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Unitary Irreducible Representations of $SU(2,2)$

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Abstract. Using a Lie algebra method based on works by Harish-Chandra, several series of unitary, irreducible representations of the group $SU(2,2)$ are obtained.

1. Introduction

The group called the conformal group is commonly defined as the pseudo-orthogonal group $SO(2,4)$ in the real six-dimensional space, which leaves the form $x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2$ invariant. It has a covering group which is the pseudo-unitary group $SU(2,2)$ leaving invariant the complex form $|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2$. The conformal group was first introduced into physics by Bateman and Çunningham [1], [2]. Many authors used it in connection with general relativity (it contains the De Sitter groups $SO(1,4)$ and $SO(2,3)$ as subgroups), and also in electromagnetism [3], [4], [5], [6]. It is the largest transformation group on space-time leaving the Maxwell equations in flat space invariant. More recently it was shown that a relation seems to exist between the conformal group and the ultraviolet singularities of quantum field theory [7]. In the present attempts to use non-compact groups to unify in a non-trivial way the Poincaré group and an internal symmetry group of strongly interacting particles, the conformal group might play a role [8], [9]. It is the smallest semi-simple group containing the Poincaré group. Finally since there has been much interest in the group $SU(6,6)$ as a strong interaction symmetry group [10] we may learn something about this group by studying first the group $SU(2,2)$.

This short review shows that it might be interesting to study the conformal group from various points of view. In all quantum mechanical applications the unitary irreducible representations are of foremost importance. Some results on the determination of these infinite dimensional representations are known in the literature. Graev [11], [12] has given three fundamental series of unitary irreducible representations of

$SU(2, 2)$ using global methods. MURAI [13] has determined a class of degenerate representations by the method of THOMAS [14]. In this paper we shall use a Lie-algebra method which has been described in an earlier paper [15]. It has been applied to a number of pseudo-orthogonal groups and proved to be quite a powerful tool to find the unitary irreducible representations[1]. The method is applicable to semi-simple groups and its rigorous mathematical justification follows from the works of HARISH-CHANDRA [19]. Starting from the Iwasawa decomposition [20] (p. 156ff.)

$$G = KAN$$

of any semi-simple Lie group into three subgroups one defines first a realization of the group as transformations on the quotient space $G/N \sim K \times A$ by means of left multiplication. This quotient space then serves as a carrier space for a linear space of functions. By introducing suitable scalar products into this linear function space, one obtains not only unitary representations but also irreducible ones. The calculations are however not performed for the group itself but instead we use the infinitesimal transformations of the Lie algebra. The possibility of "lifting" the algebraically irreducible Hermitian representations of the Lie algebra to unitary, irreducible representations of the group is again ascertained by HARISH-CHANDRA [19].

2. Iwasawa decomposition of the Lie algebra

The group $SU(2, 2)$ is the subgroup of $SL(4, \mathbf{C})$ leaving invariant the Hermitian form

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 .$$

Thus we have

$$SU(2, 2) = \{\mathrm{g} \in SL(4, \mathbf{C}) : \mathrm{g}^+ \beta \mathrm{g} = \beta\} \tag{2.1}$$

where β is the 4×4 diagonal matrix with non-zero elements $\beta_{11} = \beta_{22} = -\beta_{33} = -\beta_{44} = 1$ and g^+ is the adjoint transformation.

Let $\mathfrak{G}$ denote the Lie algebra over $\mathbf{R}$ of the group $SU(2, 2)$. Hence

$$\mathfrak{G} = \{x : e^{\tau x} \in SU(2, 2) \text{ for all } \tau \in \mathbf{R}\} . \tag{2.2}$$

As a consequence of (2.1) and (2.2) we can define the Lie algebra by the following conditions on its elements x:

$$\begin{aligned} \beta x^+ \beta + x &= 0 \\ \mathrm{Tr} x &= 0 . \end{aligned} \tag{2.3}$$

The 15 traceless γ matrices are linearly independent and can be chosen to fulfil (2.3). Therefore they form a basis in $\mathfrak{G}$. We shall use this basis in the sequel. Let

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \; \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \; \vec{\sigma} = \begin{pmatrix} \vec{\tau} & 0 \\ 0 & \vec{\tau} \end{pmatrix} \tag{2.4}$$

[1] For the application to $SO(1, 2)$ and $SO(2, 2)$, see Ref. [15]. The groups $SO(1, 3)$, $SO(3, 3)$ and $SO(1, 4)$ have been treated in [16], [17] and [18].

with

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \ \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \ \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathfrak{G}$ is generated by

$$\mathfrak{G} : \{i\beta, i\vec{\sigma}, i\beta\vec{\sigma}, \gamma_5, i\beta\gamma_5, \vec{\sigma}\gamma_5, i\beta\vec{\sigma}\gamma_5\}. \tag{2.5}$$

Now let $\mathfrak{U}$ be the Lie algebra over $\mathbf{R}$ associated with the group $SU(4)$

$$SU(4) = \{g \in SL(4, \mathbf{C}) : g^+ g = 1\}. \tag{2.6}$$

The Lie algebra

$$\mathfrak{U} = \{y : e^{\tau y} \in SU(4) \text{ for all } \tau \in \mathbf{R}\} \tag{2.7}$$

is generated by

$$\mathfrak{U} : \{i\beta, i\vec{\sigma}, i\beta\vec{\sigma}, i\gamma_5, \beta\gamma_5, i\vec{\sigma}\gamma_5, \beta\vec{\sigma}\gamma_5\}. \tag{2.8}$$

It is evident from (2.5) and (2.8) that $\mathfrak{U}$ is the dual compact Lie algebra with respect to the non-compact Lie algebra $\mathfrak{G}$. These two Lie algebras over $\mathbf{R}$ have isomorphic complex extensions

$$\mathfrak{G}_c \cong \mathfrak{U}_c.$$

We introduce first the Cartan decomposition (see [20], p. 156ff.)

$$\mathfrak{G} = \mathfrak{K} + \mathfrak{P} \tag{2.9}$$

where

$$\mathfrak{K} = \mathfrak{G} \cap \mathfrak{U}$$
$$\mathfrak{P} = \mathfrak{G} \cap i\mathfrak{U}.$$

From (2.5) and (2.8) follows immediately that $i\beta, i\vec{\sigma}, i\beta\vec{\sigma}$ form a basis in the vector space $\mathfrak{K}$ and $\gamma_5, i\beta\gamma_5, \vec{\sigma}\gamma_5, i\beta\vec{\sigma}\gamma_5$ form a basis in the vector space $\mathfrak{P}$. $\mathfrak{K}$ is the maximal compact subalgebra contained in $\mathfrak{G}$, associated with the maximal compact subgroup $S(U(2) \otimes U(2))$ contained in $SU(2, 2)$. It is convenient to choose new basis vectors in $\mathfrak{K}$:

$$\mathfrak{K} : \{\Omega, \vec{l}, \vec{j}\}$$

$$\begin{aligned} \Omega &= \frac{1}{i}\beta \\ \vec{l} &= \frac{1}{4i}(1+\beta)\vec{\sigma} \\ \vec{j} &= \frac{1}{4i}(1-\beta)\vec{\sigma} \end{aligned} \tag{2.10}$$

with

$$\begin{aligned} [\Omega, l_i] = 0 \quad & [l_i, l_j] = \varepsilon_{ijk} l_k \\ [\Omega, j_i] = 0 \quad & [j_i, j_j] = \varepsilon_{ijk} j_k \\ [l_i, j_k] &= 0\,. \end{aligned} \tag{2.11}$$

These relations show that $\mathfrak{K}$ is isomorphic to the Lie algebra of $SU(2) \otimes SU(2) \otimes C$, where C is the centre of $\mathfrak{K}$. The centre C itself is isomorphic to the Lie algebra of $U(1)$.

In the vector space $\mathfrak{P}$ we select a maximal Abelian subalgebra $\mathfrak{A}$, generated by

$$\mathfrak{A} : \{a_1, a_2\}$$

where

$$\begin{aligned} a_1 &= \gamma_5 \frac{1+\sigma_3}{2} \\ a_2 &= \gamma_5 \frac{1-\sigma_3}{2} \end{aligned} \tag{2.12}$$

with

$$[a_1, a_2] = 0 \,.$$

Finally the remaining basis vectors of $\mathfrak{P}$ in suitable linear combinations with the basis vectors of $\mathfrak{K}$ generate another subalgebra $\mathfrak{N}$. We choose the following basis

$$\mathfrak{N} : \{n_1, n_2, n_3, n_4, n_5, n_6\}$$

where

$$\begin{aligned} n_1 &= i(1-\gamma_5)\,\beta\sigma_1 \\ n_2 &= i(1-\gamma_5)\,\beta\sigma_2 \\ n_3 &= i(1-\gamma_5)\beta\,\frac{1+\sigma_3}{2} \\ n_4 &= i(1-\gamma_5)\beta\,\frac{1-\sigma_3}{2} \\ n_5 &= i\sigma_1 - \sigma_2\gamma_5 \\ n_6 &= i\sigma_2 + \sigma_1\gamma_5 \end{aligned} \tag{2.13}$$

with

$$\begin{aligned} [n_i, n_j] &= 0 \quad \text{for} \quad i, j = 1, 2, 3, 4 \\ [n_1, n_5] &= 0 \\ [n_1, n_6] &= 4n_4 \\ [n_2, n_5] &= -4n_4 \\ [n_2, n_6] &= 0 \\ [n_3, n_5] &= -2n_2 \\ [n_3, n_6] &= 2n_1 \\ [n_4, n_5] &= 0 \\ [n_4, n_6] &= 0 \\ [n_5, n_6] &= 0 \,. \end{aligned}$$

The subalgebra $\mathfrak{N}$ is nilpotent with $\mathfrak{N}^4 = 0$. Thus we arrive at the Iwasawa decomposition [20] of the Lie algebra $\mathfrak{G}$

$$\mathfrak{G} = \mathfrak{K} + \mathfrak{A} + \mathfrak{N} \tag{2.14}$$

where the sum of the three subalgebras has to be understood as the direct sum of vector spaces. Furthermore $\mathfrak{S} = \mathfrak{A} + \mathfrak{N}$ is a solvable Lie algebra:

$$\mathfrak{S}^{(3)} = 0$$

$$\begin{array}{ll}
[a_1, n_1] = -n_1 & [a_2, n_1] = -n_1 \\
[a_1, n_2] = -n_2 & [a_2, n_2] = -n_2 \\
[a_1, n_3] = -2n_3 & [a_2, n_3] = 0 \\
[a_1, n_4] = 0 & [a_2, n_4] = -2n_4 \\
[a_1, n_5] = n_5 & [a_2, n_5] = -n_5 \\
[a_1, n_6] = n_6 & [a_2, n_6] = -n_6 \,.
\end{array}$$

From these relations follows that $\mathfrak{N}$ is an ideal in $\mathfrak{S}$.

To the decomposition (2.14) of the Lie algebra corresponds the decomposition

$$G = KAN \tag{2.15}$$

of the group G, where K, A and N are groups with Lie algebras $\mathfrak{K}$ $\mathfrak{A}$ and $\mathfrak{N}$ respectively. As we are only interested in infinitesimal translations on the group, we can take for the groups K, A and N just the integrated Lie algebras, denoted by $\hat{K}$, A and N, and disregard discrete centres which always lie in K. Thus we obtain for an element $\hat{\mathbf{k}} \in \hat{K}$, introducing Euler angles:

$$\hat{\mathbf{k}} = \begin{pmatrix}
e^{-i\omega}\cos\frac{\beta}{2}e^{-i\frac{\alpha+\gamma}{2}} & -ie^{-i\omega}\sin\frac{\beta}{2}e^{-i\frac{\alpha-\gamma}{2}} & 0 & 0 \\
-ie^{-i\omega}\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}} & e^{-i\omega}\cos\frac{\beta}{2}e^{i\frac{\alpha+\gamma}{2}} & 0 & 0 \\
0 & 0 & e^{i\omega}\cos\frac{\theta}{2}e^{-i\frac{\varphi+\psi}{2}} & -ie^{i\omega}\sin\frac{\theta}{2}e^{-i\frac{\varphi-\psi}{2}} \\
0 & 0 & -ie^{i\omega}\sin\frac{\theta}{2}e^{i\frac{\varphi-\psi}{2}} & e^{i\omega}\cos\frac{\theta}{2}e^{i\frac{\varphi+\psi}{2}}
\end{pmatrix}. \tag{2.16}$$

In A and N we introduce canonical co-ordinates of the second kind and get for the elements $\mathbf{a}$ and $\mathbf{n}$ of the groups A and N respectively

$$\mathbf{a} = e^{\lambda a_1} e^{\mu a_2}$$

$$\mathbf{n} = e^{p n_3} e^{q n_4} e^{r n_1} e^{s n_2} e^{t n_5} e^{u n_6}$$

which lead to

$$\mathbf{a} = \begin{pmatrix}
\cosh\lambda & 0 & \sinh\lambda & 0 \\
0 & \cosh\mu & 0 & \sinh\mu \\
\sinh\lambda & 0 & \cosh\lambda & 0 \\
0 & \sinh\mu & 0 & \cosh\mu
\end{pmatrix} \tag{2.17}$$

and

$$\mathrm{n} = \begin{pmatrix} 1+ip & (1+2ip)(u+it)+s+ir & ip & (1+2ip)(u+it)+s+ir \\ -s+ir-u+it & 1+iq+2(u+it)(ir-s) & u-it+ir-s & iq+2(u+it)(ir-s) \\ -ip & -s-ir+(1-2ip)(u+it) & 1-ip & -s-ir+(1-2ip)(u+it) \\ s-ir+u-it & -iq+2(s-ir)(u+it) & s-ir-u+it & 1-iq+2(u+it)(s-ir) \end{pmatrix}. \tag{2.18}$$

Thus we arrived at a parametrization of the group element

$$\hat{\mathrm{g}} = \widehat{\mathrm{k}}\mathrm{an} \tag{2.19}$$

which will be used in the sequel to derive representations of the Lie algebra. In the following we can omit the "hat" in all considerations on the Lie algebra as already indicated.

3. Realization of the Lie algebra in the carrier space $K \times A$

Our aim is to find those representations of the Lie algebra $\mathfrak{G}$ which correspond to irreducible unitary representations of the group G. When we denote by x_l the generators of $\mathfrak{G}$ (fundamental representation by 4×4 matrices) and with $\mathfrak{X}_l$ a representation of them as operators acting in a Hilbert space

$$x_l \to \mathfrak{X}_l \tag{3.1}$$

this means, among other things, that the operators $\mathfrak{X}_l$ multiplied by i have to be self-adjoint. In order to find such representations, we shall follow the method given in Ref. [15]. We start by considering the quotient space G/N, i.e., the set of left cosets with respect to the subgroup N. Because of the Iwasawa decomposition

$$\mathrm{g} = \mathrm{kan}$$

with $\mathrm{g} \in G$; $\mathrm{k} \in K$; $\mathrm{a} \in A$; and $\mathrm{n} \in N$ we can identify every coset $\mathrm{g}N$ with ka

$$\mathrm{g}N = \mathrm{kan}N = \mathrm{ka}N\,.$$

Now left multiplication by $\mathrm{g}_0 \in G$ acts transitively in the coset space

$$\begin{aligned} \mathrm{g}_0\mathrm{g}N &= \mathrm{g}_0\mathrm{ka}N \\ &= \mathrm{k}'(\mathrm{g}_0, \mathrm{k})\,\mathrm{a}'(\mathrm{g}_0, \mathrm{k})\,\mathrm{n}'(\mathrm{g}_0, \mathrm{k})\,\mathrm{a}N \\ &= \mathrm{k}'(\mathrm{g}_0, \mathrm{k})\,\mathrm{a}'(\mathrm{g}_0, \mathrm{k})\,\mathrm{a}N \end{aligned} \tag{3.2}$$

In performing the last step we used the fact that A together with N forms a group with N as an invariant subgroup. Thus we have

$$\mathrm{ka} \xrightarrow{\mathrm{g}_0} \mathrm{k}'(\mathrm{g}_0, \mathrm{k})\,\mathrm{a}'(\mathrm{g}_0, \mathrm{k})\,\mathrm{a}\,. \tag{3.3}$$

Now it is important to remark that the change in the parameters induced by the left translation depends only on g_0 and on k but not on a. This is obvious for k' and follows for $\mathrm{a}'\mathrm{a}$ from the fact that A is an Abelian subgroup. Furthermore a translation by $\mathrm{k}_0 \in K$ changes only k. We therefore consider functions forming a linear space over $K \times A$ of the special form [15]

$$\mathring{f}(\mathbf{k}, \mathbf{a}) = e^{a\lambda + b\mu} f(\omega; \alpha, \beta, \gamma; \varphi, \theta, \psi) . \tag{3.4}$$

This specification is in fact a step in the direction of selecting the irreducible representation spaces. The parameters a and b will then serve to label these irreducible spaces. They will be fixed later together with the introduction of a scalar product in the linear space of the functions f. We define a representation of G by

$$T_{\mathbf{g}} \mathring{f}(\mathbf{k}, \mathbf{a}) = \mathring{f}(\mathbf{k}', \mathbf{a}'\mathbf{a}) \tag{3.5}$$

where $\mathbf{k}'$ and $\mathbf{a}'$ are given by the formula (3.3) replacing $\mathbf{g}_0$ by $\mathbf{g}^{-1}$.

Consider now the operators corresponding to infinitesimal transformations. Let

$$\chi_l(t) = e^{t x_l}$$

then, because of (3.1) and (3.5), we have

$$\begin{aligned} \mathfrak{X}_l \mathring{f}(\mathbf{k}, \mathbf{a}) &= \lim_{t \to 0} \frac{T_{\chi_l(t)} - 1}{t} \mathring{f}(\mathbf{k}, \mathbf{a}) \\ &= \sum_i \frac{\partial \mathring{f}}{\partial \nu_i} \left(\frac{\partial \nu_i'}{\partial t} \right)_{t=0} \end{aligned} \tag{3.6}$$

where ν_i with $i = 1$ to 9 stands for the variables $\lambda, \mu, \omega, \alpha, \beta, \gamma, \varphi, \theta, \psi$. We obtain the derivative of the changed variables ν_i', which depend also on the index l of the transforming x_l, from the fundamental representation using its decomposition (2.19)

$$\mathbf{g}' = e^{-t x_l} \mathbf{g} . \tag{3.7}$$

By differentiation, this gives

$$\sum_i \frac{\partial \mathbf{g}}{\partial \nu_i} \left(\frac{\partial \nu_i'}{\partial t} \right)_{t=0} = -x_l \mathbf{g} \tag{3.8}$$

which is a 4×4 matrix equation determining $\left(\frac{\partial \nu_i'}{\partial t} \right)_{t=0}$ for every l. When we denote by corresponding capital letters the operators $\mathfrak{X}_l$, we get for those of the maximal compact subalgebra according to Eq. (3.6)

$$\begin{aligned} \Omega &= -\frac{\partial}{\partial \omega} \\ L_1 &= \sin\alpha \operatorname{ctg}\beta \frac{\partial}{\partial \alpha} - \cos\alpha \frac{\partial}{\partial \beta} - \frac{\sin\alpha}{\sin\beta} \frac{\partial}{\partial \gamma} \\ L_2 &= -\cos\alpha \operatorname{ctg}\beta \frac{\partial}{\partial \alpha} - \sin\alpha \frac{\partial}{\partial \beta} + \frac{\cos\alpha}{\sin\beta} \frac{\partial}{\partial \gamma} \\ L_3 &= -\frac{\partial}{\partial \alpha} \\ J_1 &= \sin\varphi \operatorname{ctg}\theta \frac{\partial}{\partial \varphi} - \cos\varphi \frac{\partial}{\partial \theta} - \frac{\sin\varphi}{\sin\theta} \frac{\partial}{\partial \psi} \\ J_2 &= -\cos\varphi \operatorname{ctg}\theta \frac{\partial}{\partial \varphi} - \sin\varphi \frac{\partial}{\partial \theta} + \frac{\cos\varphi}{\sin\theta} \frac{\partial}{\partial \psi} \\ J_3 &= -\frac{\partial}{\partial \varphi} . \end{aligned} \tag{3.9}$$

From the remaining generators it is sufficient to calculate one with the method stated above, as all the others can be obtained from it by commutation of this one with the compact generators (3.9). We choose

$$\begin{aligned}
A_1 = &-\frac{1}{2}\left\{\cos\frac{\beta}{2}\cos\frac{\theta}{2}\sin\zeta_1 + \sin\frac{\beta}{2}\sin\frac{\theta}{2}\sin\zeta_2\right\}\frac{\partial}{\partial\omega} \\
&-\frac{\sin\frac{\theta}{2}}{\sin\frac{\beta}{2}}\sin\zeta_2\frac{\partial}{\partial\alpha} \\
&+2\cos\frac{\beta}{2}\sin\frac{\theta}{2}\cos\zeta_2\frac{\partial}{\partial\beta} \\
&+\left\{\frac{\sin\frac{\theta}{2}\cos^2\frac{\beta}{2}}{\sin\frac{\beta}{2}}\sin\zeta_2 - \cos\frac{\beta}{2}\cos\frac{\theta}{2}\sin\zeta_1\right\}\frac{\partial}{\partial\gamma} \\
&+\frac{\sin\frac{\beta}{2}}{\sin\frac{\theta}{2}}\sin\zeta_2\frac{\partial}{\partial\varphi} \\
&+2\cos\frac{\theta}{2}\sin\frac{\beta}{2}\cos\zeta_2\frac{\partial}{\partial\theta} \\
&+\left\{\cos\frac{\beta}{2}\cos\frac{\theta}{2}\sin\zeta_1 - \frac{\sin\frac{\beta}{2}}{\sin\frac{\theta}{2}}\cos^2\frac{\theta}{2}\sin\zeta_2\right\}\frac{\partial}{\partial\psi} \\
&-a\cos\frac{\beta}{2}\cos\frac{\theta}{2}\cos\zeta_1 \\
&-b\sin\frac{\beta}{2}\sin\frac{\theta}{2}\cos\zeta_2
\end{aligned} \tag{3.10}$$

where we introduced

$$\begin{aligned}
\zeta_1 &= \frac{1}{2}(4\omega + \alpha + \gamma - \varphi - \psi) \\
\zeta_2 &= \frac{1}{2}(4\omega + \alpha - \gamma - \varphi + \psi)\,.
\end{aligned} \tag{3.11}$$

4. Explicit construction of the Hilbert space

Our next step is to introduce suitable scalar products in the linear space of the functions f in order to construct the irreducible unitary representations of the group. Here we are only interested in the classification of the representations, and we use the explicit representation of some generators only as an auxiliary tool. It is sufficient to consider only the compact generators and one of the non-compact ones. Indeed it can be shown that the whole group can be generated from the compact generators and their commutators with any one of the non-compact ones, say A_1 in formula (3.10). Thus we are free to consider, besides the whole compact algebra $\mathfrak{K}$ either the generator A_1, or any commutator it appears in. All conclusions concerning the series of representations will be independent of this choice.

For the functions f, we now choose a basis which is adapted to the splitting of $\hat{K}$ into $U(1) \otimes SU(2) \otimes SU(2)$, i.e., we set

$$f(\omega, \alpha, \beta, \gamma, \varphi, \theta, \psi) = \sum_{\varkappa \ldots k} F(\varkappa, l, m, n, j, h, k)\, e^{2i\varkappa\omega}\, \mathscr{D}^l_{mn}(\alpha, \beta, \gamma)\, \mathscr{D}^j_{hk}(\varphi, \theta, \psi) \tag{4.1}$$

where $\mathscr{D}^l_{mn}$ and $\mathscr{D}^j_{hk}$ are Wigner functions. Their properties are recalled in the Appendix.

When f is a one-valued function on $\hat{K}$, $\varkappa$ has to be integer or half-integer and the sets l, m, n and j, h, k take the usual integer or half-integer values. Let us introduce the notation

$$|\phi_i\rangle \equiv |\varkappa, l, m, n, j, h, k\rangle \equiv e^{2i\varkappa\omega}\, \mathscr{D}^l_{mn}(\alpha, \beta, \gamma)\, \mathscr{D}^j_{hk}(\varphi, \theta, \psi)\,. \tag{4.2}$$

Now it follows from Eq. (3.9)

$$\begin{aligned}
i\Omega|\varkappa, l, m, n, j, h, k\rangle &= 2\varkappa|\varkappa, l, m, n, j, h, k\rangle\\
iL_3|\varkappa, l, m, n, j, h, k\rangle &= m|\varkappa, l, m, n, j, h, k\rangle\\
(L_1 \pm iL_2)|\varkappa, l, m, n, j, h, k\rangle &= -\sqrt{(l \mp m)(l \pm m + 1)}\,|\varkappa, l, m \pm 1, n, j, h, k\rangle\\
iJ_3|\varkappa, l, m, n, j, h, k\rangle &= h|\varkappa, l, m, n, j, h, k\rangle\\
(J_1 \pm iJ_2)|\varkappa, l, m, n, j, h, k\rangle &= -\sqrt{(j \mp h)(j \pm h + 1)}\,|\varkappa, l, m, n, j, h \pm 1, k\rangle\,.
\end{aligned} \tag{4.3}$$

Thus the compact generators take the form of matrices acting on the basis vectors.

Let us now introduce a suitable non-compact operator. If we take the commutator $\tilde{A} = -\frac{1}{2}[\Omega, A_1]$ and form the conjugate combinations $A_\pm = A_1 \pm i\tilde{A}$, these have the form:

$$\begin{aligned}
A_\pm = &\mp \frac{i}{2}\left\{\cos\frac{\beta}{2}\cos\frac{\theta}{2}e^{\mp i\zeta_1} + \sin\frac{\beta}{2}\sin\frac{\theta}{2}e^{\mp i\zeta_2}\right\}\frac{\partial}{\partial\omega} \mp\\
&\mp i\frac{\sin\frac{\theta}{2}}{\sin\frac{\beta}{2}}e^{\mp i\zeta_2}\frac{\partial}{\partial\alpha} + 2\cos\frac{\beta}{2}\sin\frac{\theta}{2}e^{\mp i\zeta_2}\frac{\partial}{\partial\beta} \pm\\
&\pm i\left\{\frac{\sin\frac{\theta}{2}\cos^2\frac{\beta}{2}}{\sin\frac{\beta}{2}}e^{\mp i\zeta_2} - \cos\frac{\beta}{2}\cos\frac{\theta}{2}e^{\mp i\zeta_1}\right\}\frac{\partial}{\partial\gamma} \pm\\
&\pm i\frac{\sin\frac{\beta}{2}}{\sin\frac{\theta}{2}}e^{\mp i\zeta_2}\frac{\partial}{\partial\varphi} + 2\cos\frac{\theta}{2}\sin\frac{\beta}{2}e^{\mp i\zeta_2}\frac{\partial}{\partial\theta} \pm\\
&\pm i\left\{\cos\frac{\beta}{2}\cos\frac{\theta}{2}e^{\mp i\zeta_1} - \frac{\sin\frac{\beta}{2}\cos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}}e^{\mp i\zeta_2}\right\}\frac{\partial}{\partial\psi} -\\
&- a\cos\frac{\beta}{2}\cos\frac{\theta}{2}e^{\mp i\zeta_1} - b\sin\frac{\beta}{2}\sin\frac{\theta}{2}e^{\mp i\zeta_2}\,.
\end{aligned} \tag{4.4}$$

Now we can apply these operators to the functions $|\phi_i\rangle$ of Eq. (4.2) and we see, taking into account the properties of the Wigner functions, that they transform a given $|\phi_i\rangle$ into a linear combination of such functions. Then the operators $A_\pm$ take also the form of matrices acting on the labelled vectors:

$$
\begin{aligned}
&(2l+1)(2j+1)A_+|\varkappa, l, m, n, j, h, k\rangle \\
&= \{\varkappa - a + n - k\} \sqrt{(j+h+1)(j+k+1)(l-m+1)(l-n+1)} \times \\
&\qquad \times \left|\varkappa - 1, l + \tfrac{1}{2}, m - \tfrac{1}{2}, n - \tfrac{1}{2}, j + \tfrac{1}{2}, h + \tfrac{1}{2}, k + \tfrac{1}{2}\right\rangle + \\
&+ \{\varkappa - a + n - k\} \sqrt{(l-m+1)(l-n+1)(j-h)(j-k)} \times \\
&\qquad \times \left|\varkappa - 1, l + \tfrac{1}{2}, m - \tfrac{1}{2}, n - \tfrac{1}{2}, j - \tfrac{1}{2}, h + \tfrac{1}{2}, k + \tfrac{1}{2}\right\rangle + \\
&+ \{\varkappa - a + n - k\} \sqrt{(l+m)(l+n)(j+h+1)(j+k+1)} \times \\
&\qquad \times \left|\varkappa - 1, l - \tfrac{1}{2}, m - \tfrac{1}{2}, n - \tfrac{1}{2}, j + \tfrac{1}{2}, h + \tfrac{1}{2}, k + \tfrac{1}{2}\right\rangle + \\
&+ \{\varkappa - a + n - k\} \sqrt{(l+m)(l+n)(j-h)(j-k)} \times \\
&\qquad \times \left|\varkappa - 1, l - \tfrac{1}{2}, m - \tfrac{1}{2}, n - \tfrac{1}{2}, j - \tfrac{1}{2}, h + \tfrac{1}{2}, k + \tfrac{1}{2}\right\rangle + \\
&+ \{-\varkappa + b + 2(l+j) - (n-k)\} \times \\
&\times \sqrt{(l-m+1)(l+n+1)(j+h+1)(j-k+1)} \times \qquad (4.5) \\
&\qquad \times \left|\varkappa - 1, l + \tfrac{1}{2}, m - \tfrac{1}{2}, n + \tfrac{1}{2}, j + \tfrac{1}{2}, h + \tfrac{1}{2}, k - \tfrac{1}{2}\right\rangle + \\
&+ \{\varkappa - b + 2(l-j) + n - k + 2\} \times \\
&\times \sqrt{(l+m)(l-n)(j+h+1)(j-k+1)} \times \\
&\qquad \times \left|\varkappa - 1, l - \tfrac{1}{2}, m - \tfrac{1}{2}, n + \tfrac{1}{2}, j + \tfrac{1}{2}, h + \tfrac{1}{2}, k - \tfrac{1}{2}\right\rangle + \\
&+ \{\varkappa - b - 2(l-j) + n - k + 2\} \times \\
&\times \sqrt{(l-m+1)(l+n+1)(j-h)(j+k)} \times \\
&\qquad \times \left|\varkappa - 1, l + \tfrac{1}{2}, m - \tfrac{1}{2}, n + \tfrac{1}{2}, j - \tfrac{1}{2}, h + \tfrac{1}{2}, k - \tfrac{1}{2}\right\rangle + \\
&+ \{-\varkappa + b - 2(l+j) - (n-k) - 4\} \times \\
&\times \sqrt{(l+m)(l-n)(j-h)(j+k)} \times \\
&\qquad \times \left|\varkappa - 1, l - \tfrac{1}{2}, m - \tfrac{1}{2}, n + \tfrac{1}{2}, j - \tfrac{1}{2}, h + \tfrac{1}{2}, k - \tfrac{1}{2}\right\rangle .
\end{aligned}
$$

The corresponding expression for A_- is obtained by changing the signs of $\varkappa$, m, n, h, k in the above coefficients and performing in the vectors $|\phi_i\rangle$ the substitutions $\varkappa - 1 \to \varkappa + 1$; $m - \frac{1}{2} \to m + \frac{1}{2}$; $n \pm \frac{1}{2} \to$ $\to n \mp \frac{1}{2}$; $h + \frac{1}{2} \to h - \frac{1}{2}$ and $k \pm \frac{1}{2} \to k \mp \frac{1}{2}$.

We remark that the indices m, n, h, k transform in such a way that the sums $m + h$, $n + k$ are unaffected. If we look at the compact operators, we see that m and h transform independently, but n and k are unchanged. This shows that, under the considered operators, the three numbers a, b and $c = n + k$ are constant. But as all other operators are commutators of the above ones, this conservation holds for the whole group, so that *the three numbers a, b, c label the irreducible representations.*

Now our problem is to endow the above linear space with a *metric*, in such a way that the group is represented by unitary matrices. In other words, all generators of its Lie algebra (multiplied by i) must be self-adjoint with respect to this metric. It will be sufficient to fulfil this condition for the compact operators and for A_1, as the other ones can be built up with commutators. The choice we have made for the function f in Eq. (3.4) on the manifold A is related to our aim of reducing the function space over $K \times A$ into *irreducible* spaces. Thus we need only to define the metric inside these irreducible spaces and we do this as general as possible. Moreover as the functions f which span these spaces are defined only on the compact subgroup K, we can restrict ourselves to integration over K [15].

Thus, we put for any two functions f_1, f_2 on K:

$$(f_1, f_2) = \int\int f_1(\mathrm{k}_1)^* \, M(\mathrm{k}_1, \mathrm{k}_2) \, f_2(\mathrm{k}_2) \, d\mathrm{k}_1 \, d\mathrm{k}_2 \tag{4.6}$$

where k_1, k_2 represent two different "points" in the compact manifold K, $d\mathrm{k}_1$, $d\mathrm{k}_2$ are the invariant volume elements on K, and M is a *kernel* defining an invariant bilocal measure. According to formula (4.1), we now have

$$(f_1, f_2) = \sum_{i,j} F_1^*(\phi_i) \, M\langle\phi_i|\phi_j\rangle \, F_2(\phi_j) \tag{4.7}$$

where $M\langle\phi_i|\phi_j\rangle$ are the matrix elements of the kernel in Eq. (4.6).

The metric must be Hermitian: $(f_2, f_1) = (f_1, f_2)^*$, which yields: $M\langle\phi_j|\phi_i\rangle = M\langle\phi_i|\phi_j\rangle^*$, that is $M = M^+$. It must also be positive definite: $(f, f) \geqq 0$ for any f, which implies that the real numbers

$$\sum_{i,j} F^*(\phi_i) \, M\langle\phi_i|\phi_j\rangle \, F(\phi_j)$$

are positive for any $F(\phi_i)$. Since any self-adjoint operator $i\mathfrak{X}$ has to fulfil $(f_1, \mathfrak{X} f_2) + (\mathfrak{X} f_1, f_2) = 0$. We get

$$\sum_{ijk} F_1^*(\phi_i)\, M\langle\phi_i|\phi_j\rangle\, \mathfrak{X}\langle\phi_j|\phi_k\rangle\, F_2(\phi_k) + \\ + \sum_{ijk} F_1^*(\phi_i)\, \mathfrak{X}^*\langle\phi_j|\phi_i\rangle\, M\langle\phi_j|\phi_k\rangle\, F_2(\phi_k) = 0$$

where $\mathfrak{X}\langle\phi_j|\phi_k\rangle$ are the matrix elements of $\mathfrak{X}$. That is, finally:

$$\sum_j M\langle\phi_i|\phi_j\rangle\, \mathfrak{X}\langle\phi_j|\phi_k\rangle + \sum_j \mathfrak{X}^*\langle\phi_j|\phi_i\rangle\, M\langle\phi_j|\phi_k\rangle = 0\,. \tag{4.8}$$

Now, what we have to do is to write down explicitly the above condition for each of the generators of the group, and to derive the conclusions on the allowed structure of the matrix M. For instance, if we consider only, for the sake of brevity, the indices l, m of one of the compact $SU(2)$ subalgebras, we have the relation

$$\sum_{l,m} M\langle l', m'|l, m\rangle\, L_+\langle l, m|l'', m''\rangle + \\ + \sum_{l,m} L_-^*\langle l, m|l', m'\rangle\, M\langle l, m|l'', m''\rangle = 0$$

$$L_\pm = L_1 \pm iL_2\,.$$

Using the expression (4.3) for $L_\pm$, we get

$$M\langle l', m'|lm\rangle = \tilde{\mathscr{M}}(l)\, \delta\langle l'|l\rangle\, \delta\langle m'|m\rangle \tag{4.9}$$

where the elements on the right-hand side depend on l, but not on m. This can be extended to the whole compact subalgebra $\mathfrak{K}$.

By considering explicitly the whole set of indices, it is more suitable to introduce the following notation:

$$c = n + k \quad N = n - k \quad J_+ = j + l \quad J_- = j - l\,. \tag{4.10}$$

The above result on the compact generators allows us to write:

$$M\langle\phi_i|\phi_j\rangle \\ = \tilde{\mathscr{M}}(\varkappa, l, j)\, \langle N|N'\rangle\, \delta\langle\varkappa|\varkappa'\rangle\, \delta\langle l|l'\rangle\, \delta\langle j|j'\rangle\, \delta\langle m|m'\rangle\, \delta\langle h|h'\rangle\,.$$

Now, considering the non-compact operators $A_\pm$, equation (4.8) reads

$$\sum_N \tilde{\mathscr{M}}(\varkappa', l', j')\, \langle N'|N\rangle\, A_+\langle\varkappa', l', j', m', h', N|\varkappa'', l'', j'', m'', h'', N''\rangle + \\ + \sum_N A_-^*\langle\varkappa'', l'', j'', m'', k'', N|\varkappa', l', j', m', h', N'\rangle \times \\ \times \tilde{\mathscr{M}}(\varkappa'', l'', j'')\, \langle N|N''\rangle = 0\,. \tag{4.11}$$

By looking at the matrix elements of $A_\pm$ deduced from the above formula (4.5) we see it is suitable to put them in the form

$$A_+\langle \varkappa, l, j, m, h, N \mid \varkappa', l', j', m', h', N' \rangle$$
$$= \mathscr{A}_+(a, b, c, \varkappa, m, h) \langle l, j, N \mid l', j', N' \rangle \times$$
$$\times \frac{\delta\langle \varkappa - 1 \mid \varkappa' \rangle \, \delta\left\langle m - \frac{1}{2} \,\middle|\, m' \right\rangle \delta\left\langle h + \frac{1}{2} \,\middle|\, h' \right\rangle}{(2l+1)(2j+1)}$$

$$A_-\langle \varkappa, l, j, m, h, N \mid \varkappa', l', j', m', h', N' \rangle$$
$$= \mathscr{A}_-(a, b, c, \varkappa, m, h) \langle l, j, N \mid l', j', N' \rangle \times$$
$$\times \frac{\delta\langle \varkappa + 1 \mid \varkappa' \rangle \, \delta\left\langle m + \frac{1}{2} \,\middle|\, m' \right\rangle \delta\left\langle h - \frac{1}{2} \,\middle|\, h' \right\rangle}{(2l+1)(2j+1)}$$

and then to set

$$\tilde{\mathscr{M}}(\varkappa, l, j) \langle N \mid N' \rangle = \frac{1}{(2l+1)(2j+1)} \mathscr{M}(\varkappa, l, j) \langle N \mid N' \rangle .$$

This yields

$$\sum_N \mathscr{M}(\varkappa', l', j') \langle N' \mid N \rangle \mathscr{A}_+(a, b, c, \varkappa, m, h) \langle l', j', N \mid l'', j'', N'' \rangle +$$
$$+ \sum_N \mathscr{A}_-^*(a, b, c, \varkappa, m, h) \langle l'', j'', N \mid l', j', N' \rangle \mathscr{M}(\varkappa'', l'', j'') \langle N \mid N'' \rangle = 0 ,$$

from which we get a rather complicated matrix equation which decomposes into four independent ones, because of the Kronecker deltas.

For instance, grouping the coefficients of

$$\delta\left\langle l' + \frac{1}{2} \,\middle|\, l'' \right\rangle \delta\left\langle j + \frac{1}{2} \,\middle|\, j'' \right\rangle$$

and those of

$$\delta\left\langle l' \,\middle|\, l'' - \frac{1}{2} \right\rangle \delta\left\langle j' \,\middle|\, j'' - \frac{1}{2} \right\rangle$$

we get, changing the primed variables to unprimed ones and replacing the double primes by a hat:

$$0 = (\varkappa + N - a^*) \sqrt{\left(j + 1 + \frac{c - N}{2}\right)\left(l + 1 - \frac{c + N}{2}\right)} \times$$
$$\times \mathscr{M}\left(\varkappa - 1, l + \frac{1}{2}, j + \frac{1}{2}\right) \langle N - 1 \mid \hat{N} \rangle +$$
$$+ (-\varkappa + b^* + 2J_+ - N) \sqrt{\left(l + 1 + \frac{c + N}{2}\right)\left(j + 1 - \frac{c - N}{2}\right)} \times$$
$$\times \mathscr{M}\left(\varkappa - 1, l + \frac{1}{2}, j + \frac{1}{2}\right) \langle N + 1 \mid \hat{N} \rangle +$$
$$+ (-\varkappa - \hat{N} + 1 - a) \sqrt{\left(l + \frac{1}{2} - \frac{c + \hat{N}}{2}\right)\left(j + \frac{1}{2} + \frac{c - \hat{N}}{2}\right)} \times$$
$$\times \mathscr{M}(\varkappa, l, j) \langle N \mid \hat{N} + 1 \rangle +$$
$$+ (\varkappa - 2J_+ + \hat{N} - 7 + b) \sqrt{\left(l + \frac{1}{2} + \frac{c + \hat{N}}{2}\right)\left(j + \frac{1}{2} - \frac{c - \hat{N}}{2}\right)} \times$$
$$\times \mathscr{M}(\varkappa, l, j) \langle N \mid \hat{N} - 1 \rangle .$$

Likewise

$$
\begin{aligned}
0 = {} & (\varkappa + N - a^*) \sqrt{\left(l + 1 - \frac{c+N}{2}\right)\left(j - \frac{c-N}{2}\right)} \times \\
& \times \mathscr{M}\left(\varkappa - 1, l + \frac{1}{2}, j - \frac{1}{2}\right) \langle N - 1 | \hat{N} \rangle + \\
& + (\varkappa - 2J_- + N + 2 - b^*) \sqrt{\left(l + 1 + \frac{c+N}{2}\right)\left(j + \frac{c-N}{2}\right)} \times \\
& \times \mathscr{M}\left(\varkappa - 1, l + \frac{1}{2}, j - \frac{1}{2}\right) \langle N + 1 | \hat{N} \rangle + \\
& + (-\varkappa - \hat{N} + 1 - a) \sqrt{\left(l + \frac{1}{2} - \frac{c+\hat{N}}{2}\right)\left(j + \frac{1}{2} - \frac{c-\hat{N}}{2}\right)} \times \\
& \times \mathscr{M}(\varkappa, l, j) \langle N | \hat{N} + 1 \rangle + \\
& + (-\varkappa + 2J_- - \hat{N} + 5 - b) \sqrt{\left(l + \frac{1}{2} + \frac{c+\hat{N}}{2}\right)\left(j + \frac{1}{2} + \frac{c-\hat{N}}{2}\right)} \times \\
& \times \mathscr{M}(\varkappa, l, j) \langle N | \hat{N} - 1 \rangle \\
0 = {} & (\varkappa + N - a^*) \sqrt{\left(l + \frac{c+N}{2}\right)\left(j + 1 + \frac{c-N}{2}\right)} \times \\
& \times \mathscr{M}\left(\varkappa - 1, l - \frac{1}{2}, j + \frac{1}{2}\right) \langle N - 1 | \hat{N} \rangle + \\
& + (\varkappa + 2J_- + N + 2 - b^*) \sqrt{\left(l - \frac{c+N}{2}\right)\left(j + 1 - \frac{c-N}{2}\right)} \times \\
& \times \mathscr{M}\left(\varkappa - 1, l - \frac{1}{2}, j + \frac{1}{2}\right) \langle N + 1 | \hat{N} \rangle + \\
& + (-\varkappa - \hat{N} + 1 - a) \sqrt{\left(l + \frac{1}{2} + \frac{c+\hat{N}}{2}\right)\left(j + \frac{1}{2} + \frac{c-\hat{N}}{2}\right)} \times \\
& \times \mathscr{M}(\varkappa, l, j) \langle N | \hat{N} + 1 \rangle + \\
& + (-\varkappa - 2J_- - \hat{N} + 5 - l) \sqrt{\left(l + \frac{1}{2} - \frac{c+\hat{N}}{2}\right)\left(j + \frac{1}{2} - \frac{c-\hat{N}}{2}\right)} \times \\
& \times \mathscr{M}(\varkappa, l, j) \langle N | \hat{N} - 1 \rangle \\
0 = {} & (\varkappa + N - a^*) \sqrt{\left(l + \frac{c+N}{2}\right)\left(j - \frac{c-N}{2}\right)} \times \\
& \times \mathscr{M}\left(\varkappa - 1, l - \frac{1}{2}, j - \frac{1}{2}\right) \langle N - 1 | \hat{N} \rangle + \\
& + (-\varkappa - 2J_+ - N - 4 + b^*) \sqrt{\left(l - \frac{c+N}{2}\right)\left(j + \frac{c-N}{2}\right)} \times \\
& \times \mathscr{M}\left(\varkappa - 1, l - \frac{1}{2}, j - \frac{1}{2}\right) \langle N + 1 | \hat{N} \rangle + \\
& + (-\varkappa - \hat{N} + 1 - a) \sqrt{\left(l + \frac{1}{2} + \frac{c+\hat{N}}{2}\right)\left(j + \frac{1}{2} - \frac{c-\hat{N}}{2}\right)} \times \\
& \times \mathscr{M}(\varkappa, l, j) \langle N | \hat{N} + 1 \rangle + \\
& + (\varkappa + 2J_+ + \hat{N} - 3 + b) \sqrt{\left(l + \frac{1}{2} - \frac{c+\hat{N}}{2}\right)\left(j + \frac{1}{2} + \frac{c-\hat{N}}{2}\right)} \times \\
& \times \mathscr{M}(\varkappa, l, j) \langle N | \hat{N} - 1 \rangle .
\end{aligned}
\tag{4.12}
$$

These four separate equations are still matrix equations because of the appearance of the non-diagonal $\mathcal{M}\langle N'|N''\rangle$ elements and at this stage, the resolution of such relations seems to bring into play very serious difficulties. Now as a first step towards a complete solution of the problem of the unitary representations of $SU(2, 2)$, we shall restrict ourselves to a simplifying assumption: we suppose that the matrix M is also diagonal in the label N, namely:

$$\mathcal{M}(\varkappa', l', j')\,\langle N'|N''\rangle = \mathcal{B}(\varkappa', J'_+, J'_-, N')\,\delta\langle N'|N''\rangle\,. \qquad (4.13)$$

This assumption brings a further splitting of our four matrix equations and we are left with a rather simple system of eight complex basic "recursion relations".

In order to write them down in the most manageable form, we introduce once more a new notation putting in evidence the real and imaginary parts of the two complex continuous labels:

$$a = 1 + a_2 + i a_1 \quad b = 3 + b_2 + i b_1\,. \qquad (4.14)$$

Furthermore, as the matrix M is now completely diagonal, the Hermiticity condition for the metric amounts to having the matrix elements $\mathcal{B}$ *real.* Then the eight complex relations are easily split into sixteen real ones. We are finally left with the following system

$$\begin{aligned}
&(\varkappa + N - 1 - a_2)\,\mathcal{B}(\varkappa - 1, J_+ + 1, J_-, N - 1)\\
&\qquad\qquad = (\varkappa + N - 1 + a_2)\,\mathcal{B}(\varkappa, J_+, J_-, N)\\
&a_1\mathcal{B}(\varkappa - 1, J_+ + 1, J_-, N - 1)\\
&\qquad\qquad = a_1\mathcal{B}(\varkappa, J_+, J_-, N)\\
&(\varkappa + N - 1 - a_2)\,\mathcal{B}(\varkappa - 1, J_+, J_- - 1, N - 1)\\
&\qquad\qquad = (\varkappa + N - 1 + a_2)\,\mathcal{B}(\varkappa, J_+, J_-, N)\\
&a_1\mathcal{B}(\varkappa - 1, J_+, J_- - 1, N - 1)\\
&\qquad\qquad = a_1\mathcal{B}(\varkappa, J_+, J_-, N)\\
&(\varkappa + N - 1 - a_2)\,\mathcal{B}(\varkappa - 1, J_+, J_- + 1, N - 1)\\
&\qquad\qquad = (\varkappa + N - 1 + a_2)\,\mathcal{B}(\varkappa, J_+, J_-, N)\\
&a_1\mathcal{B}(\varkappa - 1, J_+, J_- + 1, N - 1)\\
&\qquad\qquad = a_1\mathcal{B}(\varkappa, J_+, J_-, N)\\
&(\varkappa + N - 1 - a_2)\,\mathcal{B}(\varkappa - 1, J_+ - 1, J_-, N - 1)\\
&\qquad\qquad = (\varkappa + N - 1 + a_2)\,\mathcal{B}(\varkappa, J_+, J_-, N)\\
&a_1\mathcal{B}(\varkappa - 1, J_+ - 1, J_-, N - 1)\\
&\qquad\qquad = a_1\mathcal{B}(\varkappa, J_+, J_-, N)
\end{aligned}$$

$$
\begin{aligned}
&(\varkappa - 2J_+ + N - 3 - b_2)\, \mathscr{B}(\varkappa - 1, J_+ + 1, J_-, N + 1) \\
&\qquad = (\varkappa - 2J_+ + N - 3 + b_2)\, \mathscr{B}(\varkappa, J_+, J_-, N) \\
&b_1 \mathscr{B}(\varkappa - 1, J_+ + 1, J_-, N + 1) \\
&\qquad = b_1 \mathscr{B}(\varkappa, J_+, J_-, N) \\
&(\varkappa + 2J_- + N - 1 - b_2)\, \mathscr{B}(\varkappa - 1, J_+, J_- - 1, N + 1) \\
&\qquad = (\varkappa + 2J_- + N - 1 + b_2)\, \mathscr{B}(\varkappa, J_+, J_-, N) \\
&b_1 \mathscr{B}(\varkappa - 1, J_+, J_- - 1, N + 1) \\
&\qquad = b_1 \mathscr{B}(\varkappa, J_+, J_-, N) \\
&(\varkappa - 2J_- + N - 1 - b_2)\, \mathscr{B}(\varkappa - 1, J_+, J_- + 1, N + 1) \\
&\qquad = (\varkappa - 2J_- + N - 1 + b_2)\, \mathscr{B}(\varkappa, J_+, J_-, N) \\
&b_1 \mathscr{B}(\varkappa - 1, J_+, J_- + 1, N + 1) \\
&\qquad = b_1 \mathscr{B}(\varkappa, J_+, J_-, N) \\
&(\varkappa + 2J_+ + N + 1 - b_2)\, \mathscr{B}(\varkappa - 1, J_+ - 1, J_-, N + 1) \\
&\qquad = (\varkappa + 2J_+ + N + 1 + b_2)\, \mathscr{B}(\varkappa, J_+, J_-, N) \\
&b_1 \mathscr{B}(\varkappa - 1, J_+ - 1, J_-, N + 1) \\
&\qquad = b_1 \mathscr{B}(\varkappa, J_+, J_-, N)\,. \qquad (4.15)
\end{aligned}
$$

5. Discussion of the series of unitary irreducible representations

The determination of the unitary irreducible representations of $SU(2, 2)$ or rather its universal covering group amounts to a determination of all solutions of Eqs. (4.12). However due to computational difficulties we have not been able to solve these equations without making the additional assumption that

$$\mathscr{M}(\varkappa, l, j)\, \langle N | N' \rangle = \mathscr{B}(\varkappa, J_+, J_-, N)\, \delta \langle N | N' \rangle\,.$$

The equations which then result have been given in Eqs. (4.15). In this section we shall find the solutions of these latter equations. Before doing this let us however discuss the ranges of the quantum numbers $\varkappa$, l, j and N. These ranges are different for the different groups which have the same Lie algebra. If we consider the universal covering group $\overline{SU(2, 2)}$ then the only restriction on $\varkappa$ is that it is real. The numbers l and j are of course always half-integer or integer since they are attached to the representations of $SU(2)$ groups. Furthermore

$$l + j \geqq |c| \qquad (5.1)$$

since $c = n + k$ and $|n| \leqq l$, $|k| \leqq j$. For N the allowed values are different in different (l, j) subspaces. Since

$$
\begin{aligned}
N &= n - k \\
|n| &\leqq l \\
|k| &\leqq j \\
n + k &= c\,,
\end{aligned}
$$

N can take the values

$$2l - |c|, 2l - |c| - 1, \ldots, |c| - 2j \tag{5.2}$$

if $j \geqq l$ and $j - l \leqq |c|$ and the values

$$2l - |c|, 2l - |c| - 1, \ldots, -2l - |c| \tag{5.3}$$

if $j \geqq l$ and $j - l > |c|$. When $l \geqq j$ one has to interchange l and j in these relations. Fig. 1 shows the allowed N-values. If one now wants to find the

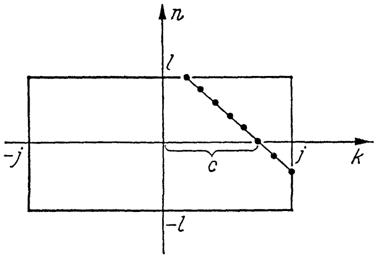

Fig. 1: Allowed values for N in an (l, j) subspace.

representations of $SU(2, 2)$ itself, one has to impose further conditions on the numbers $\varkappa$, l, j and N. $SU(2, 2)$ has a discrete centre of order four which belongs to the maximal compact subgroup.

Now our carrier space of an irreducible representation is actually not K but

$$\hat{K} = U(1) \otimes SU(2) \otimes SU(2) \tag{5.4}$$

and $\hat{K}$ covers K twice. The points $(\omega = \pi, \varphi = 2\pi, \alpha = 2\pi, \theta = \psi = \beta = \gamma = 0)$ and $(\omega = \varphi = \alpha = \theta = \psi = \beta = \gamma = 0)$ of $\hat{K}$ should be identified with the unity of K. Thus a rotation π in ω accompanied by a rotation 2π in φ and α should be mapped on the unit operator. For the basis elements

$$|\varkappa; l, m, n; j, h, k\rangle = e^{2i\varkappa\omega} \mathscr{D}^l_{mn}(\alpha, \beta, \gamma) \mathscr{D}^j_{hk}(\varphi, \theta, \psi)$$

this means that

$$\varkappa + l + j = \text{integer} \tag{5.5}$$

and thus $\varkappa$ can only be integer or half-integer in a representation of $SU(2, 2)$.

This analysis can be continued to even smaller groups. The group $SO(2, 4)$ is also locally isomorphic to $SU(2, 2)$. It has a centre of order two and thus one has

$$SO(2, 4) = SU(2, 2)/Z_2 .$$

To select the representations of $SO(2, 4)$ one has therefore to choose them in such a way that they map Z_2 on the unit operator. Now one knows that a representation of $SO(4)$ has either both l and j integer or both half-integer. Therefore a representation of $SO(2, 4)$ is characterized by

$$\varkappa = \text{integer} \tag{5.6}$$

in addition to the restrictions which hold for $SU(2, 2)$. Now as pointed out above $SO(2, 4)$ has a centre of order two. Therefore one can also consider the factor group with respect to this centre. The representations of this group are characterized by

$$\varkappa + 2j \text{ even}\,. \tag{5.7}$$

This can be shown by means of a parametrization of $SO(2, 4)$ in terms of Euler angles.

After this discussion of various related groups let us go back to the Lie algebra. Since it gives the representations of the universal covering group and of all "smaller" groups as well the restrictions

$$\begin{gathered} \varkappa = \text{integer} \\ l + j = \text{integer} \\ \varkappa + 2j \text{ even} \end{gathered}$$

must in particular be compatible with the recursion relations. In fact, $\varkappa$ and $J_+ = l + j$ are constant modulo 1, while $\varkappa + 2j$ is constant modulo 2.

In the rest of this paper we shall confine ourselves to the group $SU(2, 2)$. The extension to $\overline{SU(2, 2)}$ involves nothing principally new. According to the analysis above it must then be possible to divide the representation space of $SU(2, 2)$ into four different invariant subspaces characterized by $l + j$ either integer or half-integer and $\varkappa + 2j$ modulo 2. By looking at the recursion relations one finds even more. Each one of these four subspaces can be divided into two according to whether

$$\begin{gathered} \varkappa + N \text{ even} \\ \text{or} \\ \varkappa + N \text{ odd} \end{gathered} \tag{5.8}$$

Let us denote the subspaces $H_1, \ldots H_8$. Table 1 defines their properties.

Table 1. *Invariant representation spaces for $SU(2, 2)$*

	$l + j$ integer $\varkappa + 2j$ even	$l + j$ integer $\varkappa + 2j$ odd	$l + j$ half-integer $^1/_2 + \varkappa + 2j$ even	$l + j$ half-integer $^1/_2 + \varkappa + 2j$ odd
$\varkappa + N$ even	H_1	H_2	H_5	H_6
$\varkappa + N$ odd	H_3	H_4	H_7	H_8

We have to discuss the solutions of Eqs. (4.15) in each of these subspaces. The analysis is divided into two cases $a_1 \neq 0$ and $a_1 = 0$.

Case I — $a_1 \neq 0$. When $a_1 \neq 0$ one has from the "a_1" equations $\mathscr{B}(\varkappa - 1, J_+ - 1, J_-, N - 1) = \mathscr{B}(\varkappa, J_+, J_-, N)$. If this equality is inserted into the corresponding "a_2" equation one finds $a_2 = 0$. Let us now examine the two equations

$$\begin{aligned}(\varkappa + 2J_+ + N + 1 - b_2)\,\mathscr{B}(\varkappa - 1, J_+ - 1, J_-, N + 1)\\ = (\varkappa + 2J_+ + N + 1 + b_2)\,\mathscr{B}(\varkappa, J_+, J_-, N)\\ (\varkappa - 2J_+ + N - 3 - b_2)\,\mathscr{B}(\varkappa - 1, J_+ + 1, J_-, N + 1)\\ = (\varkappa - 2J_+ + N - 3 + b_2)\,\mathscr{B}(\varkappa, J_+, J_-, N)\,.\end{aligned} \tag{5.9}$$

From the a_1 equations we have

$$\begin{aligned}\mathscr{B}(\varkappa - 1, J_+ - 1, J_-, N + 1) = \mathscr{B}(\varkappa - 1, J_+ + 1, J_-, N + 1)\\ = \mathscr{B}(\varkappa, J_+, J_-, N + 2)\end{aligned} \tag{5.10}$$

so that one can subtract the Eqs. (5.9) to get

$$4(J_+ + 1)\,\mathscr{B}(\varkappa, J_+, J_-, N + 2) = 4(J_+ + 1)\,\mathscr{B}(\varkappa, J_+, J_-, N)\,.$$

Now since $J_+ + 1 > 0$ one has

$$\mathscr{B}(\varkappa - 1, J_+ - 1, J_-, N + 1) = \mathscr{B}(\varkappa, J_+, J_-, N + 2) = \mathscr{B}(\varkappa, J_+, J_-, N)$$

which inserted into the first of Eqs. (5.9) gives $b_2 = 0$. Therefore as soon as $a_1 \neq 0$ the diagonal matrix $\mathscr{B}(\varkappa, J_+, J_-, N)$ is independent of its arguments and the ranges of these arguments are restricted only to those valid within the subspaces $H_1, \ldots, H_8$. By keeping $a_2 = 0$ and $b_2 = 0$ one can now add the point $a_1 = 0$ if the "a_2" equations can fulfil the same mission as the "a_1" equations. This is evidently possible if $\varkappa + N - 1$ never becomes zero. In four of the subspaces this is so while in the other four subspaces this representation is reducible into two representations characterized by $\varkappa + N \geqq 1$ or $\varkappa + N \leqq -1$. To summarize we have found a continuous series of representations characterized by c, integer or half-integer, $-\infty < a_1 < \infty$, $a_2 = 0$ and $-\infty < b_1 < \infty$, $b_2 = 0$ where the only restrictions on the quantum numbers $\varkappa$, j, l and N are those which define the invariant subspaces $H_1, \ldots, H_8$. In the subspaces H_3, H_4, H_7 and H_8 the point $a_1 = 0$ has to be omitted.

Case II — $a_1 = 0$. To begin with let us put $b_1 \neq 0$. Then $b_2 = 0$ and the b_1 equations determine the $\mathscr{B}(\varkappa, J_+, J_-, N)$ in the direction $\varkappa - N$. If furthermore $\varkappa + N$ is even and $|a_2| < 1$ then the "a_2" equations determine successively positive values of $\mathscr{B}$ in the direction $\varkappa + N$. Thus the contents of these representations are the same as in case I but the measure matrix $\mathscr{B}$ is dependent on its arguments. We call this series a continuous supplementary series.

The absolute value $|a_2|$ may however be larger than 1 but then it is connected to a lower or upper bound $\underline{\varkappa + N}$ and $\overline{\varkappa + N}$ respectively for

$\varkappa + N$,

$$a_2 = -\underline{\varkappa + N} + 1$$

or (5.11)

$$a_2 = \overline{\varkappa + N} + 1\,.$$

In this case $\mathscr{B}(\varkappa, J_+, J_-, N)$ terminates with zero and thus projects out a subspace of the spaces $H_1, \ldots, H_8$. It is also found that $\underline{\varkappa + N} > 0$ and $\overline{\varkappa + N} < 0$ in order that the non-zero values of $\mathscr{B}$ all have the same sign. The series so obtained are called discrete series. One may ask what happens in the limiting case $a_2 = 1$ of the supplementary series. It is found that it is possible to have non-negative measure $\mathscr{B}$ only if it is different from zero only for $\varkappa + N = 0$. This is then a degenerate series since it is described by the two parameters b and c only. To all series discussed so far under case II one can now adjoin the point $b_1 = 0$ without any restriction. Fig. 2 illustrates the bounds on the variables $\varkappa$ and N for the discrete and degenerate series.

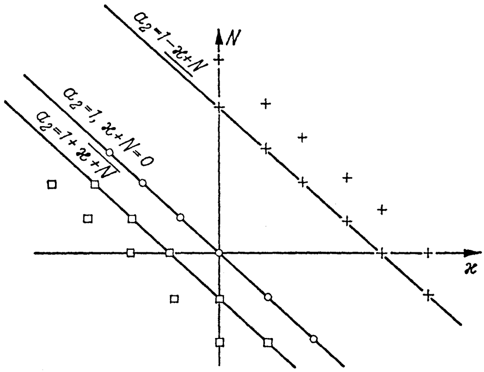

Fig. 2: Bounds on $\varkappa + N$ for discrete and degenerate series.

The possibilities are not exhausted yet. When $b_1 = 0$, $a_2 = 1$ and $\varkappa + N \equiv 0$ there can appear new degenerate series characterized by non-vanishing b_2. When J_- is integer, one can in fact have $|b_2| < 1$ without any restriction on J_-. Furthermore there appear the solutions

$$J_- \geqq \underline{J_-} = \frac{1 - b_2}{2} > 0$$

or

$$J_- \leqq \overline{J_-} = \frac{b_2 - 1}{2} < 0$$

i.e., lower or upper bounds on J_-. Finally there is also the possibility that $b_2 = 1$ and $J_- \equiv 0$. Fig. 3 shows the restrictions on J_- for some of these degenerate series.

In Table 2, we have summarized the series of representations. In connection with this Table we want to stress two things. This Table does not contain all unitary irreducible representations of $SU(2, 2)$ since we

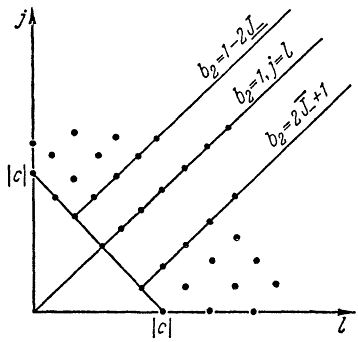

Fig. 3: Bounds on J_- for some degenerate series.

had to introduce a simplifying assumption in order to solve the recursion relations (4.12). Now this assumption of diagonality of the measure matrix is dependent on the parametrization of K. Thus one will get new series by choosing another parametrization of K. It seems however that even after variation of this parametrization one will not get all representations. The other thing to be stressed is that some representations of Table 2 may be unitarily equivalent.

We can also compare our results with those of Refs. [11], [12] and [13]. It seems as if the series 1 and 3 correspond to two of Graev's three series. The series 4 to 7 should all appear in Murai's work since they are all degenerate. We have however not found the limitation $|l - j| \leqq |c|$ which all the representations of Ref. [13] obey.

Appendix

The functions

$$\mathscr{D}^l_{mn}(\alpha, \beta, \gamma) \quad \begin{array}{l} 2l = 0, 1, 2, \ldots \\ -l \leqq m, n \leqq l \end{array}$$

have the following properties

$$\mathscr{D}^l_{mn}(\alpha, \beta, \gamma) = e^{im\alpha} P^l_{mn}(\cos\beta) e^{in\gamma}$$

where

$$P^l_{mn}(z) = \frac{(-1)^{l-n}}{2^l (l-n)!} \sqrt{\frac{(l-n)!\,(l+m)!}{(l+n)!\,(l-m)!}}$$

$$(1-z)^{-\frac{m-n}{2}} (1+z)^{-\frac{m+n}{2}} \left(\frac{d}{dz}\right)^{l-m} [(1-z)^{l-n} (1+z)^{l+n}] .$$

Table 2. *Unitary irreducible representations of* $SU(2,2)$

Type of series		Subspaces H_1, H_2 c integer $J_+ = \|c\|, \|c\| + 1, \ldots$ $\varkappa + N$ even	Subspaces H_3, H_4 c integer $J_+ = \|c\|, \|c\| + 1, \ldots$ $\varkappa + N$ odd	Subspaces H_5, H_6 c half integer $J_+ = \|c\|, \|c\| + 1, \ldots$ $\varkappa + N$ even	Subspaces H_7, H_8 c half integer $J_+ = \|c\|, \|c\| + 1, \ldots$ $\varkappa + N$ odd
1. Continuous (main) series) $-\infty < a_1 < \infty, a_2 = 0$ $-\infty < b_1 < \infty, b_2 = 0$		No further restrictions on parameters or subspace	$a_1 \neq 0$	No further restrictions on parameters or subspace	$a_1 \neq 0$
2. Continuous supplementary series $a_1 = 0, 0 < \|a_2\| < 1$ $-\infty < b_1 < \infty, b_2 = 0$		No further restrictions on parameters or subspace	Empty	No further restrictions on parameters or subspace	Empty
3. Discrete series $a_1 = 0$	$a_2 = -1, -3, \ldots$	$\varkappa + N \geqq 1 - a_2$ or $\varkappa + N \leqq a_2 - 1$	Empty	$\varkappa + N \geqq 1 - a_2$ or $\varkappa + N \leqq a_2 - 1$	Empty
$-\infty < b_1 < \infty$ $b_2 = 0$	$a_2 = 0, -2, \ldots$	Empty	$\varkappa + N \geqq 1 - a_2$ or $\varkappa + N \leqq a_2 - 1$	Empty	$\varkappa + N \geqq 1 - a_2$ or $\varkappa + N \leqq a_2 - 1$
4. Degenerate (main) series $a_1 = 0,\ a_2 = 1$ $-\infty < b_1 < \infty, b_2 = 0$		$\varkappa + N \equiv 0$	Empty	$\varkappa + N \equiv 0$	Empty
5. Degenerate supplementary series $a_1 = 0,\ a_2 = 1$ $b_1 = 0,\ 0 < \|b\|_2 < 1$		$\varkappa + N \equiv 0$	Empty	Empty	Empty
6. Degenerate discrete series	$b_2 = -1, -3, \ldots$	$\varkappa + N \equiv 0;$ $J_- \geqq {}^1/_2\,(1 - b_2)$ or $J_- \leqq {}^1/_2\,(b_2 - 1)$	Empty	Empty	Empty
$a_1 = 0,\ a_2 = 1$ $b_1 = 0$	$b_2 = 0, -2, \ldots$	Empty	Empty	$\varkappa + N \equiv 0;$ $J_- \geqq {}^1/_2\,(1 - b_2)$ or $J_- \leqq {}^1/_2\,(b_2 - 1)$	Empty
7. Maximally degenerate series $a_1 = 0,\ a_2 = 1$ $b_1 = 0,\ b_2 = 1$		$\varkappa + N \equiv 0$ $J_- \equiv 0$	Empty	Empty	Empty

The following recursion relations are valid for $P^l_{mn}(\cos\beta)$:

$$\left[\frac{d}{d\beta} - \frac{m - n\cos\beta}{\sin\beta}\right] P^l_{mn}(\cos\beta) = \sqrt{(l+n)(l-n+1)}\, P^l_{mn-1}(\cos\beta)$$

$$\left[\frac{d}{d\beta} + \frac{m - n\cos\beta}{\sin\beta}\right] P^l_{mn}(\cos\beta) = -\sqrt{(l-n)(l+n+1)}\, P^l_{mn+1}(\cos\beta)\,.$$

Using $P^l_{mn}(z) = (-1)^{n-m} P^l_{nm}(z)$ these relations can be transformed into recursion relations involving the left lower index.

From the addition of an angular momentum l and an angular momentum $\frac{1}{2}$ we obtain the following relations

$$(2l+1)\cos\frac{\theta}{2}\, P^l_{mn}(\cos\theta) = \sqrt{(l+m+1)(l+n+1)}\, P^{l+\frac{1}{2}}_{m+\frac{1}{2}\,n+\frac{1}{2}}(\cos\theta) + \sqrt{(l-m)(l-n)}\, P^{l-\frac{1}{2}}_{m+\frac{1}{2}\,n+\frac{1}{2}}(\cos\theta)$$

$$(2l+1)\cos\frac{\theta}{2}\, P^l_{mn}(\cos\theta) = \sqrt{(l-m+1)(l-n+1)}\, P^{l+\frac{1}{2}}_{m-\frac{1}{2}\,n-\frac{1}{2}}(\cos\theta) + \sqrt{(l+m)(l+n)}\, P^{l-\frac{1}{2}}_{m-\frac{1}{2}\,n-\frac{1}{2}}(\cos\theta)$$

$$(2l+1)\sin\frac{\theta}{2}\, P^l_{mn}(\cos\theta) = \sqrt{(l-m+1)(l+n+1)}\, P^{l+\frac{1}{2}}_{m-\frac{1}{2}\,n+\frac{1}{2}}(\cos\theta) + - \sqrt{(l+m)(l-n)}\, P^{l-\frac{1}{2}}_{m-\frac{1}{2}\,n+\frac{1}{2}}(\cos\theta)$$

$$(2l+1)\sin\frac{\theta}{2}\, P^l_{mn}(\cos\theta) = -\sqrt{(l+m+1)(l-n+1)}\, P^{l+\frac{1}{2}}_{m+\frac{1}{2}\,n-\frac{1}{2}}(\cos\theta) + \sqrt{(l-m)(l+n)}\, P^{l-\frac{1}{2}}_{m+\frac{1}{2}\,n-\frac{1}{2}}(\cos\theta)\,.$$

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