Commun. math. Phys. 3, 323—333 (1966)

Dynamical Groups and Spherical Potentials in Classical Mechanics

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Abstract. The one particle problem in a spherical potential is examined in Classical Mechanics from a group theorical point of view. The constants of motion are classified according to their behaviour under the rotation group $SO(3)$, i.e. according to the irreducible representations D_j of $SO(3)$ (section 1).

The Lie algebras of $SO(4)$ and $SO(3)$ are explicitly built in terms of Poisson brackets for an arbitrary potential, from global considerations. The Kepler and the 3 dimensional oscillator problems are shown to play particular roles with respect to these groups (sections 2 and 3).

In the last section, the Kepler problem is analyzed with the aid of the $SO(4)$ group instead of the Lie algebra. It is proved that the transformations generated by the angular momentum and the Runge-Lenz vector form indeed a group of canonical transformations isomorphic to $SO(4)$. Consequences with respect to the quantization problem are examined.

Introduction

Since $SU(6)$ has been proposed as a group of symmetry for hadrons, dynamical groups in Quantum Mechanics have been re-investigated in different ways. Two kinds of spherical potentials are interesting from a group theoretical point of view, namely the Coulomb and the harmonic oscillator potentials. In these two cases there is a degeneracy of the bound levels which is called "accidental" because it is not due to the invariance of the Hamiltonian h under rotations. It has been shown by several authors that h is invariant under a larger group (a dynamical group) which is the 4-dimensional rotation group $SO(4)$ in the Coulomb case [1] and the 3-dimensional unitary unimodular group $SU(3)$ in the harmonic case [2]. It would be interesting to know if there exist other spherical potentials for which accidental degeneracies occur; in the present work, we intend to give only a first approach to this problem by investigating some aspects of the corresponding classical problem.

Dynamical groups in Quantum Mechanics are usually discussed in terms of Lie algebras (commutators). The corresponding Lie algebra in

Classical Mechanics is the Poisson algebra the elements of which generate canonical transformations. Diverse procedures for replacing the Poisson brackets by commutators are known [3], [4] (the so called quantization problem) but it seems to us that the dynamical groups have never been discussed in Classical Mechanics.

1. Constants of motion and the rotation group

Consider the classical Hamiltonian of a particle of unit mass in a spherical potential $V(r)$

$$h = \frac{\mathbf{p}^2}{2} + V(r)\,. \tag{1.1}$$

It is well known that there exist in the one particle problem six functionnally independent constants of motion. We choose, for convenience, the following set

1) h, the total energy,
2) l, the length of the angular momentum $\mathbf{L}$:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1.2}$$

$$l^2 = r^2 p^2 - (\mathbf{r} \cdot \mathbf{p})^2\,, \tag{1.3}$$

3) $\mathbf{u}_0$, the unit vector in the direction of a characteristic point of the trajectory (for instance the perihelion)[1],
4) $\mathbf{w}_0$, the unit vector in the direction of $\mathbf{L}$ [2]

$$\mathbf{w}_0 = \frac{\mathbf{L}}{l}\,, \tag{1.4}$$

5) t_0, the time when the particle is at the characteristic point.

If we take into account of the orthogonality relation

$$\mathbf{u}_0 \cdot \mathbf{w}_0 = 0 \tag{1.5}$$

it can be easily verified that this set does contain six independent constants of motion.

We are interested in classifying all constants of motion which are invariant under time translations i.e. all functions $f(H, l, \mathbf{u}_0, \mathbf{w}_0)$, according to their behaviour under rotations around the origin $r = 0$. In other words, every such function can be decomposed into a sum of *irreducible functions*, each of them belonging to an irreducible representation D_j of the rotation group. Let us recall that an irreducible function of class D_j is a $(2j + 1)$ component function $f_{i_1 i_2 \ldots i_j}$ which is completely symmetric with respect to its indices and which vanishes for any number of contractions. Moreover, it satisfies the following Poisson bracket

[1] Every point of a trajectory can be considered as a characteristic point, except of course if the trajectory is a circle.

[2] It cannot be defined if the trajectory is rectilinear.

relation

$$\{L_k, f_{i_1 i_2 \ldots i_j}\} = \varepsilon_{k i_1 p}\, f_{p i_2 \ldots i_j} + \varepsilon_{k i_2 p}\, f_{i_1 p i_3 \ldots i_j} + \ldots + \varepsilon_{k i_j p}\, f_{i_1 \ldots i_{j-1} p} \tag{1.6}$$

where the Poisson bracket is defined as follows

$$\{f(\mathbf{r}, \mathbf{p}), g(\mathbf{r}, \mathbf{p})\} = \sum_i \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial x^i} \right) \tag{1.7}$$

where the x_i's and the p_i's are taken at an arbitrary time.

A function $f(h, l, \mathbf{u}_0, \mathbf{w}_0)$ will be called a scalar constant of motion if its Poisson brackets with the components of $\mathbf{L}$ are zero. This implies that f depends only on h and l. It follows immediately that every Poisson bracket of two scalar constants of motion vanishes.

The irreducible constants of motion of class D_j are the spherical harmonics of order j. They can be built in the following way. Let us define the complex vector $\mathbf{z}_0$ in the plane of the trajectory

$$\mathbf{z}_0 = \mathbf{u}_0 + i\mathbf{v}_0 \tag{1.8}$$

with

$$\mathbf{v}_0 = \mathbf{w}_0 \times \mathbf{u}_0 \,. \tag{1.9}$$

The three vectors $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{w}_0$ form an orthonormal basis. Every real irreducible constant of motion of class D_1 (vector) is of the form

$$T^{(1)} = \mathscr{R}\{\varphi \mathbf{z}_0\} + \psi \mathbf{w}_0 \tag{1.10}$$

where $\mathscr{R}$ means "real part" and φ and ψ are arbitrary scalar constants of motion (ψ is real but φ is complex). More generally it can be shown that every irreducible constant of motion of class D_j can be put in the form[3]

$$T^{(j)} = \mathscr{R}\{\varphi \mathbf{z}_0 \otimes \mathbf{z}_0 \otimes \cdots \otimes \mathbf{z}_0\} + \overline{\mathbf{w}_0 \otimes T^{(j-1)}} \tag{1.11}$$

where the brackets contain a tensor product of j times the vector $\mathbf{z}_0$ and the bar denotes a symmetrization. Eq. (1.11) allows us to build recurrently the irreducible constants of motion of any order j. For $j = 2$, one gets

$$T^{(2)} = \mathscr{R}\{\varphi \mathbf{z}_0 \otimes \mathbf{z}_0\} + \mathbf{w}_0 \otimes T^{(1)} + T^{(1)} \otimes \mathbf{w}_0 \,. \tag{1.12}$$

2. The dynamical group $SO(4)$

The Lie algebra of $SO(4)$ is, in terms of Poisson brackets

$$\{\mathbf{L}, \mathbf{L}\} = \mathbf{L} \tag{2.1}$$

$$\{\mathbf{L}, \mathbf{A}\} = \mathbf{A} \tag{2.2}$$

$$\{\mathbf{A}, \mathbf{A}\} = \mathbf{L} \,. \tag{2.3}$$

If $\mathbf{A}$ is a constant of motion satisfying (2.2) and (2.3), the group $SO(4)$

[3] Note that the tensors $A^{(j)} = \mathscr{R}\{\varphi \mathbf{z}_0 \otimes \mathbf{z}_0 \otimes \cdots \mathbf{z}_0\}$ are irreducible *real* representations of the two dimensional rotation group (rotations in the plane of the trajectory).

leaves the Hamiltonian invariant and is called a dynamical group. According to (1.10), the vector $\mathbf{A}$ is of the form

$$\mathbf{A} = \mathscr{R}\{\varphi \mathbf{z}_0\} + \psi \mathbf{w}_0 \tag{2.4}$$

where φ and ψ are scalar constants of motion. Eq. (A.10) given in the appendix and eq. (2.3) lead to the conditions

$$\frac{\psi}{l} + \frac{\partial \psi}{\partial l} = 0 \tag{2.5}$$

$$\frac{\psi^2}{l} - \frac{1}{2}\frac{\partial |\varphi|^2}{\partial l} = 0\,, \tag{2.6}$$

the solutions of which are

$$l^2 + |\varphi|^2 + \psi^2 = \mathbf{L}^2 + \mathbf{A}^2 = \alpha(h) \tag{2.7}$$

$$l\psi = \mathbf{L}\cdot\mathbf{A} = \beta(h)\,, \tag{2.8}$$

where $\alpha(h)$ and $\beta(h)$ are *arbitrary functions of* h.

If we compare these results to the usual ones for the Kepler problem [5], [6] they seem very surprising. In fact in the usual treatment of the hydrogen atom, one has $\mathbf{L}^2 + \mathbf{A}^2 = -\frac{1}{2h}$ and $\mathbf{L}\cdot\mathbf{A} = 0$ whereas in (2.7) and (2.8), these two quantities are arbitrary functions of h. Nevertheless it is necessary to recall that our way of constructing the irreducible constants of motion is not valid for rectilinear and circular orbits (notes 1 and 2 in section 1). For rectilinear trajectories, $\mathbf{L} = 0$ and if $\mathbf{A}$ is still defined[4], eq. (2.8) leads to

$$\beta(h) = \mathbf{L}\cdot\mathbf{A} = 0\,. \tag{2.9}$$

A circular trajectory is invariant under rotations around $\mathbf{w}_0$. So must be the constants of motion and particularly $\mathbf{A}$. Taking into account of (2.9), one is led to the condition

$$\mathbf{A} = 0 \quad \text{for circular orbits}\,. \tag{2.10}$$

This relation furnishes a restriction on the function $\alpha(h)$. In fact, for a circular trajectory, one has

$$h = \frac{1}{2} r V' = V \tag{2.11}$$

$$l^2 = r^3 V'\,. \tag{2.12}$$

In eliminating r between h and l^2, one finds the function $l^2(h) = \alpha(h)$ according to (2.10) and (2.7). Let us consider, as an example a potential of the kind

$$V = \mu r^n \tag{2.13}$$

where $n\mu > 0$ (attractive force) one gets immediately

$$\alpha(h) = \mu n \left[\frac{h}{\mu\left(\frac{n}{2} + 1\right)}\right]^{1+\frac{2}{n}}. \tag{2.14}$$

[4] Rectilinear trajectories exist for an arbitrary value of h.

In the special case of the Kepler problem ($n = -1$, $\mu = -1$), this formula becomes

$$\mathbf{L}^2 + \mathbf{A}^2 = \alpha(h) = -\frac{1}{2h}, \tag{2.15}$$

a condition which implies $h < 0$ (elliptic orbits). Taking into account of the equation of the trajectory

$$\cos\theta = \frac{1}{\sqrt{1 + 2l^2 h}}\left(1 - \frac{l^2}{r}\right) \tag{2.16}$$

and

$$\mathbf{z}_0 = \mathbf{z}\, e^{i\theta} \tag{2.17}$$

one readily obtains

$$\mathbf{A} = \frac{1}{\sqrt{-2h}}\left(\mathbf{L} \times \mathbf{p} + \frac{\mathbf{r}}{r}\right) \tag{2.18}$$

a vector proportional to the well-known Runge-Lenz vector [5].

It is exceptional that the vector $\mathbf{A}$ can be expressed as a one-valued function of initial conditions $\mathbf{r}$ and $\mathbf{p}$ at any time. This is due to the fact that there exists on each trajectory a privileged point, namely the perihelion.

It can be shown that this exceptional case occurs only in the Kepler case. In fact for the harmonic oscillator, the orbits are closed but they possess two equivalent privileged points (symmetric trajectories) and $\mathbf{A}$ is a two-valued function of $\mathbf{r}$ and $\mathbf{p}$.

3. The dynamical group $SU(3)$

The Lie algebra of $SU(3)$ is generated by the $M_{ij} = -M_{ji} = \varepsilon_{ijk} L^k$ and the five components of a symmetric traceless tensor $N_{ij} = N_{ji}$. The commutation rules are the following ones.

$$\{M_{ij}, M_{kl}\} = -\delta_{jk} M_{il} + \delta_{jl} M_{ik} + \delta_{ik} M_{jl} - \delta_{il} M_{jk} \tag{3.1}$$

$$\{M_{ij}, N_{kl}\} = -\delta_{jk} N_{il} - \delta_{jl} N_{ik} + \delta_{ik} N_{jl} + \delta_{il} N_{jk} \tag{3.2}$$

$$\{N_{ij}, N_{kl}\} = \delta_{jk} M_{il} + \delta_{jl} M_{ik} + \delta_{ik} M_{jl} + \delta_{il} M_{jk}\,. \tag{3.3}$$

A well known representation of this algebra consists in

$$M_{ij} = x_i p_j - x_j p_i \tag{3.4}$$

$$N_{ij} = x_i x_j - p_i p_j\,. \tag{3.5}$$

We are interested in constructing the M_{ij}'s and N_{ij}'s as some irreducible constants of motion. We choose $M_{ij} = \varepsilon_{ijk} L^k$. The most general tensor N can be written in the form

$$N = \mathscr{R}\{\Phi z_0\} + \mathscr{R}\{\varphi R_0\} + \psi W_0 \tag{3.6}$$

where

$$Z_0 = \mathbf{z}_0 \otimes \mathbf{z}_0 \tag{3.7}$$

$$R_0 = \mathbf{w}_0 \otimes \mathbf{z}_0 + \mathbf{z}_0 \otimes \mathbf{w}_0 \tag{3.8}$$

$$W_0 = \frac{1}{3}\mathbf{1} - \mathbf{w}_0 \otimes \mathbf{w}_0 \tag{3.9}$$

$$\mathbf{1} = \mathbf{u}_0 \otimes \mathbf{u}_0 + \mathbf{v}_0 \otimes \mathbf{v}_0 + \mathbf{w}_0 \otimes \mathbf{w}_0 \tag{3.10}$$

and Φ, φ and ψ are scalar constants of motion.

With the above definitions, eq. (3.3) takes the form

$$\{N, N\} = i l Z_0 \otimes Z_0^* + i \frac{l}{2} R_0 \wedge R_0^* \tag{3.11}$$

and, according to (A,12), this condition implies

$$\left.\begin{aligned}
-\frac{1}{2}\frac{\partial |\Phi|^2}{\partial l} + 2\frac{|\varphi|^2}{l} &= l\\
\frac{1}{2l}\left[\psi^2 - |\Phi|^2\right] - \frac{\partial |\varphi|^2}{\partial l} &= \frac{l}{2}\\
\frac{1}{2}\varphi\frac{\partial \Phi}{\partial l} - \Phi\frac{\partial \varphi}{\partial l} - \frac{\Phi\varphi}{2l} &= 0\\
\varphi^*\frac{\partial \Phi}{\partial l} + 2\Phi\frac{\partial \varphi^*}{\partial l} + \frac{2\varphi\psi}{l} &= 0\\
6\frac{\varphi^2}{l} - \Phi\frac{\partial \psi}{\partial l} &= 0\\
\frac{3}{l}(\Phi\varphi^* - \psi\varphi) - \varphi\frac{\partial \psi}{\partial l} &= 0\,.
\end{aligned}\right\} \tag{3.12}$$

It can be easily seen that φ is necessarily zero and that we are left with the two following conditions

$$|\Phi|^2 + l^2 = \psi^2 = \gamma(h) \tag{3.13}$$

where $\gamma(h)$ is an arbitrary function of h.

It is now natural to ask what happens for rectilinear and circular orbits. The rectilinear ones impose no condition on $\gamma(h)$. The invariance of N under rotations around $\mathbf{w}_0$ for circular trajectories implies

$$\Phi = 0 \quad \text{for circular trajectories} \tag{3.14}$$

and consequently $\gamma(h) = l^2(h)$. This is the same condition which obeys the function $\alpha(h)$ of section 2. Eq. (2.14) gives us

$$\gamma(h) = h^2 \tag{3.15}$$

for the potential $V = \frac{1}{2} r^2$ of the harmonic oscillator.

A simple calculation using the equation of the trajectory

$$\cos 2\theta = \frac{1}{\sqrt{h^2 - l^2}}\left(h^2 - \frac{l^2}{r^2}\right)$$

leads to the relations

$$\Phi = h - \frac{l^2}{r^2} - i l \frac{\mathbf{r} \cdot \mathbf{p}}{r^2}$$

$$\psi = h = \frac{1}{2}(r^2 + p^2)$$

and consequently

$$N = \mathbf{r} \otimes \mathbf{r} + \mathbf{p} \otimes \mathbf{p} - \frac{2h}{3} \mathbf{1}$$

which is a well known result.

The above results can be applied to the hydrogen atom case. We note that the tensor N is uniquely defined but it cannot permit to distinguish between the perihelion and the aphelion. In other words, to each couple $(\mathbf{L}, N)$ of constants of motion correspond two symmetric trajectories. This corresponds to the inverse situation of that considered at the end of the preceding section.

4. Global investigation of a dynamical group

We intend to examine in the case $V = -\frac{1}{r}$ alone, the finite transformations corresponding to the Lie algebra of $\mathbf{A}$ and $\mathbf{L}$.

These transformations act on the 6-dimensional manifold $\mathscr{V}$ of the negative energy motions defined by the 3 Kepler laws; they are canonical in the sense that they leave invariant the symplectic structure of $\mathscr{V}$ defined by the Poisson or Lagrange brackets.

It is possible to parametrize $\mathscr{V}$ using the orthogonal vectors $\mathbf{L}$ and $\mathbf{A}$ and a time t when the moving body is at the perihelion. We will use here two unit vectors $\mathbf{M}$ and $\mathbf{N}$ and two numbers ϱ and τ such that

$$\mathbf{A} = \frac{\varrho}{2}[\mathbf{M} + \mathbf{N}], \quad \mathbf{L} = \frac{\varrho}{2}[\mathbf{M} - \mathbf{N}], \quad t = \varrho^3 \tau; \tag{4.1}$$

since the period of the motion equals $2\pi\varrho^3$, τ is defined modulo 2π.

The domain of validity of these parameters cover all motions except a 4-dimensional sub-manifold corresponding to the circular motions $(\mathbf{M} = -\mathbf{N})$.

Any element of the above Lie algebra is a dynamical variable

$$\frac{\varrho}{2}[\mathbf{Y}.\,\mathbf{M} - \mathbf{Z}.\,\mathbf{N}], \tag{4.2}$$

where $\mathbf{Y}$ and $\mathbf{Z}$ are constant vectors. The associated infinitesimal transformation of $\mathscr{V}$ is

$$\delta\mathbf{M} = \mathbf{Y} \times \mathbf{M}, \quad \delta\mathbf{N} = \mathbf{Z} \times \mathbf{N}, \quad \delta\tau = \frac{[\mathbf{Z} - \mathbf{Y}] \cdot [\mathbf{M} + \mathbf{N}]}{[\mathbf{M} + \mathbf{N}]^2}. \tag{4.3}$$

The corresponding finite transformations will be obtained in integrating the associate differential system

$$\frac{d\mathbf{M}}{ds} = \delta\mathbf{M}, \quad \frac{d\mathbf{N}}{ds} = \delta\mathbf{N}, \quad \frac{d\varrho}{ds} = 0 \quad \frac{d\tau}{ds} = \delta\tau. \tag{4.4}$$

This integration is elementary but somewhat troublesome. In order to present the results it is useful to recall some algebraic properties.

If we put, for two vectors $\mathbf{U}$ and $\mathbf{V}$ or R^3

$$\mathbf{U}\mathbf{V} = \mathbf{U} \times \mathbf{V} - \mathbf{U} \cdot \mathbf{V} \tag{4.5}$$

we define on the set of symbols $\lambda + \mathbf{U}$ (λ real) a structure of a *field* isomorphic to the quaternion field; the corresponding complexified field is an irreducible algebra, isomorphic to that of the 2×2 complex matrices.

The notation which can be naturally associated permits us to write the general integral of the differential system (4.3; 4.4) in the form

$$\begin{aligned} \mathbf{M} &= \eta \mathbf{M}_0 \eta^{-1} \\ \mathbf{N} &= \zeta \mathbf{N}_0 \zeta^{-1} \\ \varrho &= \varrho_0 \\ \tau &= \tau_0 + \operatorname{Arg} \operatorname{Tr} (\zeta^{-1} \eta [1 + i\mathbf{M}_0] [1 + i\mathbf{N}_0]) \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \eta &= \exp\left(\frac{s}{2}\mathbf{Y}\right) = \cos\left(\frac{s|\mathbf{Y}|}{2}\right) + \sin\left(\frac{s|\mathbf{Y}|}{2}\right)\frac{\mathbf{Y}}{|\mathbf{Y}|} \\ \zeta &= \exp\left(\frac{s}{2}\mathbf{Z}\right). \end{aligned} \tag{4.7}$$

Note that the argument of a complex number is defined, as τ, in the additive group $R/2\pi$. The parameters η and ζ each describe the set of unitary quaternions, namely the group $SU(2)$; if we denote the correspondence (4.6) between the initial point $(\mathbf{M}_0, \mathbf{N}_0, \varrho_0, \tau_0)$ of $\mathscr{V}$ and the point $(\mathbf{M}, \mathbf{N}, \varrho, \tau)$ by the notation $\varphi(\eta, \zeta)$, it is fairly easy to verify directly that φ is an homomorphism of the group $SU(2) \times SU(2)$ in the group of the global canonical transformations of $\mathscr{V}$ (the case of the circular motions could be treated by continuity); one sees immediately that the kernel of this homomorphism is the group of two elements $\eta = \zeta = \pm 1$; consequently, the image of φ is a group of canonical transformations which is isomorphic to $\frac{SU(2) \times SU(2)}{Z_2}$, therefore to $SO(4)$.

One can note that the differential system of the equations of the planet motions possesses an absolute integral invariant of degree one, namely

$$\mathbf{p} \cdot d\mathbf{r} - h\,dt + d[3ht - 2\mathbf{p} \cdot \mathbf{r}]. \tag{4.8}$$

The existence of this invariant comes from the suitably generalized Noether theorem [9] applied to the invariance of the problem under the substitutions

$$r \to \lambda^2 r, \quad t \to \lambda^3 t, \quad \mathscr{A} \to \lambda \mathscr{A} \tag{4.9}$$

where $\mathscr{A}$ denotes the Hamiltonian action.

This 1-form (4.8) is therefore the reciprocical image of a 1-form ϖ of $\mathscr{V}$; $\bar{\omega}$ is a "potential" of $\mathscr{V}$ in the sense of reference [9].

One can now verify that $\bar{\omega}$ is invariant under the transformation (4.6); consequently the lifting of this group on the "espace fibré quantifiant" built above $\mathscr{V}$ is isomorphic to the direct product of $SO(4)$ by $U(1)$; in the geometrical quantization à la Souriau, the space of states is a representation space for the group $SO(4)$; so is the space of states in the hydrogen atom treatment by the Schroedinger equation [1].

Conclusion

In constructing dynamical groups in Classical Mechanics, three stages can be reached:

1. Local construction of the given Lie algebra in terms of Poisson brackets. It is not surprising that this operation is possible since the Lie algebra of the dynamical variables is very large and infinite dimensional. In fact, if this construction is possible for a given dynamical problem, it is necessarily possible for all problems with the same number of degrees of freedom, due to the fact that all $2n$-dimensional symplectic manifolds are locally isomorphic [9].

2. The same construction can be attempted in the global sense. This stage allowed us a) to make precise the functions $\alpha(h)$ and $\beta(h)$ (eqs. 2.9 and 2.15) for the Lie algebra of $SO(4)$ and $\gamma(h)$ for the Lie algebra of $SU(3)$, b) eliminate all multiform dynamical variables, discarding in this way all potentials except the Newtonian and harmonic ones.

3. Once the Lie algebra is obtained, it is necessary to know if the finite canonical transformations which it generates form a group and in that case, what is the structure of this group. It is remarkable that in the Kepler case one gets in fact a group and that this group is effectively $SO(4)$[5].

This last result could seem natural to a physicist who knows that the quantum states of the non relativistic hydrogen atom span a representation space of the group $SO(4)$. It is a wonder that there exists such a parallelism between the classical and quantum mechanical treatments of this problem. It is a positive argument in favour of the quantization methods.

The same problems can be investigated in Relativistic Classical Mechanics. The first stage is of course possible and the Lie algebras of $SO(4)$ and $SU(3)$ are exactly the same as in the non-relativistic case except that we have to replace the momentum and energy by their relativistic equivalents. Nevertheless the second and *a fortiori* the third stage are radically different when we consider the two particular cases of Newtonian and harmonic potentials. The reason is that the orbits are no

[5] *A priori*, one can expect to find an extension of $SU(2) \times SU(2)$ by an infinite discrete group.

longer closed and consequently the global structure of the manifold of motions is completely changed.

It would be very interesting to know if a group theoretical method can be invented to explain the relativistic splitting of levels.

Appendix

We give in this appendix, without proofs, some useful formulae.

$$\{f(h, l, r),\ g(h, l, r)\} = \left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial h} - \frac{\partial g}{\partial r}\frac{\partial f}{\partial h}\right)\frac{l}{r^2}\left(\frac{\partial\theta}{\partial r}\right)^{-1}. \tag{A1}$$

In particular,

$$\{f(h, l), \theta(h, l, r)\} = -\frac{\partial f}{\partial h}\frac{l}{r^2}. \tag{A2}$$

On the other hand

$$\{f(h, l, r),\ \mathbf{z}\} = i\left(\frac{\partial f}{\partial h}\frac{l}{r^2} + \frac{\partial f}{\partial l}\right)\mathbf{z} \tag{A3}$$

$$\{f(h, l, r), \mathbf{z}_0\} = i\frac{\partial f}{\partial l}\mathbf{z}_0. \tag{A4}$$

The following relations are valid for $\mathbf{z}_0$ and $\mathbf{w}$ as well as for $\mathbf{z}$ and $\mathbf{w}$:

$$\{\mathbf{z}, \mathbf{z}\} = 0 \tag{A5}$$

$$\{\mathbf{z}, \mathbf{z}\} = -2i\frac{\mathbf{w}\otimes\mathbf{w}}{l} \tag{A6}$$

$$\{\mathbf{z}, \mathbf{w}\} = i\frac{\mathbf{w}\otimes\mathbf{z}}{l} \tag{A7}$$

$$\{\mathbf{w}, \mathbf{w}\} = \frac{\mathbf{w}}{l} = \frac{i}{2l}\mathbf{z}\times\mathbf{z}^* \tag{A8}$$

Poisson bracket of a vector constant of motion with itself

$$\mathbf{A} = \mathscr{R}\{\varphi\mathbf{z}_0\} + \psi\mathbf{w}_0 \tag{A9}$$

$$\{\mathbf{A}, \mathbf{A}\} = \left(\frac{\psi^2}{l} - \frac{1}{2}\frac{\partial|\varphi|^2}{\partial l}\right)\mathbf{w}_0 + \left(\frac{\psi}{l} + \frac{\partial\psi}{\partial l}\right)\mathscr{R}\{\varphi\mathbf{z}_0\}. \tag{A10}$$

Poisson bracket of a symmetric traceless tensor of rank two constant of motion with itself

$$N = \mathscr{R}\{\Phi Z_0\} + \mathscr{R}\{\varphi R_0\} + \psi W_0 \tag{A11}$$

$$\begin{aligned}\{N, N\} = &\left[-\frac{i}{2}\frac{\partial|\Phi|^2}{\partial l} + \frac{2i|\varphi|^2}{l}\right] Z_0\wedge Z_0^* + \left[\frac{i}{2l}(\psi^2 - |\Phi|^2) - i\frac{\partial|\varphi|^2}{\partial l}\right] R_0\wedge R_0^*\\ &+ \left[\frac{i}{2}\frac{\partial\Phi}{\partial l}\varphi - i\Phi\frac{\partial\varphi}{\partial l} - \frac{i}{2l}\Phi\varphi\right] Z_0\wedge R_0 + \text{im. conj.} +\\ &+ \left[-\frac{i}{2}\varphi^*\frac{\partial\Phi}{\partial l} - i\frac{\partial\varphi^*}{\partial l}\Phi - \frac{i}{l}\varphi\psi\right] Z_0\wedge R_0^* + \text{im. conj.} +\\ &+ \left[\frac{6i}{l}\varphi^2 - i\Phi\frac{\partial\psi}{\partial l}\right] Z_0\wedge W_0 + \text{im. conj.} +\\ &+ \left[\frac{3i}{l}(\Phi\varphi^* - \psi\varphi) - i\varphi\frac{\partial\psi}{\partial l}\right] R_0\wedge W_0 + \text{im. conj.}.\end{aligned} \tag{A12}$$

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