-1}(a, \Lambda)$ *is defined. One has*

$$U(a, \Lambda)\, A(x)\, U^{-1}(A, \Lambda) = A(\Lambda x + a)\ .$$

These statements will be derived from the more general or more detailed propositions to be proved below.

II. Proofs

a) The space $H_I(X; \mu)$

Let μ be a positive Radon measure on a locally compact space X.

Denote by H_0 the Hilbert space $L^{(2)}(X; \mu)$ of (classes of) complex-valued functions defined on X, measurable and square integrable with respect to μ, with the obvious scalar product.

Let $r(k)$ $(k \in X)$ be a *continuous* function defined on X taking *strictly positive* values. Denote by I the set of all such functions with the natural partial order: $r \geqq p$ means $r(k) \geqq p(k)$ for all $k \in X$. Define in I the order-reversing involution $r \leftrightarrow \bar{r}$ where $\bar{r}(k) = 1/r(k)$. The reader should keep in mind this somewhat unusual notation.

For every $r \in I$, consider the measure $r^{-2}\mu$ (the product of the continuous function $r^{-2}(k) = \bar{r}^2(k)$ and of the measure μ) Denote by H_r the Hilbert space $L^{(2)}(X; r^{-2}\mu)$.

If $r \geqq p$, then $H_r \supseteqq H_p$; the natural embedding of H_p into H_r is a nesting E_{rp} which satisfies

$$\|E_{rp}\| \leqq 1\ . \tag{2.1}$$

2.1. Proposition. *The algebraic inductive limit of the Hilbert spaces* H_r $(r \in I)$ *with respect to the natural embeddings is a nested Hilbert space which will be denoted by* $H_I(X; \mu)$.

Proof. The verification of (NH_1) of Section 3a of [3] is immediate. (Take $p(k) = \min(r(k), q(k))$. In order to see that (NH_2) holds, notice first that the adjoint of E_{rp} is the operator of multiplication by $p^2(k)/r^2(k)$. Define $u_{\bar{r}r}$ as the operator of multiplication by $r^{-2}(k)$. It is a unitary map from H_r onto $H_{\bar{r}}$ which satisfies the equation (3.2b) of [3] q.e.d.

b) The spaces $H_I^{(n)}(X; \mu)$

Let X and μ be as above. Define $H_I^{(1)}(X;\mu)$ as the space $H_I(X;\mu)$ of Proposition 2.1. For $n > 1$, define $H_I^{(n)}(X;\mu)$ as $H_I(X^n, \mu^n)$ where (X^n, μ^n) is the cartesian product of n copies of X, with the measure $\mu \times \cdots \times \mu$. The set $I^{(n)}$ consists of all positive continuous functions

$r^{(n)}(k) = r^{(n)}(k_1 \ldots k_n)$ $(k_i \in X, i = 1, 2 \ldots n)$. Notice that $o^{(n)} \in I^{(n)}$ is identically equal to one.

If $r^{(m)} \in I^{(m)}$ and $q^{(n)} \in I^{(n)}$, define $r^{(m)} \times q^{(n)}$ by

$$(r^{(m)} \times q^{(n)})(k_1 \ldots k_{m+n}) = r^{(m)}(k_1 \ldots k_m)\, q^{(n)}(k_{m+1} \ldots k_{m+n}) \,. \tag{2.2}$$

Then

$$H^{(m+n)}_{r \times q} = H^{(m)}_r \otimes H^{(n)}_q \tag{2.3}$$

where $\otimes$ denotes the (completed) tensor product of Hilbert spaces.

Furthermore: If $z^{(m)} \geqq r^{(m)}$ (in $I^{(m)}$) and if $s^{(n)} \geqq q^{(n)}$ (in $I^{(n)}$) then $z^{(m)} \times s^{(n)} \geqq r^{(m)} \times q^{(n)}$ (in $I^{(m+n)}$) and

$$E^{(m+n)}_{z \times s\, r \times q} = E^{(m)}_{zr} \otimes E^{(n)}_{sq} \,. \tag{2.4}$$

Here $\otimes$ denotes the tensor product of operators; the $E^{(n)}$-s are the natural embeddings between the Hilbert spaces of $H^{(n)}_I$.

The nested Hilbert space $H^{(0)}_I$ ("the vacuum state") is one-dimensional; the set $I^{(0)}$ consists of all positive numbers $r^{(0)}$, and the Hilbert space $H^{(0)}_r$ is the set of complex numbers with the scalar product $\{z, w\} \to$ $\to z^* w/(r^{(0)})^2$.

c) *The space T_I*

2.2. Proposition. *The spaces $H^{(n)}_I$ satisfy the conditions (DS_1), (DS_2) of* [4] *so that the direct sum $\bigoplus_{n=0}^{\infty} H^{(n)}_I$ is defined.*

Proof. (a) The condition (DS_1) follows from (2.1), which is valid for all n.

(b) In order to verify (DS_2), define $p^{(n)} \leqq r^{(n)}, q^{(n)}$ by

$$p^{(n)}(k) = \min\,[r^{(n)}(k), q^{(n)}(k)]$$

for every k. Let $E^{(n)}_{[rq]p}$ be the operator which maps every $f^{(n)}_p \in H^{(n)}_p$ into

$$E^{(n)}_{[rq]p} f^{(n)}_p = \begin{pmatrix} E^{(n)}_{rp} f^{(n)}_p \\ E^{(n)}_{qp} f^{(n)}_p \end{pmatrix} \tag{2.5}$$

By Theorem 3.9 of [3] this operator is bijective. Now it will be shown that the bound norm of its inverse does not exceed one, independently of n; this will verify the condition (DS_2). It is sufficient to show that $E^{(n)}_{[rq]p}$ increases all norms. The square of the norm of the vector (2.5) is

$$(f^{(n)}_r, f^{(n)}_r) + (f^{(n)}_q, f^{(n)}_q) = \int \{[r^{(n)}(k)]^{-2} + [q^{(n)}(k)]^{-2}\}\, |f(k)|^2\, d\mu(k) \geqq$$
$$\geqq \int [p^{(n)}(k)]^{-2}\, |f(k)|^2\, d\mu(k) = (f^{(n)}_p, f^{(n)}_p) \,.$$

This proves the proposition.

Define $T_I = T_I(X; \mu)$ as the direct sum

$$T_I = \bigoplus_{n=0}^{\infty} H^{(n)}_I \,.$$

In accordance with the notations of [4], denote by $\mathcal{J}^{(n)}$ the natural embedding from $H_I^{(n)}$ into T_I and by $\mathcal{P}^{(n)}$ the adjoint of $\mathcal{J}^{(n)}$ which projects T_I onto $H_I^{(n)}$. A vector $f \in T_I$ can be identified to a sequence $\{f^{(0)}, \ldots f^{(n)}, \ldots\}$ where $f^{(n)} = \mathcal{P}^{(n)} f$ is the "n-particle component" of f. We shall now show that, — under our assumptions — there are no restrictions on $f^{(n)}$ as $n \to \infty$. This is due to the fact that the set $I^{(n)}$ contains, together with any $q^{(n)}(k)$, also every multiple $\lambda q^{(n)}(k)$ $(\lambda > 0)$ and is in contrast with the familiar situation in the case of Hilbert spaces.

2.3. Proposition. *Let* $f^{(0)}, f^{(1)}, \ldots f^{(m)}, \ldots$ *be arbitrary;* $(f^{(m)} \in H_I^{(m)}$, $m = 0, 1, 2, \ldots)$. *Then there exists an* $\underline{r} \in \underline{I}$ *such that*

$$\sum_{m=0}^{\infty} (f_r^{(m)}, f_r^{(m)}) < \infty . \tag{2.6}$$

It follows by Proposition 3.2 *of* [4] *that* $f = \{f^{(0)}, \ldots f^{(m)}, \ldots\}$ *belongs to* T_I.

Proof. For every m, let $q^{(m)} \in J(f^{(m)})$ so that the representative $f_q^{(m)}$ exists.

Let $\sum_m \delta_m$ be any convergent series of positive numbers. Define $r^{(m)}(k)$ by

$$[r^{(m)}(k)]^{-2} = \delta_m [(f_q^{(m)}, f_q^{(m)})]^{-1} [q^{(m)}(k)]^{-2} .$$

Then

$$\begin{aligned}\sum_m (f_r^{(m)}, f_r^{(m)}) &= \sum_m \int [r^{(m)}(k)]^{-2} |f^{(m)}(k)|^2 \, d\mu(k) \\ &= \sum_m \delta_m [(f_q^{(m)}, f_q^{(m)})]^{-1} \int [q^{(m)}(k)]^{-2} |f^{(m)}(k)|^2 \, d\mu(k) \\ &= \sum_m \delta_m < \infty\end{aligned}$$

which proves the proposition.

d) The operator $|f^{(m)}\rangle^{(I)}$

This section is devoted to the operator $|f^{(m)}\rangle^{(I)}$ which acts in T_I and "creates m particles described by the vector $f^{(m)} \in H_I^{(m)}$". (Usually one has $m = 1$.) Symmetrization will be discussed later.

Our main result concerns the set $J(|f^{(m)}\rangle^{(I)})$ which describes the "goodness" of the operator $|f^{(m)}\rangle^{(I)} \in L(T_I; T_I)$. It reads:

2.4. Proposition. *Let* $f^{(m)} \in H_I^{(m)}$ *be arbitrary. Let* $\underline{q} \in \underline{I}$ *be arbitrary and let* $r^{(m)}$ *be any element of* $J(f^{(m)})$. *(That is,* $r^{(m)} \in I^{(m)}$ *is such that the vector* $f^{(m)} \in H_I^{(m)}$ *has a representative in the Hilbert space* $H_r^{(m)}$*). Define* $\underline{s} \in \underline{I}$ *by*

$$\begin{aligned} &s^{(0)}, s^{(1)} \ldots s^{(m-1)} \quad \textit{arbitrary} \\ &s^{(m+n)} = r^{(m)} \times q^{(n)} \quad (n = 0, 1, 2, \ldots) . \end{aligned} \tag{2.7}$$

Then $\{\underline{q}, \underline{s}\} \in J(|f^{(m)}\rangle^{(I)})$. *In particular,*

$$D(|f^{(m)}\rangle^{(I)}) = \underline{I} . \tag{2.8}$$

The intuitive meaning of this is very simple: Starting with any state in $T_{\underline{I}}$, one can create additional particles in arbitrary states. One obtains, however, a state that is "no better than its ingredients".

In order to prove the proposition, one must of course give a precise definition of $|f^{(m)}\rangle^{(\underline{I})}$. This is done in three steps:

1°: Given any $q^{(n)} \in I^{(n)}$ and any vector $f_r^{(m)}$ in the Hilbert space $H_r^{(m)}$, consider the mapping $|f_r^{(m)}\rangle_{r\times q}^{(m+n;n)}{}_q$ (from $H_q^{(m)}$ into $H_{r\times q}^{(m+n)}$) defined by

$$|f_r^{(m)}\rangle_{r\times q}^{(m+n;n)}{}_q\, g_q^{(n)} = f_r^{(m)} \otimes g_q^{(n)} \quad (g_q^{(n)} \in H_q^{(n)})\,. \tag{2.9}$$

It is clear from (2.9) that the bound norm of $|f_r^{(m)}\rangle_{r\times q}^{(m+n;n)}{}_q$ is $\|f_r^{(m)}\|$, independently of n and of $q^{(n)}$.

2°: Let $f^{(m)}$ be any vector in the nested Hilbert space $H_{\underline{I}}^{(m)}$. Then, for every n, there exists an operator $|f^{(m)}\rangle^{(n)} \in L(H_{\underline{I}}^{(n)}; H_{\underline{I}}^{(m+n)})$ such that, for every $r^{(m)} \in J(f^{(m)})$ and for every $q^{(n)} \in I^{(n)}$, the representative of $|f^{(m)}\rangle^{(n)}$ between $H_q^{(n)}$ and $H_{r\times q}^{(m+n)}$ is $|f_r^{(m)}\rangle_{r\times q}^{(m+n;n)}{}_q$. Here $f_r^{(m)}$ is the representative of $f^{(m)}$ in $H_r^{(m)}$.

In order to prove this statement, let $r^{(m)} \in J(f^{(m)})$, $\hat{r}^{(m)} \in J(f^{(m)})$, $q^{(n)} \in I^{(n)}$ and $\hat{q}^{(n)} \in I^{(n)}$. Let $s^{(n)} \geqq q^{(n)}, \hat{q}^{(n)}$; let $z^{(m)} \geqq r^{(m)}, \hat{r}^{(m)}$ and $p^{(n)} \leqq q^{(n)}, \hat{q}^{(n)}$. By (2.4) and (2.9) one has then

$$E_{z\times s}^{(m+n)}{}_{r\times q}\, |f_r^{(m)}\rangle_{r\times q}^{(m+n;n)}{}_q\, E_{qp}^{(n)} = E_{z\times s}^{(m+n)}{}_{\hat{r}\times\hat{q}}\, |f_{\hat{r}}^{(m)}\rangle_{\hat{r}\times\hat{q}}^{(m+n;n)}{}_{\hat{q}}\, E_{\hat{q}p}^{(n)} \quad \text{q.e.d.}$$

We have just shown that the set $J(|f^{(m)}\rangle^{(n)})$ contains all the pairs $\{q^{(n)}, r^{(m)} \times q^{(n)}\}$ where $q^{(n)} \in I^{(n)}$ is arbitrary and where $r^{(m)} \in J(f^{(m)})$.

3°: The operator $|f^{(m)}\rangle^{(\underline{I})} \in L(T_{\underline{I}}; T_{\underline{I}})$ is now defined by

$$|f^{(m)}\rangle^{(\underline{I})} = \sum_{n=0}^{\infty} \mathscr{I}^{(m+n)}\, |f^{(m)}\rangle^{(n)}\, \mathscr{P}^{(n)}\,. \tag{2.10}$$

The operators $\mathscr{I}$ and $\mathscr{P}$ have been defined in Section 2c; their properties were studied in [4]. The assertion of Proposition 2.4 follows now immediately from the definition (2.10).

Remark. If ${}^{(\underline{I})}\langle f^{(m)}|$ denotes the adjoint of $|f^{(m)}\rangle^{(\underline{I})}$ then Proposition 2.4, (together with the general results of [3]) shows that the product $|f^{(m)}\rangle^{(\underline{I})}\,{}^{(\underline{I})}\langle g^{(m)}|$ is defined for any two $f^{(m)} \in H_{\underline{I}}^{(m)}$, $g^{(m)} \in H_{\underline{I}}^{(m)}$, but that the product ${}^{(\underline{I})}\langle g^{(m)}\,|\,f^{(m)}\rangle^{(\underline{I})}$ need to be defined. This is already essentially the well-known result [2] that Wick products are "better" than others. The picture is not changed by symmetrization and by normalization, as will be seen below.

Remark. If $f = \sum_m \mathscr{I}^{(m)} f^{(m)} \in T_{\underline{I}}$ is arbitrary, one can study the operator $|f\rangle^{(\underline{I})} = \sum_m |f^{(m)}\rangle^{(\underline{I})}$ and show that it satisfies $D(|f\rangle^{(\underline{I})}) = \underline{I}$. So $T_{\underline{I}}$ is actually an algebra, in contrast again to the case of Hilbert spaces.

e) Functions of the number operator

Since the space $T_{\underline{I}}$ contains all the sums $\sum_{n=0}^{\infty} \mathscr{I}^{(n)} f^{(n)}$ ($f^{(n)} \in H_{\underline{I}}^{(n)}$) (see Proposition 2.3) it is clear that the "particle number operator" N is defined on all of $T_{\underline{I}}$. A slightly more precise statement will be proved below.

Let $\varphi(n)$ be a sequence satisfying $\varphi(n) \geqq 1$ for $n = 0, 1, 2, \ldots$. If $\underline{r} \in \underline{I}$ is arbitrary, define $\varphi \underline{r} \in \underline{I}$ by

$$(\varphi r)^{(n)}(k) = \varphi(n)\, r^{(n)}(k) \quad (n = 0, 1, 2, \ldots)\,.$$

Notice that $\varphi \underline{r} \geqq \underline{r}$. Consequently $E^{(n)}_{\varphi r\, r}$ is defined for all n.

2.5. Proposition. *Let φ be as above. Then there exists a unique $\varphi(N) \in$ $\in L(T_{\underline{I}}; T_{\underline{I}})$ such that $\mathscr{P}^{(n)} \varphi(N) \mathscr{I}^{(n)} = \varphi(n)\, 1^{(n)}$ for $n = 0, 1, 2, \ldots$. For every $\underline{r} \in \underline{I}$, the pair $\{\underline{r}, \varphi \underline{r}\}$ belongs to $J(\varphi(N))$. In particular, $D(\varphi(N)) = \underline{I}$ and $R(\varphi(N)) = \underline{I}$.*

Proof. (a) For every n and for every $r^{(n)} \in I^{(n)}$ one has

$$\|\varphi(n) E^{(n)}_{\varphi r\, r}\| = 1\,. \tag{2.11}$$

Indeed, one has

$$\|E^{(n)}_{\varphi r\, r} f_r^{(n)}\| = [\varphi(n)]^{-1} \{\textstyle\int [r^{(n)}(k)]^{-2}\, |f^{(n)}(k)|^2\, d^n \mu(k)\}^{1/2} = [\varphi(n)]^{-1}\, \|f_r^{(n)}\|$$

for every $f_r^{(n)} \in H_r^{(n)}$.

(b) Consider the orthogonal sum $\bigoplus_{n=0}^{\infty} \varphi(n) E^{(n)}_{\varphi r\, r}$ of operators. By (2.11), it is a bounded operator from $H_{\underline{r}}$ into $H_{\varphi \underline{r}}$, to be denoted by $(\varphi(N))_{\varphi \underline{r}\, \underline{r}}$.

(c) Verify that the operator $\varphi(N)$, defined by (b) satisfies the conditions of the proposition.

f) Symmetrization

Let π be any permutation of $1, 2, \ldots n$. For every $r^{(n)} \in I^{(n)}$, denote by $\pi r^{(n)}$ the element of $I^{(n)}$ defined by

$$(\pi r^{(n)})\,(k_1, \ldots, k_n) = r^{(n)}(k_{\pi 1}, \ldots, k_{\pi n})\,.$$

Denote by $\Pi^{\pi}_{\mathrm{o\,o}}$ the unitary mapping of $H_{\mathrm{o}}^{(n)}$ onto itself defined by

$$(\Pi^{\pi}_{\mathrm{o\,o}} f)\,(k_1, \ldots, k_n) = f(k_{\pi 1}, \ldots, k_{\pi n}) \quad (f \in H_{\mathrm{o}}^{(n)})\,.$$

Let $\Pi^{\pi} \in L(H_I^{(n)}; H_I^{(n)})$ be the operator having $\Pi^{\pi}_{\mathrm{o\,o}}$ as $\{\mathrm{o}, \mathrm{o}\}$-representative.

2.6. Proposition. *The operator Π^{π} is a unitary automorphism of $H_I^{(n)}$.*

Proof. (a) Verify that, for every $r^{(n)} \in I^{(n)}$, the representative $\Pi^{\pi}_{\pi r\, r}$ exists and is unitary. (The superscript n has been omitted for the sake of typographical simplicity.)

(b) Notice that $\pi \bar{r} = \overline{\pi r}$ for every $r \in I^{(n)}$. It follows that

$$\Pi^{\pi} \in \mathrm{Hom}\,(H_I^{(n)}, H_I^{(n)})$$

q.e.d.

Remark. The set $J(\Pi^\pi)$ does *not* contain, in general, the diagonal of $I^{(n)} \times I^{(n)}$. If $\pi r^{(n)} \neq r^{(n)}$ then $J(\Pi^\pi)$ need not contain the pair $\{r^{(n)}, r^{(n)}\}$.

Define now $S^{(n)} \in L(H_I^{(n)}; H_I^{(n)})$ by

$$S^{(n)} = \frac{1}{n!} \sum_\pi \Pi^\pi ,$$

so that $S_{0\,0}^{(n)}$ is the usual symmetrization operator in $H_0^{(n)}$. Notice that $S^{(n)}$ is in general *not* a homomorphism.

The symmetrization operator in $T_{\underline{I}}$ is defined by

$$S = \sum_n \mathscr{I}^{(n)} S^{(n)} \mathscr{P}^{(n)} .$$

2.7. Proposition. *Let $\underline{r} \in \underline{I}$ be arbitrary. Let $\underline{s} \in \underline{I}$ be such that, for every n, $s^{(n)}$ is a common successor of the $n!$ elements $\pi r^{(n)}$. Then the pair $\{\underline{r}, \underline{s}\}$ belongs to $J(S)$. In particular, $D(S) = \underline{I}$.*

Proof. (a) Verify that, for every n, the representative $S_{sr}^{(n)}$ exists and satisfies

$$\|S_{sr}^{(n)}\| \leqq 1 . \tag{2.12}$$

(b) It follows from (2.12) that the orthogonal sum $\bigoplus_{n=0}^{\infty} S_{sr}^{(n)} = S_{\underline{s}\underline{r}}$ is a bounded operator from $H_{\underline{r}}$ into $H_{\underline{s}}$ q.e.d.

Remark. The repetition of a well-known argument [5] shows that

$$S\,|f^{(m)}\rangle^{(I)} S = S\,|f^{(m)}\rangle^{(I)} . \tag{2.13}$$

It is clear that S commutes with $\varphi(N)$.

g) The operators $a^+(f^{(m)})$ and $a^-(f^{(m)})$

Since $D(|f^{(m)}\rangle^{(I)}) = D(S) = D(\varphi(N)) = \underline{I}$, the product

$$a^+(f^{(m)}) = S\,|f^{(m)}\rangle^{(I)}\, \varphi(N) S = S\,|f^{(m)}\rangle^{(I)}\, \varphi(N) \in L(T_{\underline{I}};\, T_{\underline{I}})$$

is defined for every $f^{(m)} \in H_I^{(m)}$. Its "goodness" is described by

2.8. Proposition. *Let $\underline{q} \in \underline{I}$ be arbitrary and let $r^{(m)} \in J(f^{(m)})$. Define $\underline{u} \in \underline{I}$ by*

$$u^{(0)}, \ldots, u^{(m-1)} \quad \text{are arbitrary}$$

$$u^{(m+n)} = \varphi(n)\,(r^{(m)} \times q^{(n)}) \quad (n = 0, 1, 2, \ldots) .$$

Let $\underline{s} \in \underline{I}$ be such that, for $l = 0, 1, 2, \ldots$ and for every permutation π, $s^{(l)} \geqq \pi u^{(l)}$. Then

$$\{\underline{q}, \underline{s}\} \in J(a^+(f^{(m)})) \tag{2.14}$$

and

$$\|a_{\underline{s}\underline{q}}^+(f^{(m)})\| \leqq \|f_r^{(m)}\| . \tag{2.15}$$

Proof. The assertions are obtained by a straightforward application of results of Sections 2a—2f.

It is important to notice that the definition of $\underline{u}$ involves the products

$$r^{(m)}(k_1, \ldots, k_m)\, q^{(n)}(k_{m+1}, \ldots, k_{m+n})$$

in which the arguments of $r^{(m)}$ are different from the arguments of $q^{(n)}$. If $r^{(m)}$ is "large" in $I^{(m)}$ (e.g. if it grows fast at infinity) then no choice of $\underline{q}$ can make $\underline{u}$ arbitrarily small in $\underline{I}$. If $f^{(m)}$ is "singular", then $r^{(m)}$ has to be "large" since the set $J(f^{(m)})$ does not extend "far to the left" in $I^{(m)}$.

The correspondence $f^{(m)} \to a^+(f^{(m)})$ is obviously linear. In order to discuss further its properties, we need the concept of integrable family of vectors (or operators) in a nested Hilbert space.

Let Ω be a measure space (not to be confused with X). For almost every $x \in \Omega$, let $f(x)$ (resp. $A(x)$) be a vector (resp. an operator) in a nested Hilbert space H_I. (The space H_I need not be of the special kind discussed in this paper.) We shall say that the family $f(x)$ (resp. $A(x)$) is *integrable* if there exists *at least one* $r \in I$ (resp. at least one pair $\{r, r'\}$) such that the family of representatives $f_r(x)$ (resp. $A_{r'r}(x)$) is integrable— considered as a function from x to the Hilbert space H_r (resp. the Banach space $L(r, r')$).

If $f(x)$ is integrable, then the integral $\int f(x)\, d\Omega(x)$ is defined by

$$\int f(x)\, d\Omega(x) = E_{Ir} \int f_r(x)\, d\Omega(x)\, .$$

It is unique.

We return now to the operators $a^+(f^{(m)})$.

Let $f^{(m)}(x)$ be an integrable family of vectors in $H_I^{(m)}$. Then the operator family $a^+(f^{(m)}(x))$ is integrable and

$$\int a^+(f^{(m)}(x))\, d\Omega(x) = a^+(\int f^{(m)}(x)\, d\Omega(x))\, . \tag{2.16}$$

This assertion is an immediate consequence of (2.15).

We consider next families of operators depending on $z \in C^\varkappa$ i.e. on $\varkappa$ complex arguments ($\varkappa \geqq 1$).

Let H_I and $F_{I'}$ be nested Hilbert spaces and $B(\hat{z})$ a family[3] of elements of $L(H_I; F_{I'})$. For each $z_1 \in C^\varkappa$, define a (possibly empty) subset $J^{an}(B(z_1)) \subseteqq J(B(z_1)) \subseteqq I \times I'$ by:

A pair $\{r, r'\}$ belongs to $J^{an}(B(z_1))$ if and only if there exists an open neighbourhood $O(z_1)$ of z_1 in $C^\varkappa$ such that

1°: $\{r, r'\} \in J(B(z))$ for every $z \in O(z_1)$.

2°: The family $B_{r'r}(\hat{z})$ ($\hat{z} \in O(z_1)$) of representatives — which is a function from $O(z_1)$ to the fixed Banach space $L(r, r')$ — is holomorphic in $O(z_1)$.

The family $B(\hat{z})$ is said to be holomorphic at the point z_1 if $J^{an}(B(z_1))$ is not empty. It is called holomorphic in an open set $\Delta \subseteqq C^\varkappa$ if it is holomorphic at every point of Δ.

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[3] Dummy variables have carets above them if there is a possibility of confusion.

Notice that we do not require the existence of one pair $\{r, r'\}$ such that $B_{r'r}(\hat{z})$ be holomorphic in all of Δ. The existence of such a pair is obvious only if Δ can be covered by a finite number of sets $O(z)$ considered above.

The principle of analytic continuation holds: A holomorphic family is determined by its values in any open set.

The definition of holomorphic vector families is of course entirely analogous.

Concerning the product of ν holomorphic operator families we have the following statement

2.9. Proposition. *Let $B^{(1)}(\hat{z}_1), \ldots, B^{(\nu)}(\hat{z}_\nu)$ be families of operators belonging*[4] *to $L(H_I; H_I)$. Let $z_1, \ldots, z_\nu$ be points in $C^\varkappa$. Assume that there exist in I elements $q_1, \ldots, q_{\nu+1}$ such that*

$$\{q_{j+1}, q_j\} \in J^{an}(B^{(j)}(z_j)) \quad (j = 1, \ldots \nu)\,.$$

Then the product of ν factors $B^{(1)}(z_1) \ldots B^{(\nu)}(z_\nu)$ is defined and the family $B^{(1)}(\hat{z}_1) \ldots B^{(\nu)}(\hat{z}_\nu)$ is holomorphic at the point $\{z_1, \ldots, z_\nu\} \in C^{\varkappa\nu}$.

Proof. It is sufficient to notice that the cartesian product of open neighbourhoods $O(z_j)$ is an open neighbourhood in $C^{\varkappa\nu}$.

We return now again to the operators $a^+(f^{(m)})$.

2.10. Proposition. *Let $f^{(m;z)}$ $(z \in \Delta \subseteqq C^\varkappa)$ be a holomorphic family of vectors. Then the operator family $a^+(f^{(m;z)})$ is also holomorphic in Δ. If $z_1 \in \Delta$, if $r^{(m)} \in J^{an}(f^{(m;\,z_1)})$ and if $\{\underline{q}, \underline{s}\}$ is defined as in Proposition* 2.8, *then $\{\underline{q}, \underline{s}\} \in J^{an}(a^+(f^{(m;\,z_1)}))$.*

Proof. The correspondence $z \to a^+_{\underline{s}\underline{q}}(f^{(m;z)})$ is the product of the holomorphic map $z \to f_r^{(m;z)}$ (from $O(z_1)$ into $H_r^{(m)}$) and of the bounded linear map $f_r^{(m)} \to a^+_{\underline{s}\underline{q}}(f^{(m)})$ (from $H_r^{(m)}$ into $L(\underline{q}, \underline{s})$).

It is convenient to define $a^-(f^{(m)})$ in such a way that $a^-(f^{(m;z)})$ is analytic (rather than antianalytic) in z whenever $f^{(m;z)}$ is.

For every $f^{(m)} \in H_I^{(m)}$ define $a^-(f^{(m)})$ by

$$a^-(f^{(m)}) = (a^+(f^{(m)*}))^* \tag{2.17}$$

where the function $f^{(m)*}(k)$ is the complex conjugate of $f^{(m)}(k)$ and where $(a^+)^*$ is the adjoint of a^+.

Notice that $J(f^{(m)*}) = J(f^{(m)})$ since the norm in any $H_r^{(m)}$ involves only $|f^{(m)}(k)|^2$.

It is seen from Proposition 2.8 that $D(a^+(f^{(m)})) = \underline{I}$ for every $f^{(m)} \in H_I^{(m)}$ but that $R(a^+(f^{(m)})) \neq \underline{I}$ in general. Consequently (see [3]) one has $R(a^-(f^{(m)})) = \underline{I}$ but $D(a^-(f^{(m)})) \neq \underline{I}$ in general. So: If $f^{(m)}$ and $g^{(m)}$ are any two vectors in $H_I^{(m)}$ then the products $a^+(f^{(m)})\, a^-(g^{(m)})$, $a^+(f^{(m)})\, a^+(g^{(m)})$ and $a^-(f^{(m)})\, a^-(g^{(m)})$ are all defined. The product $a^-(f^{(m)})\, a^+(g^{(m)})$ need not be defined.

[4] The consideration of operators between spaces would only complicate notations.

We shall need more detailed information about $J(a^-(f^{(m)}))$ in order to study the existence and analyticity of multiple products.

2.11. Proposition. *Let $\underline{q} \in \underline{I}$ be arbitrary and let $r^{(m)} \in \bar{J}(f^{(m)})$. Define $\underline{u} \in \underline{I}$ by*

$$u^{(0)}, \ldots u^{(m-1)} \quad \text{are arbitrary}$$

$$u^{(m+n)} = (\varphi(n))^{-1}\,(r^{(m)} \times q^{(n)})\,.$$

Let $\underline{s} \in \underline{I}$ be such that, for $l = 0, 1, 2, \ldots$ and for every permutation π, $s^{(l)} \leqq \pi u^{(l)}$. Then

$$\{\underline{s}, \underline{q}\} \in J(a^-(f^{(m)}))$$

and

$$\|a^-_{\underline{q}\,\underline{s}}(f^{(m)})\| \leqq \|f^{(m)}_{\underline{r}}\|\,.$$

Proof. Since $J(f^{(m)}*) = J(f^{(m)})$, the assertions follow from Proposition 2.8 and the general results on adjoints (see [3]) with the substitutions $\underline{r} \leftrightarrow \bar{\underline{r}}$, $\underline{u} \leftrightarrow \bar{\underline{u}}$ and $\underline{s} \leftrightarrow \bar{\underline{s}}$.

One sees, just as in Proposition 2.10, that $r^{(m)} \in \bar{J}^{an}(f^{(m;\hat{z})})$ gives $\{\underline{s}, \underline{q}\} \in J^{an}(a^-(f^{(m;\hat{z})}))$ where $\{\underline{s}, \underline{q}\}$ is defined as in Proposition 2.11.

2.12. Proposition. *Let $f^{(m;\hat{z})}$ be a family of vectors holomorphic in an open set $\Delta \subseteqq C^{\varkappa}$. Consider the product of ν factors*

$$a^+(f^{(m;\hat{z}_1)}) \ldots a^-(f^{(m;\hat{z}_\nu)}) \tag{2.18}$$

where all the creation operators a^+ are to the left of all the annihilation operators a^-. Then (2.18) is holomorphic in the cartesian product

$$\Delta \times \Delta \cdots \times \Delta \subseteqq C^{\varkappa\nu}\,.$$

Proof. Assume that in (2.18) the first β factors from the left are creation operators $(0 \leqq \beta \leqq \nu)$. Let $\{z_1, \ldots, z_\nu\}$ be any point of $\Delta \times \cdots \times \Delta$. In order to apply Proposition 2.9, we have to find a "chain" of elements $\underline{q}_1, \ldots \underline{q}_{\nu+1}$, such that

$$\{\underline{q}_{j+1}, \underline{q}_j\} \in J^{an}(a^+(f^{(m;z_j)})) \quad (j = 1, \ldots \beta)$$

and

$$\{\underline{q}_{j+1}, \underline{q}_j\} \in J^{an}(a^-(f^{(m;z_j)})) \quad (j = \beta + 1, \ldots \nu)\,.$$

Choose $\underline{q}_{\beta+1}$ to be an arbitrary element of $\underline{I}$. If $\beta < \nu$, define $\underline{q}_{\beta+2}$ as the element $\underline{s}$ in Proposition 2.11, with $r^{(m)} \in \bar{J}^{an}(f^{(m;z_{\beta+1})})$. Repeat the procedure, if necessary, to obtain $\underline{q}_{\beta+3}$ (which can be done since the element $\underline{q}$ in Proposition 2.11 is arbitrary) and continue in this way to $\underline{q}_{\nu+1}$. Similarly, if $\beta \geqq 1$, use Proposition 2.8 to obtain $\underline{q}_\beta, \ldots \underline{q}_1$.

h) The automorphism $U(\Lambda)$

We shall now show that certain transformations of X induce unitary automorphisms of $T_{\underline{I}}(X;\mu)$. It is convenient to begin with a general criterion.

2.13. Proposition. *Let H_I and $F_{I'}$ be any nested Hilbert spaces (they need not be of the special kind studied in this paper). Let γ be a mapping from I into I' such that*

$$\overline{(\gamma r)} = \gamma \bar{r} .$$

Let $A \in L(H_I; F_{I'})$ be such that

$$\{r, \gamma r\} \in J(A) \tag{2.19}$$

for every $r \in I$. Then $A \in \mathrm{Hom}(H_I; F_{I'})$.

Proof. Let $r \in I$ be arbitrary; then $\{r, \gamma r\} \in J(A)$ by assumption. Consequently $\{\bar{r}, \overline{\gamma r}\} = \{\bar{r}, \gamma\bar{r}\} \in \bar{J}(A)$. Since $\{\bar{r}, \gamma\bar{r}\}$ is again of the form (2.19), it belongs to $J(A)$. Consequently $\{r, \gamma r\} \in \bar{J}(A) \cap J(A)$ for every $r \in I$, which verifies the condition (Hom) of [4].

Remark. If γ is surjective, one has also $A^* \in \mathrm{Hom}(F_{I'}; H_I)$.

2.14. Proposition. *Let $H_I(X; \mu)$ be the nested Hilbert space defined in Section 2a. Let Λ be a bijective bicontinuous map of X onto itself, such that the measure μ is preserved by Λ. Let $U_{00}(\Lambda)$ be the unitary operator in H_0, defined by*

$$(U_{00}(\Lambda) f_0)(k) = f_0(\Lambda^{-1} k) .$$

Let $U(\Lambda) \in L(H_I; H_I)$ be the operator having $U_{00}(\Lambda)$ as $\{0, 0\}$-representative. Then $U(\Lambda)$ is a unitary automorphism of $H_I(X; \mu)$.

Proof. It is sufficient to show that $U(\Lambda)$ is a homomorphism. For every $r \in I$, define $\Lambda r \in I$ by $(\Lambda r)(k) = r(\Lambda^{-1}k)$. (We are using here the continuity of Λ^{-1}.)

It is easy to verify that the $\{r, \Lambda r\}$-representative of U exists (and is unitary). Since $\overline{\Lambda r} = \Lambda \bar{r}$, the assertion follows from Proposition 2.13.

Notice that $U(\Lambda)$ does *not*, in general, define a unitary mapping of every space H_r onto itself.

Let $T_{\underline{I}}$ be the nested Hilbert space of Section 2b. Define $\Lambda^{(n)}$ and $U^{(n)}$ in the obvious fashion, and let $\underline{U}(\Lambda) \in L(T_{\underline{I}}; T_{\underline{I}})$ be the orthogonal sum $\underline{U}(\Lambda) = \bigoplus_{n=0}^{\infty} U^{(n)}(\Lambda)$.

It follows then from Proposition 2.11 that $\underline{U}(\Lambda)$ is a unitary automorphism of $T_{\underline{I}}$.

For $n = 0, 1, 2 \ldots$ let $e^{(n)}(k)$ $(k \in X^n)$ be a continuous complex-valued function such that $|e^{(n)}(k)| = 1$. Then multiplication by $e^{(n)}(k)$ is a unitary automorphism of $H_I^{(n)}(X; \mu)$. The orthogonal sum of these operators is a unitary automorphism of $T_{\underline{I}}$.

i) Field operators at a point

The results of the preceding sections can be specialized to the case where

1°: X is the positive hyperboloid

$$k^2 = M^2 \geqq 0; \quad k^0 \geqq 0 .$$

2°: μ is the invariant measure

$$d\mu(k) = \frac{d^3 k}{2k^0}$$

on X.

3°: $\varphi(n)$ is equal to $(n+1)^{1/2}$.

4°: The integer m is equal to 1.

5°: The vectors $e^{(z)} = f^{(1;z)} \in H_I^{(1)}$ are

$$e^{(z)}(k) = (2\pi)^{-3/2} \exp(ikz) . \tag{2.20}$$

Here k varies over X and z is a fixed four-vector which may be complex. The superscript 1 has been omitted in $e^{(z)}$. We write

$$z = x + iy \tag{2.21}$$

where x and y are real.

If $r^{(1)} \in I^{(1)}$ is such that

$$\int [r^{(1)}(k)]^{-2} \exp(-2ky)\, d\mu(k) < \infty \tag{2.22}$$

then $r^{(1)} \in J^{an}(e^{(z)})$.

The sum

$$a^+(e^{(z)}) + a^-(e^{(-z)}) \tag{2.23}$$

is denoted by $A(z)$ and called the (neutral scalar free) field operator at the point z. Notice the sign of z in the second term. For real z, it compensates the complex conjugation of $e^{(z)}$ in the definition (2.17) of a^-, so that $A(x)$ is the usual formal field operator.

We proceed now to prove the assertions of Theorem 1.

The statements (i) and (ii) are immediate consequences of Proposition 2.12 since the sum of a finite number of holomorphic families is holomorphic.

The statement (iv) follows from the results of Section 2h.

There remains (iii).

Notice first that Proposition 2.8 and 2.10 give

2.15. Lemma. *Let z be a complex four-vector. Let $r^{(1)} \in I^{(1)}$ satisfy (2.22). Define $\underline{q} \in \underline{I}$ by*

$$q^{(0)} = 1 \tag{2.24}$$

$$\cdots\cdots\cdots$$

$$q^{(n)}(k_1, \ldots k_n) = (n!)^{1/2} r^{(1)}(k_1) \ldots r^{(1)}(k_n) .$$

$$\cdots\cdots\cdots$$

Then $\{\underline{q}, \underline{q}\} \in J^{an}(a^+(e^{(z)}))$.

It is convenient to denote by $\underline{K}(z)$ the subset of $\underline{I}$ consisting of the elements of the form (2.24), where $r^{(1)}$ satisfies (2.22). So Lemma 2.15 states that $\underline{q} \in \underline{K}(z)$ gives $\{\underline{q}, \underline{q}\} \in J^{an}(a^+(e^{(z)}))$. By the definition (2.17) we have also

2.16. Lemma. *If* $\underline{q} \in \overline{\underline{K}}(-z) = \overline{\underline{K}}(z^*)$ *then* $\{\underline{q}, \underline{q}\} \in J^{an}(a^-(e^{(-z)}))$.

About $A(z) = a^+(e^{(z)}) + a^-(e^{(-z)})$ we obtain

2.17. Lemma. *Let* $\underline{q} \in \overline{\underline{K}}(-z)$ *and* $\underline{q}' \in \underline{K}(z)$ *be such that* $\underline{q}' > \underline{q}$. *Then* $\{\underline{q}, \underline{q}'\} \in J^{an}(A(z))$.

Proof. Notice that $\{\underline{q}, \underline{q}'\}$ is a common successor, in[5] $(\underline{I} \times \underline{I})_2$, of $\{\underline{q}, \underline{q}\}$ and of $\{\underline{q}', \underline{q}'\}$. Consequently $\{\underline{q}, \underline{q}'\}$ belongs both to $J^{an}(a^+(e^{(z)}))$ and to $J^{an}(a^-(e^{(-z)}))$ so that the assertion follows.

Let y and y' be real four-vectors. Write $y > y'$ to denote $y - y' \in V_+$ where V_+ is the interior of the forward light-cone. If $r^{(1)}(k) = e^{ky}$, $r^{(1)\prime}(k) = e^{ky'}$ and if $y > y'$, then $r^{(1)} > r^{(1)\prime}$ in $I^{(1)}$. Consequently also $\underline{q} > \underline{q}'$ in $\underline{I}$, where $\underline{q}$ and $\underline{q}'$ are defined by (2.24).

Let $z_1, \ldots z_\nu$ be complex four-vectors such that $y_\nu > \cdots > y_2 > y_1$ (see (2.21)). We shall show that the product $A(z_1) \ldots A(z_\nu)$ is defined and holomorphic at $\{z_1, \ldots, z_\nu\}$. Choose an arbitrary four-vector y_0 such that $y_1 > y_0$ and an arbitrary $y_{\nu+1}$ such that $y_{\nu+1} > y_\nu$. Define $r_1^{(1)}, \ldots r_{\nu+1}^{(1)}$ in $I^{(1)}$ by

$$r_j^{(1)}(k) = \exp\left(-\frac{1}{2}(y_{j-1} + y_j)k\right) \quad (j = 1, \ldots \nu + 1)\,.$$

Define $\underline{q}_1, \ldots \underline{q}_{\nu+1}$ by (2.24). Then $\underline{q}_1 > \underline{q}_2 > \cdots > \underline{q}_{\nu+1}$ in $\underline{I}$.

By Proposition 2.9, it is enough to prove that

$$\{\underline{q}_{j+1}, \underline{q}_j\} \in J^{an}(A(z_j)) \quad (j = 1, 2, \ldots \nu)\,. \tag{2.25}$$

By Lemma 2.17, (2.25) follows from $\underline{q}_{j+1} \in \overline{K}(-z_j)$ and $\underline{q}_j \in K(z_j)$. This means

$$\int [r_{j+1}^{(1)}(k)]^2 \exp(2ky_j)\, d\mu(k) = \int \exp((y_j - y_{j+1})k)\, d\mu(k) < \infty$$

and

$$\int [r_j^{(1)}(k)]^{-2} \exp(-2ky_j)\, d\mu(k) = \int \exp((y_{j-1} - y_j)k)\, d\mu(k) < \infty$$

which are satisfied. So the assertion (iii) of Theorem 1 is proved.

We have so obtained for operator products (rather than vacuum expectation values) the initial analyticity domain of the Wightman theory.

Notice that we have not defined the product $A(x)\,A(x')$ for real x and x'. Equation (2.16) makes it easy, however, to find suitable smearing functions $f^{(1)}(x)$ such that $A(x)\,A(f^{(1)})$ is defined.

The spaces used in this work are much more elementary than, say, the space of tempered distributions. This corresponds to the elementary character of the field studied here. Other spaces are needed e.g. in the *reformulation of Wightman axioms in which fields are operators in a* suitable nested Hilbert space. Still, the explicit example of free fields can be used as a guideline in this reformulation. It seems also likely that similar methods are useful in the study of current algebras.

[5] See Section 2e of [3].

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