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States of Physical Systems

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Abstract. States of physical systems may be represented by states on B^*-algebras, satisfying certain requirements of physical origin. We discuss such requirements as are associated with the presence of unbounded observables or invariance under a group. It is possible in certain cases to obtain a unique decomposition of states invariant under a group into extremal invariant states. Our main results is such a decomposition theorem when the group is the translation group in ν dimensions and the B^*-algebra satisfies a certain locality condition. An application of this theorem is made to representations of the canonical anticommutation relations.

1. Introduction: B^*-algebras and states

The main purpose of this paper is to prove a theorem yielding an integral representation of invariant states on a B^*-algebra in terms of extremal invariant states. The theorem and related results are presented in Section 3 to which the reader may proceed directly[1]. This first and the second sections are devoted to motivation and some background information. Sections 4 and 5 contain the proof of the theorem of Section 3 and Section 6 an application to canonical anticommutation relations. Other applications, to the states of equilibrium statistical mechanics, will be presented in a forthcoming paper.

The use of C^*-algebras in physics, proposed by SEGAL and HAAG, has been mostly restricted to the study of canonical commutation relations and field theory. Other domains, like statistical mechanics, are however potential fields of application.

[1] After a first version of this paper was completed, I benefited from conversations with KASTLER and ROBINSON. These authors and DOPLICHER ([4], Section 5) have obtained, independently, results corresponding roughly to Corollary 2, Section 3, of the present paper. Furthermore, ROBINSON [6] has obtained important generalizations of Lemma 4, Section 4, and Corollaries 1 and 2, section 3. Contrary to what is done here, ROBINSON makes systematic use of Hilbert space methods. I am greatly indebted to KASTLER and ROBINSON for discussing with me their results, a large part of which is not yet written down [6]. These discussions have prompted me to make a few changes to the original version of this paper, notably by replacing "local" by "asymptotically Abelian" [4] B^*-algebras and appending two remarks (after the theorem in Section 3 and after Lemma 4, Section 4) which relate the present work to the forthcoming paper [6] of KASTLER and ROBINSON.

If "observables" of a physical system are given as a collection of bounded self-adjoint operators A_i in some Hilbert space, one may think of describing a physical state of the system by giving the expectation values $\langle A_i \rangle$ which it associates with these observables. In concrete examples it is seen however that the expectation values $\langle A_1 \ldots A_l \rangle$ of products are also physically relevant and describe correlations between measurements. Naturally enough the expectation value of the identity $\mathbf{1}$ is 1 and the expectation value of a positive operator is positive. A state is thus described as a linear functional $\langle . \rangle$, positive and normalized by $\langle \mathbf{1} \rangle = 1$, on the algebra $\tilde{\mathfrak{A}}$ with identity $\mathbf{1}$ generated by the A_i. The uniform closure of $\tilde{\mathfrak{A}}$ is a self-adjoint uniformly closed algebra $\mathfrak{A}$ of bounded operators in a complex Hilbert space, i.e. a C^*-algebra. A state $\langle . \rangle$ on $\tilde{\mathfrak{A}}$ continues uniquely to a functional ϱ on $\mathfrak{A}$ which is again linear, positive and normalized and such a functional is called a state on the C^*-algebra $\mathfrak{A}$, it is necessarily continuous.

An interesting feature of C^*-algebras is that they can — as B^*-algebras — be described abstractly i.e. without reference to operators acting on a Hilbert space. A B^*-algebra $\mathfrak{A}$ with or without an identity is an algebra over the complex numbers with a norm: $A \to \|A\|$ and an involution $A \to A^*$ satisfying the following conditions:

1. As a normed vector space $\mathfrak{A}$ is a Banach space.
2. $\|AB\| \leqq \|A\| \,.\, \|B\|$, i.e. $\mathfrak{A}$ is a Banach algebra.
3. The involution is a conjugate linear $((A + \lambda B)^* = A^* + \lambda^* B^*)$, involutary $(A^{**} = A)$, antiautomorphism $((AB)^* = B^* A^*)$ which preserves the norm $(\|A^*\| = \|A\|)$.
4. For all $A \in \mathfrak{A}: \|A^* A\| = \|A\|^2$.

It is seen easily that a C^*-algebra is a B^*-algebra. Conversely it can be shown that a B^*-algebra may always be realized as a C^*-algebra of bounded operators on a complex Hilbert space.

If L is a locally compact space, the algebra $\mathscr{C}_0(L)$ of complex continuous functions vanishing at infinity on L is, with respect to the uniform norm and involution given by complex conjugation, an Abelian B^*-algebra. Conversely, every Abelian B^*-algebra $\mathfrak{A}$ is isomorphic to $\mathscr{C}_0(L)$ for suitable L (this is the Gel'fand isomorphism). L is compact if and only if $\mathfrak{A}$ has an identity.

One defines an order on a B^*-algebra $\mathfrak{A}$ by writing $A \geqq 0$ if $A = B^* B$. A state ϱ on $\mathfrak{A}$ is then a positive linear functional on $\mathfrak{A}$ such that

$$\|\varrho\| = \sup_{\|A\| \leqq 1} |\varrho(A)| = 1 \,.$$

If $\mathfrak{A}$ has an identity $\mathbf{1}$, this last condition is equivalent to $\varrho(\mathbf{1}) = 1$.

Given a state ϱ on $\mathfrak{A}$, the standard construction of Gel'fand-Segal yields the following results

1. A complex Hilbert space $\mathfrak{H}_\varrho$.

2. A homomorphism π_ϱ of $\mathfrak{A}$ into the bounded operators on $\mathfrak{H}_\varrho$.

3. A normalized vector $\Omega \in \mathfrak{H}_\varrho$ such that $\pi_\varrho(\mathfrak{A})\Omega$ is dense in $\mathfrak{H}_\varrho$ (cyclicity) and for all $A \in \mathfrak{A}$:

$$\varrho(A) = (\Omega, \pi_\varrho(A)\Omega) .$$

The states on an Abelian B^*-algebra correspond via the Gel'fand isomorphism to the positive measures of norm (= total mass) 1 on a locally compact space[2].

While it is natural and useful to represent states of a physical system by states on a C^*- or B^*-algebra, in general only part of the mathematical states are of physical interest for a given problem. For instance, a physical theory has in general an invariance group and one may like to restrict one's attention to invariant states. This problem is considered in the next section.

Another type of restriction on physical states comes about if some observables of the physical theory are represented by unbounded self-adjoint operators. Let H be such an unbounded operator on $\mathscr{H}$, and let $\mathfrak{A}$ be the C^*-algebra on $\mathscr{H}$ generated by the observables. We may assume that for every complex continuous function f vanishing at infinity on the real line, $f \in \mathscr{C}_0(R)$, the operator $A_f = f(H)$ belongs to $\mathfrak{A}$. We have then a homomorphism h of $\mathscr{C}_0(R)$ into $\mathfrak{A}$

$$h : f \to A_f .$$

Any state ϱ on $\mathfrak{A}$ defines then, by restriction to $h(\mathscr{C}_0(R))$, a positive linear functional ϱh on $\mathscr{C}_0(R)$. This is a positive measure μ on the spectrum of H, with obvious probabilistic interpretation. However, while it is clear that $\|\mu\| \leqq 1$ it is quite possible that $\|\mu\| < 1$, i.e. that the total probability of finding some point of the spectrum be strictly less than 1. This occurs for instance if H is a particle number operator and ϱ describes a system with an infinite number of particles.

It may be that for a given physical problem one is interested only in the states ϱ which correspond to a measure of norm 1 on the spectrum of the unbounded operator H. If this is the case one has to require that the restriction of ϱ to the subalgebra $h(\mathscr{C}_0(R))$ has norm 1. If instead of a C^*-algebra one has a B^*-algebra $\mathfrak{A}$, one may think of defining an unbounded operator "abstractly" by a homomorphism $h : f \to A_f$ of $\mathscr{C}_0(R)$ into $\mathfrak{A}$. One may then ask if, for a state ϱ with restriction of norm 1 to $h(\mathscr{C}_0(R))$, there exists a self-adjoint unbounded operator H_ϱ in $\mathfrak{H}_\varrho$ such that

$$\pi_\varrho(h(f)) = f(H_\varrho)$$

where π_ϱ is the canonical homomorphism of $\mathfrak{A}$ into the bounded operators

[2] For a more detailed introduction to the mathematical theory of B^*-algebras the reader is referred to §§ 1., 2. of the book [2] of DIXMIER or to lectures [5] by KADISON.

on $\mathfrak{H}_\varrho$. Under certain assumptions a positive answer to this question is given in the Appendix.

2. Invariant states

Let $\mathfrak{A}$ be a B^*-algebra with an identity 1. The set E of all states on $\mathfrak{A}$ is a convex weakly compact subset of the dual $\mathfrak{A}'$ of $\mathfrak{A}$ (see [2]). We remind the reader that an extremal point ϱ of E is one which cannot be written in the form $\varrho = \frac{1}{2}\varrho_1 + \frac{1}{2}\varrho_2$ with $\varrho_1, \varrho_2 \in E$ and $\varrho_1 \neq \varrho_2$. The theorem of Krein-Milman asserts that a convex compact subset of a locally convex topological vector space is the closure of the convex hull of its extremal points. This applies in particular to E, the extremal points of E, or extremal states, are called pure states.

Let G be a topological group with identity e. We assume that for every $g \in G$ an automorphism τ_g of $\mathfrak{A}$ is given such that

1. *The mapping $g \to \tau_g$ is a homomorphism of G into the automorphisms of $\mathfrak{A}$.*

2. *If $A \in \mathfrak{A}$, $g \to e$ implies $\|\tau_g A - A\| \to 0$.*

If $f \in \mathfrak{A}'$ is a continuous linear functional on $\mathfrak{A}$, we define $\tau'_g f$ by

$$\tau'_g f(A) = f(\tau_g A), \quad \text{all} \quad A \in \mathfrak{A}$$

If $\tau'_g f = f$ for all $g \in G$ we say that f is G-invariant.

Let $\mathfrak{L}$ be the subspace of $\mathfrak{A}$ generated by the elements of the form $A - \tau_g A$ and

$$\mathfrak{L}^\perp = \{f \in \mathfrak{A}' : A \in \mathfrak{L} \Rightarrow f(A) = 0\}.$$

Then, an element f of $\mathfrak{A}'$ is G-invariant if and only if it belongs to the weakly closed subspace $\mathfrak{L}^\perp$. Notice that if $\overline{\mathfrak{L}}$ is the closure of $\mathfrak{L}$ in $\mathfrak{A}$, $\mathfrak{L}^\perp$ is isomorphic as Banach space to the dual of $\mathfrak{A}/\overline{\mathfrak{L}}$. The G-invariant states on $\mathfrak{A}$ are precisely the elements of the convex (weakly) compact set $E \cap \mathfrak{L}^\perp$.

If one performs the Gel'fand-Segal construction starting with an invariant state $\varrho \in E \cap \mathfrak{L}^\perp$, it can be shown[3] that there exists in $\mathfrak{H}_\varrho$ a strongly continuous unitary representation U of G such that for all $g \in G$, $A \in \mathfrak{A}$

$$U(g)\Omega = \Omega, \quad U(g)\,\pi_\varrho(A)\,U(g)^{-1} = \pi_\varrho(\tau_g A).$$

One can see that if G is compact or Abelian, $E \cap \mathfrak{L}^\perp$ is not empty. If G is compact and $\varrho \in E$ the state $\tilde{\varrho}$ defined by

$$\tilde{\varrho}(A) = \varrho\left[\int_G \tau_g A\, dg\right] \quad \text{all} \quad A \in \mathfrak{A}$$

is indeed G-invariant. If G is Abelian, the existence of a G-invariant state follows from the theorem of MARKOV-KAKUTANI (see DOPLICHER [3] Proposition I).

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[3] See SEGAL [9] or DIXMIER [2] 2.12.11.

We know by the theorem of KREIN-MILMAN that every $\varrho \in E \cap \mathfrak{L}^{\perp}$ is the weak limit of convex linear combinations of extremal points of $E \cap \mathfrak{L}^{\perp}$ (i.e. extremal invariant states). An interesting case is that in which ϱ can be written as an integral over extremal points, i.e. is the resultant of a measure over extremal points, specially if this measure is unique. In that last case the study of G-invariant states on $\mathfrak{A}$ is effectively reduced to that of extremal G-invariant states.

Let us first suppose that G is reduced to the identity so that $E \cap \mathfrak{L}^{\perp} = E$. In that case an element of E does not in general have a unique integral representation in terms of extremal states (pure states). However, if $\mathfrak{A}$ is Abelian the set of pure states is compact and identical to the space L of the Gel'fand isomorphism. In that case the desired integral representation exists and associates with $\varrho \in E$ a measure μ on L as explained in Section 1.

For a discussion of the existence and uniqueness of integral representations on a convex compact set K in a locally convex topological vector space, we refer the reader to an article by CHOQUET and MEYER (see [1]). Let us however mention some results which we shall use. An order relation $\prec$ among the positive measures on K is defined such that $\mu_1 \prec \mu_2 \Leftrightarrow \mu_1(\varphi) \leqq \mu_2(\varphi)$ for all convex continuous functions φ on K. If $\mu_1 \prec \mu_2$ then $\mu_1(\psi) = \mu_2(\psi)$ for all continuous linear functions ψ on K, in particular μ_1 and μ_2 have the same norm. Intuitively $\mu_1 \prec \mu_2$ means that μ_2 is "concentrated nearer to the boundary of K" than μ_1. We shall say that μ is maximal if it is maximal for the order $\prec$. Let δ_ϱ be the unit measure at $\varrho \in K$, then we may look for an integral representation of ϱ by trying to find a maximal μ such that $\delta_\varrho \prec \mu$. Such a μ always exist ([1], theorem 3) but need not be unique or concentrated on the extremal points of K. We shall use the fact that if K is metrizable a measure μ is maximal if and only if it is concentrated on the set $\mathscr{E}(K)$ of extremal points of K (see [1], Corollary 14) and that in any case a measure concentrated on $\mathscr{E}(K)$ is maximal ([1], proposition 15).

For the question of uniqueness of the integral representation, it is useful to consider that K is the basis of a convex cone C with apex at the origin, i.e. K is the intersection of C with a closed hyperplane not containing the origin and which intersects all the generating lines of C. Giving a convex cone C defines a (partial) order in the ambiant locally convex space ($x \geqq 0 \Leftrightarrow x \in C$). We say that K is a simplex if C is a lattice for this order (i.e. any two elements of C have a l. u. b. and a g. l. b.). The uniqueness problem is then solved by the following theorem ([1], theorem 11): K is a simplex if and only if for every $\varrho \in K$, δ_ϱ is majorized by a unique maximal measure.

Let us come back to the case where $K = E \cap \mathfrak{L}^{\perp}$ is the set of G-invariant states on a B^*-algebra $\mathfrak{A}$. We assert that *if* $\mathfrak{A}$ *is Abelian, then* *each* $\varrho \in E \cap \mathfrak{L}^{\perp}$ *is majorized by a unique maximal measure on* $E \cap \mathfrak{L}^{\perp}$. Let C be the cone of positive continuous linear functionals on $\mathfrak{A}$, H the hyperplane $H = \{f \in \mathfrak{A}' : f(1) = 1\}$, then $E = C \cap H$. Since C can be identified with the set of positive measures on a compact set L, and is thus a lattice for the ordinary order on these measures, E is a simplex. We have also $E \cap \mathfrak{L}^{\perp} = (C \cap \mathfrak{L}^{\perp}) \cap H$, and $C \cap \mathfrak{L}^{\perp}$ is again a lattice for the order it defines (the l. u. b. and g. l. b. of two G-invariant elements of C is again G-invariant), hence $E \cap \mathfrak{L}^{\perp}$ is a simplex and our assertion follows from the theorem mentioned above.

The main purpose of this paper is to prove uniqueness for a case where $\mathfrak{A}$ is not abelian, but G is now taken to be the translation group in ν dimensions: R^{ν}. The uniqueness theorem which we prove yields explicitly the integral representation of $\varrho \in E \cap \mathfrak{L}^{\perp}$ by a maximal measure, and is based on a locality assumption for $\mathfrak{A}$. We collect this new assumption together with the conditions 1 and 2 on the action of G on $\mathfrak{A}$ in the definition below of an asymptotically Abelian B^*-algebra [4] (with respect to R^{ν}).

3. Statement of results

Definition. *Let* $\mathfrak{A}$ *be a* B^**-algebra with an identity* 1. *We assume that for every* $x \in R^{\nu}$ *an automorphism* τ_x *of* $\mathfrak{A}$ *is given such that*

(A1). *The mapping* $x \to \tau_x$ *is a homomorphism of the abelian group* R^{ν} *into the automorphisms of* $\mathfrak{A}$.

(A2). *If* $A \in \mathfrak{A}$, $x \to 0$ *implies* $\|\tau_x A - A\| \to 0$.

We shall say that $\mathfrak{A}$ *is asymptotically Abelian if furthermore we have*

(A3). *If* $A_1, A_2 \in \mathfrak{A}$, $x \to \infty$ *implies* $\|[A_1, \tau_x A_2]\| \to 0$.

We adapt to the present situation $G = R^{\nu}$ the notations of Section 2: E is the convex and (weakly) compact set of states in the dual $\mathfrak{A}'$ of $\mathfrak{A}$, $\mathfrak{L}$ is the subspace of $\mathfrak{A}$ generated by the elements of the form $A - \tau_x A$ with $A \in \mathfrak{A}$, $x \in R^{\nu}$, and $\overline{\mathfrak{L}}$ its closure. We write

$$\mathfrak{L}^{\perp} = \{f \in \mathfrak{A}' : A \in \mathfrak{L} \Rightarrow f(A) = 0\}.$$

Then $E \cap \mathfrak{L}^{\perp}$ is the convex compact set of *translationally invariant* (i.e. R^{ν}-invariant) *states*. If $A \in \mathfrak{A}$ we define a complex continuous function $\hat{A}$ on $E \cap \mathfrak{L}^{\perp}$ by

$$\hat{A}(\varrho) = \varrho(A), \quad \text{all} \quad \varrho \in E \cap \mathfrak{L}^{\perp}.$$

We recall that $\mathfrak{L}^{\perp} \cong (\mathfrak{A}/\overline{\mathfrak{L}})'$, therefore a translationally invariant state may be viewed as a continuous linear functional on $\mathfrak{A}/\overline{\mathfrak{L}}$. Let $\mathfrak{M} : \mathfrak{A} \to \mathfrak{A}/\overline{\mathfrak{L}}$ be the quotient mapping. The main idea of the theorem below is that $\mathfrak{M}$ can be approximated in some sense by an averaging operation $\mathfrak{M}_a : \mathfrak{A} \to \mathfrak{A}$ where $\mathfrak{M}_a A_1$ commutes in the limit with $\mathfrak{M}_a A_2$ for all $A_1, A_2 \subset \mathfrak{A}$. Here $a = (a^1, \ldots, a^{\nu}) \in R^{\nu}$ with $a^1 > 0, \ldots, a^{\nu} > 0$ and

$$\mathfrak{M}_a A = V(a)^{-1} \int_{\Lambda(a)} db\, \tau_b A$$

where
$$A(a) = \{x \in R^\nu : 0 \leqq x^i < a^i\},\ V(a) = \prod_{i=1}^{\nu} a^i .$$
The integral in the definition of $\mathfrak{M}_a A$ makes sense in view of (A2). We shall write $a \to \infty$ for $a^1 \to \infty, \ldots, a^\nu \to \infty$.

As explained in Section 2, a partial order $\prec$ is introduced among the positive measures on $E \cap \mathfrak{L}^\perp$ such that $\mu_1 \prec \mu_2$ means that for all convex continuous functions φ on $E \cap \mathfrak{L}^\perp$, $\mu_1(\varphi) \leqq \mu_2(\varphi)$. In particular $\mu_1(\hat{A}) = \mu_2(\hat{A})$ and $\|\mu_1\| = \mu_1(1) = \mu_2(1) = \|\mu_2\|$. We say that μ is maximal if it is maximal for the order $\prec$. We look for an integral representation of $\varrho \in E \cap \mathfrak{L}^\perp$ by trying to find a maximal measure $\mu_\varrho \succ \delta_\varrho$ where δ_ϱ is the unit mass at ϱ. In particular we have for all $A \in \mathfrak{A} : \varrho(A) = \hat{A}(\varrho) = \delta_\varrho(\hat{A}) = \mu_\varrho(\hat{A})$. We shall show that μ_ϱ is unique and, in the good cases (see part 5 of the theorem) concentrated on the extremal points of $E \cap \mathfrak{L}^\perp$.

Theorem. *Let the B^*-algebra $\mathfrak{A}$ be asymptotically Abelian with respect to R^ν, $\nu > 0$.*

1. *To every $\varrho \in E \cap \mathfrak{L}^\perp$ there corresponds a positive normalized measure μ_ϱ on the compact set $E \cap \mathfrak{L}^\perp$ such that if $A_1, \ldots, A_1 \in \mathfrak{A}$, then*
$$\mu_\varrho(\hat{A}_1 \cdot \ldots \cdot \hat{A}_1) = \lim_{a_1, \ldots, a_l \to \infty} \varrho(\mathfrak{M}_{a_1} A_1 \cdot \ldots \cdot \mathfrak{M}_{a_l} A_l) .$$

2. *Let $\varrho_1, \ldots, \varrho_n \in E \cap \mathfrak{L}^\perp$ and $\alpha_1, \ldots, \alpha_n$ be positive numbers such that $\sum_i \alpha_i = 1$, then if $\varrho = \sum \alpha_i \varrho_i$ we have*
$$\mu_\varrho = \sum \alpha_i \mu_{\varrho_i} .$$

3. *μ_ϱ is the unique maximal measure on $E \cap \mathfrak{L}^\perp$ which majorizes the unit mass δ_ϱ at $\varrho \in E \cap \mathfrak{L}^\perp$.*

4. *Let $(\mathfrak{A}_\alpha)$ be a countable family of self-adjoint subalgebras of $\mathfrak{A}$, and let $\mathfrak{J}$ be the subset of $E \cap \mathfrak{L}^\perp$ formed by the elements ϱ such that the restriction of ϱ to $\mathfrak{A}_\alpha$ has norm 1 for each α. Then, μ_ϱ is concentrated on $\mathfrak{J}$ if and only if $\varrho \in \mathfrak{J}$.*

5. *With the same assumptions, let there exist a countable family (A_i) of elements of $\mathfrak{A}$ such that if $\varrho \in \mathfrak{J}$ and $\sigma \in E \cap \mathfrak{L}^\perp$, $\varrho \neq \sigma$, then $\hat{A}_i(\varrho) \neq \hat{A}_i(\sigma)$ for some i. Let $\mathscr{E}(E \cap \mathfrak{L}^\perp)$ be the set of extremal points of $E \cap \mathfrak{L}^\perp$. If $\varrho \in \mathfrak{J}$, then μ_ϱ is concentrated on $\mathscr{E}(E \cap \mathfrak{L}^\perp) \cap \mathfrak{J}$. Conversely if the measure $\mu \geqq 0$ of norm 1 on $E \cap \mathfrak{L}^\perp$ is concentrated on $\mathscr{E}(E \cap \mathfrak{L}^\perp) \cap \mathfrak{J}$, then $\mu = \mu_\varrho$ for some $\varrho \in E \cap \mathfrak{L}^\perp$.*

The theorem remains true if R^ν is replaced by a closed subgroup $(\neq 0)$. The consideration of states which have a restriction of norm 1 to certain subalgebras is of interest for instance if there exist unbounded observables (see Section 1 and Appendix).

Remark. It is of course a problem of interest to extend the above theorem to more general locally compact groups G. Most of our arguments are actually independent of the assumption $G = R^\nu$ and the only delicate point is the generalization of Lemma 4 in Section 4. Such a generalization has been obtained by ROBINSON [6], permitting the extension of the machinery of our theorem to various groups of physical interest, provided that a physically meaningful "asymptotic Abelianness" may be postulated.

Before starting the proof of the theorem we mention some of its consequences.

Corollary 1. *Let $\mathfrak{A}$ be asymptotically Abelian with respect to R^ν and let $\varrho \in E \cap \mathfrak{L}^\perp$. We recall that in the space $\mathfrak{H}_\varrho$ of the Gel'fand-Segal construction there exist a representation $A \to \pi_\varrho(A)$ of $\mathfrak{A}$, a unitary representation $x \to U(x)$ of R^ν, and a normalized vector Ω cyclic for $\pi_\varrho(\mathfrak{A})$ such that*

$$\varrho(A) = (\Omega, \pi_\varrho(A)\Omega), \quad U(a)\Omega = \Omega$$

$$U(x)\,\pi_\varrho(A)\,U(-x) = \pi_\varrho(\tau_x A) .$$

We assert that the commutant of $\pi_\varrho(\mathfrak{A}) \cup U(R^\nu)$ is Abelian.

Let $A \in \mathfrak{A}$ and let C_1, C_2 commute with $\pi_\varrho(\mathfrak{A})$ and $U(R^\nu)$, we want to show that

$$(\pi_\varrho(A)\Omega, C_1 C_2 \Omega) = (\pi_\varrho(A)\Omega, C_2 C_1 \Omega)$$

where we may assume that A, C_1, C_2 are self-adjoint of norm $\leqq 1$. Given $\varepsilon > 0$ we may choose self-adjoint $A_1, A_2 \in \mathfrak{A}$ such that

$$\|[C_i - \pi_\varrho(A_i)]\Omega\| < \varepsilon, \quad \|\pi_\varrho(A_i)\Omega\| \leqq 1$$

for $i = 1, 2$, hence also

$$\|[C_i - \pi_\varrho(\mathfrak{M}_a A_i)]\Omega\| < \varepsilon .$$

We have then

$$\begin{aligned}
&|(\pi_\varrho(A)\Omega, C_1C_2\Omega) - (\pi_\varrho(A)\Omega, C_2C_1\Omega)| \\
&\qquad = |(C_1\Omega, \pi_\varrho(\mathfrak{M}_a A)C_2\Omega) - (C_2\Omega, \pi_\varrho(\mathfrak{M}_a A)C_1\Omega)| < \\
&\qquad\quad < |(\pi_\varrho(\mathfrak{M}_a A_1)\Omega, \pi_\varrho(\mathfrak{M}_a A)\,\pi_\varrho(\mathfrak{M}_a A_2)\Omega) - \\
&\qquad\quad - (\pi_\varrho(\mathfrak{M}_a A_2)\Omega, \pi_\varrho(\mathfrak{M}_a A)\,\pi_\varrho(\mathfrak{M}_a A_1)\Omega)| + 4\varepsilon \\
&\qquad = |\varrho(\mathfrak{M}_a A_1 \,.\, \mathfrak{M}_a A \,.\, \mathfrak{M}_a A_2) - \varrho(\mathfrak{M}_a A_2 \,.\, \mathfrak{M}_a A \,.\, \mathfrak{M}_a A_1)| + 4\varepsilon .
\end{aligned}$$

Since this inequality holds for all a we see, using part 1 of the theorem, that the left-hand side is $< 4\varepsilon$, hence vanishes, which proves the corollary.

Corollary 2. *With the notations and assumptions of corollary 1, let $\mathfrak{H}_\Omega$ be the subspace of $\mathfrak{H}_\varrho$ formed by the vectors invariant under $U(R^\nu)$. The following conditions are equivalent*

1. *ϱ is an extremal point of $E \cap \mathfrak{L}^\perp$.*
2. *For all l and $A_1, \ldots, A_l \in \mathfrak{A}$,*

$$\lim_{a_1, \ldots, a_l \to \infty} \varrho(\mathfrak{M}_a A_1 \cdot \ldots \cdot \mathfrak{M}_a A_l) = \prod_{i=1}^{l} \varrho(A_i) .$$

3. *For all self-adjoint $A \in \mathfrak{A}$, $\lim_{a\to\infty} \varrho((\mathfrak{M}_a A)^2) = (\varrho(A))^2$.*
4. *$\mathfrak{H}_\Omega$ is one-dimensional* (spanned by Ω).
5. *The set $\pi_\varrho(\mathfrak{A}) \cup U(R^\nu)$ is irreducible.*

Conditions 2. and 3. are weak "cluster properties" while 4. is usually referred to as "uniqueness of the vacuum" in field theory.

1. $\Leftrightarrow$ 2. From part 1 of the theorem it follows that 2. is equivalent to

$$\mu_\varrho(\hat{A}_1 \cdot \ldots \cdot \hat{A}_l) = \prod_{i=1}^{l} \varrho(A_i) = \prod_{i=1}^{l} \hat{A}_i(\varrho) = \delta_\varrho(\hat{A}_1, \ldots, \hat{A}_l)$$

hence to $\mu_\varrho = \delta_\varrho$, hence to the maximality of δ_ϱ, hence to 1.

2. $\Rightarrow$ 3. is obvious.

3. $\Rightarrow$ 4. If 4. does not hold, there exists $\psi \in \mathfrak{H}_\Omega$ such that $\|\psi\| = 1$, $(\psi, \Omega) = 0$; then for self-adjoint $A \in \mathfrak{A}$

$$|(\psi, \pi_\varrho(A)\Omega)|^2 + |(\Omega, \pi_\varrho(A)\Omega)|^2 = |(\psi, \pi_\varrho(\mathfrak{M}_a A)\Omega)|^2 + |(\Omega, \pi_\varrho(\mathfrak{M}_a A)\Omega)|^2 \leqq$$
$$\leqq (\Omega, [\pi_\varrho(\mathfrak{M}_a A)]^2 \Omega)$$

or $|(\psi, \pi_\varrho(A)\Omega)|^2 \leqq \varrho((\mathfrak{M}_a A)^2) - (\varrho(A))^2$ and 3. implies that $(\psi, \pi_\varrho(A)\Omega) = 0$, hence $\psi = 0$, a contradiction.

4. $\Rightarrow$ 5. Let C be in the commutant of $\pi_\varrho(\mathfrak{A}) \cup U(R^\nu)$; then $C\Omega \in \mathfrak{H}_\Omega$. If 4. holds, for some scalar c we have $C\Omega = c\Omega$ and, since Ω is cyclic for $\pi_\varrho(\mathfrak{A})$, C is a multiple of the identity: 5. holds.

5. $\Rightarrow$ 1. Let $\varrho = 1/2(\varrho_1 + \varrho_2)$ with $\varrho_1, \varrho_2 \in E \cap \mathfrak{L}^\perp$; there exist then (Dixmier [2], 2.5.1.) self-adjoint operators C_1, C_2 in the commutant of $\pi_\varrho(\mathfrak{A})$ such that

$$\varrho_i(A) = (\Omega, \pi_\varrho(A)\, C_i \Omega)\,.$$

Since $\varrho_i(A) = \varrho_i(\tau_{-x} A)$, we have

$$(\Omega, \pi_\varrho(A)\, C_i \Omega) = (\Omega, \pi_\varrho(A)\, U(x)\, C_i U(-x)\Omega)$$

hence C_1, C_2 belong to the commutant of $\pi_\varrho(\mathfrak{A}) \cup U(R^\nu)$. If 1. does not hold we may choose $\varrho_1 \neq \varrho_2$ hence $C_1 \neq C_2$ and 5. does not hold.

4. Preliminary lemmas

Lemma 1. *If $A_1, A_2 \in \mathfrak{A}$ and $\mathfrak{M} A_1 = \mathfrak{M} A_2$, then*

$$\lim_{a \to \infty} \|\mathfrak{M}_a A_1 - \mathfrak{M}_a A_2\| = 0\,.$$

By assumption $A_1 - A_2 \in \bar{\mathfrak{L}}$ hence, given $\varepsilon > 0$, there exists $A_0 \in \mathfrak{L}$ such that $\|A_1 - A_2 - A_0\| < \varepsilon$, and therefore

$$\|\mathfrak{M}_a A_1 - \mathfrak{M}_a A_2\| < \|\mathfrak{M}_a A_0\| + \varepsilon\,.$$

We may write A_0 as a finite sum of terms of the form $A - \tau_x A$ and if χ_a is the characteristic function of $\Lambda(a)$ we have

$$\mathfrak{M}_a(A - \tau_x A) = V(a)^{-1} \int db\, [\chi_a(b) - \chi_a(b - x)]\tau_b A$$
$$\lim_{a \to \infty} \|\mathfrak{M}_a(A - \tau_x A)\| \leqq \|A\| \lim_{a \to \infty} V(a)^{-1} |\int db\, [\chi_a(b) - \chi_a(b - x)]| = 0\,,$$

which proves the lemma[4].

[4] We may remark that if $A \in \mathfrak{A}$, then $\lim_{a \to \infty} \|\mathfrak{M}_a A\| = \|\mathfrak{M} A\|$. This follows from Lemma 1 upon noticing that $\|\mathfrak{M}_a A\| \leqq \|A\|$ and using the definition of the norm in $\mathfrak{A}/\bar{\mathfrak{L}}$.

Lemma 2. *If $A_1, A_2 \in \mathfrak{A}$ and $\varepsilon > 0$, there exists $a_0 \in R^\nu$ such that*

$$\|\mathfrak{M}_{a_1} A_1, \mathfrak{M}_{a_2} A_2]\| < \varepsilon$$

if $a_1 > a_0$, independently of a_2.

This is an immediate consequence of (A3).

Lemma 3. *Let A self-adjoint belong to $\mathfrak{A}$ and $\varrho \in E \cap \mathfrak{L}^\perp$, then the following limit exists and is finite*

$$\lim_{a_1, a_2 \to \infty} \varrho(\mathfrak{M}_{a_1} A \,.\, \mathfrak{M}_{a_2} A) \,.$$

Let us write

$$\|A\|_2 = [\varrho(A^2)]^{1/2}$$

and, for Lebesgue measurable $\Lambda \subset R^\nu$

$$A(\Lambda) = \int_\Lambda db\, \tau_b A \,.$$

For disjoint Λ_1, Λ_2 we have by the Schwarz inequality

$$\|A(\Lambda_1) + A(\Lambda_2)\|_2 \leqq \|A(\Lambda_1)\|_2 + \|A(\Lambda_2)\|_2 \,. \tag{1}$$

Define

$$X = \inf_{a > 0} V(a)^{-1} \|A(\Lambda(a))\|_2 \,. \tag{2}$$

For all $\varepsilon > 0$ there exists $a_0 > 0$ such that

$$V(a_0)^{-1} \|A(\Lambda(a_0))\|_2 < X + \varepsilon$$

and since $V(b)^{-1}\|A(\Lambda(b))\|_2$ is a continuous function of b, there exists $\delta > 0$ such that if $|b^i - .a_0^i| < \delta$ for $i = 1, \ldots, \nu$; then

$$V(b)^{-1} \|A(\Lambda(b))\|_2 < X + \varepsilon \,. \tag{3}$$

For sufficiently large a, we may choose b and integers $k^1, \ldots, k^\nu$ such that $|b^i - a_0^i| < \delta$ and $a^i = k^i b^i$. There exists then a partition of $\Lambda(b)$ into $\prod_{i=1}^{\nu} k^i$ translates Λ_α of $\Lambda(b)$, and (1) yields

$$\begin{aligned} &\|A(\Lambda(a))\|_2 \leqq \sum_\alpha \|A(\Lambda_\alpha)\|_2 \\ &= \left(\prod_{i=1}^{\nu} k^i\right) \|A(\Lambda(b))\| < \left(\prod_{i=1}^{\nu} k^i\right) V(b)\,(X + \varepsilon) = V(a)\,(X + \varepsilon) \end{aligned} \tag{4}$$

where we have used the translational invariance of ϱ and (3). From (4) and (2) we obtains thus

$$\lim_{a \to \infty} \varrho((\mathfrak{M}_a A)^2) = X^2 \,. \tag{5}$$

Denoting by $-\Lambda(a)$ the symmetric of $\Lambda(a)$ with respect to the origin of R^ν, we define

$$\mathfrak{M}_{-a} A = V(a)^{-1} \int_{-\Lambda(a)} db\, \tau_b A$$

$$\mathfrak{M}'_a A = \mathfrak{M}_{-a} \mathfrak{M}_a A \,.$$

Notice that by translational invariance of ϱ we have

$$\varrho((\mathfrak{M}_b A)^2) = \varrho(\mathfrak{M}'_b A \,.\, A) \,.$$

On the other hand

$$\lim_{a\to\infty} \|\mathfrak{M}_{-a}\mathfrak{M}_b' A - \mathfrak{M}_b' A\| = 0$$

therefore (5) gives

$$\begin{aligned}\lim_{a\to\infty}\lim_{b\to\infty} \varrho(\mathfrak{M}_b' A \,.\, \mathfrak{M}_a A) &= \lim_{a\to\infty}\lim_{b\to\infty} \varrho(\mathfrak{M}_{-a}\mathfrak{M}_b' A \,.\, A)\\ &= \lim_{b\to\infty} \varrho(\mathfrak{M}_b' A \,.\, A) = \lim_{b\to\infty} \varrho((\mathfrak{M}_a A)^2) = X^2\end{aligned} \tag{6}$$

and we have

$$\begin{aligned}\limsup_{b\to\infty} \varrho((\mathfrak{M}_b' A)^2) &\leqq \limsup_{b\to\infty} \left[\sup_{a_1,a_2} \varrho(\tau_{a_1}\mathfrak{M}_b A \,.\, \tau_{a_2}\mathfrak{M}_b A)\right] \leqq\\ &\leqq \lim_{b\to\infty} \varrho((\mathfrak{M}_b A)^2) = X^2 \,.\end{aligned} \tag{7}$$

The lemma results from the following formula derived from (5), (6), (7)

$$\begin{aligned}&\limsup_{a_1,a_2\to\infty} |\varrho(\mathfrak{M}_{a_1} A \,.\, \mathfrak{M}_{a_2} A) - X^2|\\ &\quad= \limsup_{a_1,a_2\to\infty}\lim_{b\to\infty} |\varrho[(\mathfrak{M}_{a_1} A - \mathfrak{M}_b' A) \,.\, \mathfrak{M}_{a_2} A]| \leqq\\ &\quad\leqq \lim_{a_1\to\infty}\lim_{b\to\infty} \varrho[(\mathfrak{M}_{a_1} A - \mathfrak{M}_b' A)^2]^{1/2} X = 0 \,.\end{aligned}$$

Lemma 4. *If $A_1, \ldots, A_l \in \mathfrak{A}$ and $\varrho \in E \cap \mathfrak{L}^\perp$, then the limit*

$$\lim_{a_1,\ldots,a_l\to\infty} \varrho(\mathfrak{M}_{a_1} A_1 \cdot \ldots \cdot \mathfrak{M}_{a_l} A_l) \tag{8}$$

exists, if finite, depends only on the classes $\mathfrak{M}A_1, \ldots, \mathfrak{M}A_l$ and is invariant under permutations of $A_1, \ldots, A_l$.

We may assume that $A_1, \ldots, A_l$ are self-adjoint. We write

$$X_j' = \mathfrak{M}_{a_1} A_1 \cdot \ldots \cdot \mathfrak{M}_{a_{j-1}} A_{j-1}, \quad X_j'' = \mathfrak{M}_{a_{j+1}} A_{j+1} \cdot \ldots \cdot \mathfrak{M}_{a_l} A_l \,.$$

To prove the existence of (8), it is sufficient to show that for each j and $\varepsilon > 0$ there exists $a_0 \in R^\nu$ independent of $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_l$ such that

$$|\varrho(X_j'(\mathfrak{M}_{a_j'} A_j - \mathfrak{M}_{a_j''} A_j) X_j'')| < \varepsilon \tag{9}$$

if $a_j', a_j'' > a_0$. By Lemma 2, for sufficiently large a_0 we have

$$|\varrho(X_j'(\mathfrak{M}_{a_j'} A_j - \mathfrak{M}_{a_j''} A_j) X_j'') - \varrho(X_j' X_j''(\mathfrak{M}_{a_j'} A_j - \mathfrak{M}_{a_j''} A_j))| < \varepsilon/2$$

and by Lemma 3

$$|\varrho(X_j' X_j''(\mathfrak{M}_{a_j'} A_j - \mathfrak{M}_{a_j''} A_j))| < \|X_j' X_j''\| \, \varrho((\mathfrak{M}_{a_j'} A_j - \mathfrak{M}_{a_j''} A_j)^2) < \varepsilon/2$$

which proves (9).

Lemma 1 shows that (8) depends only on the classes $\mathfrak{M}A_1, \ldots, \mathfrak{M}A_l$ and Lemma 2 that (8) is invariant under permutations of $A_1, \ldots, A_l$.

Remark. The fact that the limit (8) exists when $a_1, \ldots, a_l$ tend *independently* to ∞ and the fact that Lemma 4 holds for asymptotically Abelian (rather than local) algebras turned up during discussions with Robinson and Kastler. Robinson [7] has obtained a proof of our Lemma 4 which extends to locally compact groups more general than R^ν and to averages over regions Λ of rather general shape. Furthermore he needs that $\mathfrak{A}$ is asymptotically Abelian only to prove that (8) is invariant under permutations of $A_1, \ldots, A_l$.

Lemma 5. *Let $\varrho \in E \cap \mathfrak{L}^\perp$. For self-adjoint $A_1, \ldots, A_l \in \mathfrak{A}$ Let $\mathcal{P}$ be the algebra of complex polynomials in the indeterminates $\mathfrak{M}A_1, \ldots, \mathfrak{M}A_l$. A linear functional $\hat{\varrho}$ on $\mathcal{P}$ is defined by the condition that*

$$\hat{\varrho}\,[(\mathfrak{M}A_1)^{n_1} \cdot \ldots \cdot (\mathfrak{M}A_l)^{n_l}] = \lim_{a_1, \ldots, a_l \to \infty} \varrho\,[(\mathfrak{M}_{a_1}A_1)^{n_1} \cdot \ldots \cdot (\mathfrak{M}_{a_l}A_l)^{n_l}]\,.$$

There exists then a positive measure $m_{A_1 \ldots A_l}$ with compact support in R^l such that

$$\hat{\varrho}\,[P(\mathfrak{M}A_1, \ldots, \mathfrak{M}A_l)] = m_{A_1 \ldots A_l}(P)\,.$$

Let P be a product $P = P_1 \otimes \cdots \otimes P_l$ where P_i is a polynomial in the i-th argument of P. With respect to the norms

$$\|P_i\| = \max_{-\|A_i\| \leqq t \geqq \|A_i\|} |P_i(t)|$$

the expression

$$\hat{\varrho}\,[P(\mathfrak{M}A_1, \ldots, \mathfrak{M}A_l)] = \lim_{a \to \infty} \varrho\,[P_1(\mathfrak{M}_a A_1) \cdot \ldots \cdot P_1(\mathfrak{M}_a A_l)]$$

is a continuous multilinear form in $P_1, \ldots, P_l$ because $\|P_i(\mathfrak{M}_a A_i)\| \leqq \leqq \|P_i\|$. If $P_i > 0$ for $i = 1, \ldots, l$ there exist polynomials Q_i such that $P_i = Q_i^* Q_i$ hence, with $Q = Q_1 \otimes \cdots \otimes Q_l$, we have

$$\begin{aligned}&\hat{\varrho}\,[P(\mathfrak{M}A_1, \ldots, \mathfrak{M}A_l)]\\ &\qquad = \lim_{a \to \infty} \varrho\,[Q(\mathfrak{M}_a A_1, \ldots, \mathfrak{M}_a A_l)^* \, Q(\mathfrak{M}_a A_1, \ldots, \mathfrak{M}_a A_l)] \geqq 0\,.\end{aligned}$$

The complex continuous functions on the real line may be approximated by polynomials with respect to the norms introduced above, the positive functions by positive polynomials. This shows that the above multilinear functional extends uniquely to a multilinear functional $M(f_1, \ldots, f_l)$ on continuous functions, such that $f_1 \geqq 0, \ldots, f_l \geqq 0$ implies $M(f_1, \ldots, f_l) \geqq 0$. Separately in the i-th variable, M is thus a measure (with support in $[-\|A_i\|, +\|A_i\|]$), hence a distribution and by Schwartz' kernel theorem there exists a distribution $m_{A_1 \ldots A_l}$ in R^l such that

$$M(f_1, \ldots, f_l) = m_{A_1 \ldots A_l}(f_1 \otimes \cdots \otimes f_l)$$

if $f_1, \ldots, f_l \in \mathfrak{D}(R)$. The distribution $m_{A_1 \ldots A_l}$ has its support in the product of the intervals $[-\|A_i\|, +\|A_i\|]$. If $0 \leqq \alpha_i \in \mathfrak{D}(R)$ and α_i tends to DIRAC's δ measure for $i = 1, \ldots, l$, regularization of $m_{A_1 \ldots A_l}$ by $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_l$ yields a positive function $m_{A_1 \ldots A_l} * \alpha$ which tends weakly to m, implying that m is a positive measure and the lemma is proved.

Lemma 6. *Let $\varrho \in E \cap \mathfrak{L}^\perp$; $A_1, \ldots, A_l \in \mathfrak{A}$ be self-adjoint, and P be a complex polynomial in l arguments, then*

$$|\hat{\varrho}\,[P(\mathfrak{M}A_1, \ldots, \mathfrak{M}A_l)]| \leqq \sup_{\sigma \in E \cap \mathfrak{L}^\perp} |P(\sigma(A_1), \ldots, \sigma(A_l))|\,.$$

Let Δ be the support of $m_{A_1 \ldots A_l}$ and $x \in \Delta$. Let $\chi \geqq 0$ be a continuous function with compact support on R^l such that $\chi(x) > 0$; then

$$\alpha = m_{A_1 \ldots A_l}(\chi) > 0 \ .$$

We define a functional ϱ'_χ on the self-adjoint elements of $A \in \mathfrak{A}$ by

$$\varrho'_\chi(A) = \alpha^{-1} \int dt_1 \ldots dt_l \, dt \, m_{A_1 \ldots A_l A}(t_1, \ldots, t_l, t) \, \chi(t_1, \ldots, t_l) \, . \, t \, .$$

One checks from the definition (Lemma 5) of $m_{A_1 \ldots A_l A}$ that if $A \geqq 0$, then $m_{A_1 \ldots A_l A}$ has its support in the product of Δ by the positive semi-axis. Therefore ϱ'_χ is a positive functional. The support of the measure $m_{A_1 \ldots A_l 1}$ is $\Delta \times \{1\}$ hence $\varrho'_\chi(1) = 1$. By considering the support of $m_{A_1 \ldots A_l A' A'', A' + \lambda A''}$ for real scalar λ one sees similarly that ϱ'_χ is real linear. By linear extension to non self-adjoint elements of $\mathfrak{A}$, ϱ'_χ yields thus an element ϱ_χ of $E \cap \mathfrak{L}^\perp$. When the support of χ tends to x, we have

$$(\varrho_\chi(A_1), \ldots, \varrho_\chi(A_l)) \to x$$

and the lemma follows from the inequality

$$|\hat{\varrho}\,[P(\mathfrak{M} A_1, \ldots, \mathfrak{M} A_l)]| \leqq \sup_{x \in \Delta} |P(x)| \ .$$

5. Proof of the theorem

1°) Extension by linearity from the formula

$$\mu_\varrho(\hat{A}_1, \ldots, \hat{A}_l) = \lim_{a_1, \ldots, a_l \to \infty} \varrho(\mathfrak{M}_{a_1} A_1, \ldots, \mathfrak{M}_{a_l} A_l)$$

yields a linear functional μ_ϱ on the polynomials in the $\hat{A}_i$. Lemma 6 shows that this functional is continuous for the topology of uniform convergence on $E \cap \mathfrak{L}^\perp$. Let $\mathscr{C}(E \cap \mathfrak{L}^\perp)$ be the space of complex continuous functions on $E \cap \mathfrak{L}^\perp$ with the same topology. Since the $\hat{A}$ separate the points of $E \cap \mathfrak{L}^\perp$, the polynomials in the $\hat{A}$ are dense in $\mathscr{C}(E \cap \mathfrak{L}^\perp)$ by the theorem of Stone-Weierstrass, and μ_ϱ extends to a continuous linear functional on $\mathscr{C}(E \cap \mathfrak{L}^\perp)$, i.e. a measure on $E \cap \mathfrak{L}^\perp$, again noted μ_ϱ.

For self-adjoint $A_1, \ldots, A_l \in \mathfrak{A}$,and a complex continuous function f on R^l, Lemma 5 shows that

$$\mu_\varrho(f(\hat{A}_1, \ldots, \hat{A}_l)) = m_{A_1 \ldots A_l}(f) \ .$$

If $0 \leqq \varphi \in \mathscr{C}(E \cap \mathfrak{L}^\perp)$, one can approximate φ by functions of the form $f(\hat{A}_1, \ldots, \hat{A}_l)$ with $f \geqq 0$ (e.g. taking for f the absolute value of a polynomial) and it follows that $\mu_\varrho(\varphi) \geqq 0$. Finally $\mu_\varrho(1) = 1$, which concludes the proof of part 1 of the theorem.

2°) Part 2 of the theorem follows directly from the equation defining μ_ϱ.

3°) Let μ be a measure which majorizes δ_ϱ, i.e. such that $\mu(\hat{A}) = \varrho(A)$ for all $A \in \mathfrak{A}$. If $\varphi \in (E \cap \mathfrak{L}^\perp)$ and $\varepsilon > 0$ one can find a measure μ' with finite support: $\mu' = \sum \alpha_i \delta_{\varrho i}$, $\alpha_i > 0$ such that $|\mu(\varphi) - \mu'(\varphi)| < \varepsilon$ and

$\sum \alpha_i \varrho_i = \varrho$ (see [1], footnote p. 141). If φ is convex we have thus

$$\mu(\varphi) - \varepsilon \leqq \mu'(\varphi) = \sum \alpha_i \delta_{\varrho_i}(\varphi) \leqq \sum \alpha_i \mu_{\varrho_i}(\varphi) = \mu_\varrho(\varphi)$$

hence $\mu_\varrho \succ \mu$ which proves part 3 of the theorem.

4°) If $\mathfrak{A}_\alpha$ is a self-adjoint subalgebra of $\mathfrak{A}$, let $B_\alpha = \{A \in \mathfrak{A}_\alpha : A = A^*, \|A\| \leqq 1\}$, the subset of $E \cap \mathfrak{L}^\perp$ formed by the elements ϱ such that the restriction of ϱ to $\mathfrak{A}_\alpha$ has norm 1 is then

$$\mathfrak{J}_\alpha = \left\{\varrho \in E \cap \mathfrak{L}^\perp : \sup_{A \in B_\alpha} \varrho(A) = 1\right\}$$

and we have

$$\mathfrak{J}_\alpha = \bigcap_{m > 0} V_m$$

$$V = \bigcup_{A \in B_\alpha} \left\{\varrho \in E \cap \mathfrak{L}^\perp : \varrho(A) > 1 - \frac{1}{m}\right\}.$$

Since the V_m are open, their countable intersection $\mathfrak{J}_\alpha$ is measurable.

We prove first that if $\varrho \in \mathfrak{J}_\alpha$, then μ_ϱ is concentrated on $\mathfrak{J}_\alpha$. Let $\mu_\varrho = \mu' + \mu''$ where $\|\mu'\| + \|\mu''\| = 1$, μ' is carried by V_m and μ'' by $E \cap \mathfrak{L}^\perp - V_m$. We have for all $A \in B_\alpha$:

$$\mu''(\hat{A}) \leqq \|\mu''\| \left(1 - \frac{1}{m}\right)$$

hence

$$\varrho(A) = \mu_\varrho(\hat{A}) = \mu'(\hat{A}) + \mu''(\hat{A}) \leqq \|\mu'\| + \|\mu''\| \left(1 - \frac{1}{m}\right) = 1 - \frac{1}{m} - \|\mu''\|$$

and therefore $\|\mu''\| = 0$. For all m, μ_ϱ is thus concentrated on V_m, hence μ_ϱ is concentrated on $\mathfrak{J}_\alpha$.

Let now μ_ϱ be concentrated on $\mathfrak{J}_\alpha$, hence on V_{2m}. There exists then a compact $K \subset V_{2m}$ such that $\mu_\varrho(K) > 1 - \frac{1}{2m}$. We may suppose that $\mathfrak{A}_\alpha$ is a sub-B^*-algebra of $\mathfrak{A}$, and has thus an approximate identity (see DIXMIER [2], 1.7.2. and 2.1.5 (v)). Since $K \subset V_{2m}$ is compact one can find, using the approximate identity, $A_0 \in B_\alpha$ such that

$$\left\{\varrho \in E \cap \mathfrak{L}^\perp : \varrho(A_0) > 1 - \frac{1}{2m}\right\} \supset K .$$

Under these conditions

$$\varrho(A_0) = \mu_\varrho(\hat{A}_0) > \left(1 - \frac{1}{2m}\right) \mu_\varrho(K) > 1 - \frac{1}{m},$$

hence $\varrho \in V_m$, hence $\varrho \in \mathfrak{J}_\alpha$.

The above results generalize immediately from the case of one subalgebra $\mathfrak{A}_\alpha$ to the case of a countable family $(\mathfrak{A}_\alpha)$ because $\mathfrak{J} = \bigcap_\alpha \mathfrak{J}_\alpha$ is measurable, proving part 4 of the theorem.

5°) Suppose that the conditions of part 5 of the theorem are fulfilled and let π be the mapping which associates to $\varrho \in E \cap \mathfrak{L}^\perp$ the sequence $(\varrho(A_i))$. Considered as a mapping of $E \cap \mathfrak{L}^\perp$ into the product of a countable sequence of copies of the real line, π is continuous and its image $\pi(E \cap \mathfrak{L}^\perp)$ is thus compact. Corresponding to μ_ϱ, with $\varrho \in \mathfrak{J}$, a measure ν_ϱ on $\pi(E \cap \mathfrak{L}^\perp)$ is defined by

$$\nu_\varrho(\psi) = \mu_\varrho(\psi \circ \pi), \quad \psi \in \mathscr{C}(\pi(E \cap \mathfrak{L}^\perp)) .$$

We have then $\nu_\varrho \succ \delta_{\pi\varrho}$ on $\pi(E \cap \mathfrak{L}^\perp)$. We show that ν_ϱ is maximal. Let indeed $\nu \succ \delta_{\pi\varrho}$. If $\varepsilon > 0$ and $\psi \in \mathscr{C}(\pi(E \cap \mathfrak{L}^\perp))$, one can find a measure ν' with finite support: $\nu' = \sum \alpha_i \delta_{\sigma_i}$, $\alpha_i > 0$, such that $|\nu(\psi) - \nu'(\psi)| < \varepsilon$ and $\sum \alpha_i \sigma_i = \pi\varrho$ (see [1], footnote p. 141). Let $\varrho_i \in E \cap \mathfrak{L}^\perp$ be such that $\pi\varrho_i = \sigma_i$, then $\pi(\sum \alpha_i \varrho_i) = \pi\varrho$, hence by assumption $\varrho = \sum \alpha_i \varrho_i$. We have thus

$$\nu'(\psi) = \sum \alpha_i \psi(\pi\varrho_i) \leqq \sum \alpha_i \mu_{\varrho_i}(\psi \circ \pi) = \mu_\varrho(\psi \circ \pi) = \nu_\varrho(\psi)$$

hence $\nu_\varrho(\psi) \geqq \nu'(\psi) - \varepsilon$ which shows that ν_ϱ is maximal. Since $\pi(E \cap \mathfrak{L}^\perp)$ is metrizable, it follows (see Choquet and Meyer [1], Corollary 14) that ν_ϱ is concentrated on the set $\mathscr{E}(\pi(E \cap \mathfrak{L}^\perp))$ of extremal points of $\pi(E \cap \mathfrak{L}^\perp)$. Therefore μ_ϱ is concentrated on

$$\mathfrak{I} \cap \pi^{-1}\mathscr{E}(\pi(E \cap \mathfrak{L}^\perp)) .$$

But, using the assumptions one sees $\mathfrak{I} \cap \pi^{-1}\mathscr{E}(\pi(E \cap \mathfrak{L}^\perp)) \subset \mathscr{E}(E \cap \mathfrak{L}^\perp)$, hence μ_ϱ is concentrated on $\mathscr{E}(E \cap \mathfrak{L}^\perp) \cap \mathfrak{I}$.

Conversely let $\mu \geqq 0$ be a measure of norm 1 on $E \cap \mathfrak{L}^\perp$, there exists then $\varrho \in E \cap \mathfrak{L}^\perp$ (the resultant of μ) such that $\varrho(A) = \mu(\hat{A})$ for all $A \in \mathfrak{A}$, i.e. $\mu \succ \delta_\varrho$. If μ is concentrated on $\mathscr{E}(E \cap \mathfrak{L}^\perp)$, it follows (see Choquet and Meyer [1], proposition 15) that μ is extremal, hence by part 3 of the theorem, $\mu = \mu_\varrho$. If μ is concentrated on $\mathscr{E}(E \cap \mathfrak{L}^\perp) \cap \mathfrak{I}$, part 4 of the theorem shows that $\varrho \in \mathfrak{I}$, which concludes the proof.

6. An application to anticommutation relations

Let $\mathscr{H}$ be the Hilbert space of the Fock representation of the canonical anticommutation relations (Fock space of the CAR). We take as test-functions the real square-integrable functions on R^ν, which form a real Hilbert space $L^2_R(R^\nu)$. Let $\mathfrak{A}_0$ be the algebra of bounded operators on $\mathscr{H}$ generated by the annihilation and creation operators $a(f)$, $a^*(g)$ with $f, g \in L^2_R(R^\nu)$ and let $\mathfrak{B}_0$ be the subalgebra of $\mathfrak{A}_0$ generated by the monomials of even degree. We note $\mathfrak{A}$ (resp. $\mathfrak{B}$) the uniform closure of $\mathfrak{A}_0$ (resp. $\mathfrak{B}_0$). It is known that the states on $\mathfrak{A}$ exactly correspond (by the Gel'fand-Segal construction) to the cyclic representations of the canonical anticommutation relations.

We call even state a state on $\mathfrak{A}$ which vanishes on the monomials of odd degree in the creation and annihilation operators (only even states occur in questions of physical interest). There is then a natural one-to-one correspondance between the states on B and the even states on $\mathfrak{A}$.

If $x \in R^\nu$, an automorphism τ_x of the algebra $\mathfrak{A}$ exists such that $\tau_x a(f) = a(f_x)$ where $f_x \in L^2_R(R^\nu)$ is defined by $f_x(\xi) = f(\xi - x)$. With this definition (A1) is obviously satisfied for $\mathfrak{A}$ and $\mathfrak{B}$. As is well known, the

CAR imply $\|a(f)\| = \|a^*(f)\| = \|f\|_2$. From this (A2) follows for $\mathfrak{A}$ and $\mathfrak{B}$. It follows also that the algebra $\mathfrak{B}$ of even polynomials in the $a(f)$, $a^*(g)$, where f, g have compact support, is dense in $\mathfrak{B}$ and therefore (A3) is satisfied for $\mathfrak{B}$. Finally since $L^2_R(R^\nu)$ is separable, $\mathfrak{B}$ is separable.

The information collected above shows that part 5 of the theorem of Section 3 applies to the translationally invariant states on B. We reformulate this result as follows.

Proposition. *Let K be the convex compact set of even translationally invariant states on the algebra $\mathfrak{A}$ of the CAR. Let $\mathscr{E}(K)$ be the set of extremal points of K. There is a one-to-one correspondance between the elements ϱ of K and the measures $\mu_\varrho \geqq 0$, of total mass 1 on K concentrated on $\mathscr{E}(K)$, such that*

$$\varrho(A) = \mu_\varrho(\hat{A}) \quad \text{for all} \quad A \in \mathfrak{A} .$$

That K contains many elements follows from [8] Section VIII.

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Appendix

Let H be an unbounded self-adjoint operator in the Hilbert space $\mathscr{H}$ of a C^*-algebra $\mathfrak{A}$, and let ϱ be a state on $\mathfrak{A}$. The aim of this appendix is to give conditions under which an unbounded self-adjoint operator H_ϱ, corresponding to H, can be reconstructed in the Gel'fand-Segal Hilbert space $\mathfrak{H}_\varrho$.

We shall however state our results somewhat more generally and use the language of B^*-algebras. Throughout what follows $\mathfrak{A}$ will be a B^*-algebra and, given a state ϱ on $\mathfrak{A}$, we shall note $\mathfrak{H}_\varrho$ the Hilbert space of the Gel'fand-Segal construction and π_ϱ the canonical homomorphism of $\mathfrak{A}$ into the bounded operators on $\mathfrak{H}_\varrho$. Ω will be the normalized vector in $\mathfrak{H}_\varrho$, cyclic with respect to $\pi_\varrho(\mathfrak{A})$ such that $(\Omega, \pi_\varrho(A)\Omega) = \varrho(A)$ for all $A \in \mathfrak{A}$.

Let $\mathfrak{A}_1$ be a sub-B^*-algebra of $\mathfrak{A}$ and ϱ a state on $\mathfrak{A}$ such that its restriction to $\mathfrak{A}_1$ has norm 1. Given $\varepsilon > 0$ there exists a self-adjoint $A_1 \in \mathfrak{A}_1$ such that

$$\|A_1\| \leqq 1, \ \varrho(A_1) > 1 - \varepsilon^2/2 . \tag{A1}$$

This is a consequence of the existence of an approximate identity in $\mathfrak{A}_1$ (see DIXMIER [2] 1.7.2. and 2.1.5. (v)). Therefore

$$\|\Omega - \pi_\varrho(A_1)\Omega\| < \varepsilon . \tag{A2}$$

Proposition 1. *Let $\mathfrak{A}_1$ be a sub-B^*-algebra of $\mathfrak{A}$; the following conditions are equivalent*

(i) *The closed left ideal L (or the closed right ideal R) generated by $\mathfrak{A}_1$ is a two-sided ideal.*

(ii) *For all* $A \in \mathfrak{A}$, $A_1 \in \mathfrak{A}_1$, $\varepsilon > 0$, *there exists a self-adjoint* $B \in \mathfrak{A}_1$, *such that*

$$\|B\| \leqq 1, \ \|BAA_1 - AA_1\| < \varepsilon . \tag{A3}$$

(iii) *There exist self-adjoint subsets* $\tilde{\mathfrak{A}}$ *and* $\tilde{\mathfrak{A}}_1$ *respectively generating* $\mathfrak{A}$ *and* $\mathfrak{A}_1$ *(for their structures of* B^**-algebras) and such that for all* $A \in \tilde{\mathfrak{A}}$, $A_1 \in \tilde{\mathfrak{A}}_1$, $\varepsilon < 0$, *there exists* $B \in \mathfrak{A}_1$ *such that*

$$\|BAA_1 - AA_1\| < \varepsilon . \tag{A4}$$

(i) $\Rightarrow$ (ii) If L is a two-sided ideal, then $L \supset R$, hence $L = R^* \subset L^* = R$ and for all $\varepsilon > 0$ there exist $A_i' \in \mathfrak{A}$, $B_i' \in \mathfrak{A}_1$, $i = 1, \ldots, n$ such that

$$\|AA_1 - \sum_{i=1}^{n} B_i' A_i'\| < \varepsilon/3 . \tag{A5}$$

In view of the existence of an approximate identity in $\mathfrak{A}_1$ there exists a self-adjoint $B \in \mathfrak{A}_1$ such that

$$\|B\| \leqq 1, \ \|BB_i' - B_i'\| \,.\, \|A_i'\| < \varepsilon/3n \tag{A6}$$

and (A3) follows from (A5), (A6).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). (A4) implies that $\tilde{\mathfrak{A}}\tilde{\mathfrak{A}}_1 \subset R$, hence $\tilde{\mathfrak{A}}\mathfrak{A}_1 \subset R$, hence $\tilde{\mathfrak{A}}\mathfrak{A}_1 \subset R$, hence $L \subset R$, hence $R = L^* \subset R^* = L$, hence $L = R$ is a two-sided ideal.

Definition. *We shall say that a sub-*B^**-algebra* $\mathfrak{A}_1$ *of* $\mathfrak{A}$ *is clean if it satisfies the conditions of Proposition* 1.

Lemma. *Let* $\mathfrak{A}_1$ *be a sub-*B^**-algebra of* $\mathfrak{A}$ *and* ϱ *a state on* $\mathfrak{A}$. *If* $\mathfrak{A}_1$ *is clean and if the restriction of* ϱ *to* $\mathfrak{A}_1$ *has norm* 1 *then, for all* $\Phi \in \mathfrak{H}_\varrho$, $\varepsilon > 0$, *there exists a self-adjoint* $B \in \mathfrak{A}_1$ *such that*

$$\|B\| \leqq 1, \ \|\Phi - \pi_\varrho(B)\Phi\| < \varepsilon . \tag{A7}$$

The restriction of π_ϱ *to* $\mathfrak{A}_1$ *is thus non degenerate.*

There exists $A \in \mathfrak{A}$ such that

$$\|\Phi - \pi_\varrho(A)\Omega\| < \varepsilon . \tag{A8}$$

Choose also A_1, $B \in \mathfrak{A}_1$ satisfying respectively (A2), (A3); then (A8) and (A2) yield

$$\|\Phi - \pi_\varrho(AA_1)\Omega\| < \varepsilon(1 + \|A\|) . \tag{A9}$$

We have

$$\begin{aligned}\Phi - \pi_\varrho(B)\Phi = (\Phi - \pi_\varrho(AA_1)\Omega) + \\ + \pi_\varrho(AA_1 - BAA_1)\Omega + \pi_\varrho(B)\,(\pi_\varrho(AA_1)\Omega - \Phi) .\end{aligned} \tag{A10}$$

So that (A3) and (A9) yield

$$\|\Phi - \pi_\varrho(B)\Phi\| < \varepsilon(3 + 2\|A\|) \tag{A11}$$

proving the Lemma.

Let $\mathscr{C}_0(R)$ be the B^*-algebra of complex continuous functions vanishing at infinity on the real line and let $\mathscr{H}$ be the dense ideal of $\mathscr{C}_0(R)$ formed by the functions with compact support.

Proposition 2. *Let* h *be a homomorphism of* $\mathscr{C}_0(R)$ *into the* B^**-algebra* $\mathfrak{A}$ *and* $\mathfrak{A}_1 = h\mathscr{C}_0(R)$ *its image. Let* ϱ *be a state on* $\mathfrak{A}$. *if the restriction of*

ϱ to $\mathfrak{A}_1$ has norm 1 and if $\mathfrak{A}_1$ is clean, then $\pi_\varrho h$ is non degenerate and there is a unique self-adjoint operator H_ϱ on $\mathfrak{H}_\varrho$ such that for all $f \in \mathscr{C}_0(R)$

$$\pi_\varrho(hf) = f(H_\varrho)\,. \tag{A12}$$

It follows from the lemma that $\pi_\varrho(h\mathscr{C}_0(R))\mathfrak{H}_\varrho$ is dense in $\mathfrak{H}_\varrho$. Therefore also $D = \pi_\varrho(h\mathscr{H})\mathfrak{H}_\varrho$ is dense in $\mathfrak{H}_\varrho$. If $g \in \mathscr{H}$ we define $g' \in \mathscr{H}$ by

$$g'(t) = tg(t)\,. \tag{A13}$$

If $\pi_\varrho(hg)\Phi = \pi_\varrho(h\tilde{g})\tilde{\Phi}$, with $g, \tilde{g} \in \mathscr{H}$, $\Phi, \tilde{\Phi} \in \mathfrak{H}_\varrho$, there exists $\alpha \in \mathscr{H}$ such that $g' = \alpha g$, $\tilde{g}' = \alpha\tilde{g}$, hence

$$\pi_\varrho(hg')\,\Phi = \pi_\varrho(h\alpha)\,\pi_\varrho(hg)\,\Phi = \pi_\varrho(h\alpha)\,\pi_\varrho(h\tilde{g})\,\tilde{\Phi} = \pi_\varrho(h\tilde{g}')\tilde{\Phi}\,. \tag{A14}$$

We may thus define a linear operator H' on D by

$$H'\pi_\varrho(hg)\,\Phi = \pi_\varrho(hg')\Phi \tag{A15}$$

and, α being as above, we have

$$\|(H')^n\,\pi_\varrho(hg)\Phi\| = \|(\pi_\varrho(h\alpha))^n\,\pi_\varrho(hg)\Phi\| \leqq \|\alpha\|^n\,\|\pi_\varrho(hg)\Phi\|\,. \tag{A16}$$

The vectors in D are thus analytic vectors of H' in the sense of NELSON, and since D is dense in $\mathfrak{H}_\varrho$, H' is essentially self-adjoint (see [7], Lemma 5.1). Let H_ϱ be the closure of H'. If P is a polynomial we have

$$P(H_\varrho)\,\pi_\varrho(hg)\Phi = \pi_\varrho\big(h\big(P(\alpha)\,.\,g\big)\big)\Phi\,. \tag{A17}$$

Therefore if P tends uniformly on $(-\|h\|, \|h\|)$ to the restriction to this interval of $f \in \mathscr{C}_0(R)$, the right-hand side of (A.17) has a limit, and therefore

$$f(H_\varrho)\,\pi_\varrho(h)\Phi = \pi_\varrho\big(h(f(\alpha)\,.\,g)\big)\Phi = \pi_\varrho(h(f\,.\,g))\Phi = \pi_\varrho(hf)\,\pi_\varrho(hg)\Phi\,.$$

This shows that $f(H_\varrho) = \pi_\varrho(hf)$ on D, hence on $\mathfrak{H}_\varrho$.

Remark. If in Proposition 2 we replace $\mathscr{C}_0(R)$ by $\mathscr{C}_0(S)$, where S is a closed subset of R, a self-adjoint operator H_ϱ satisfying (A12) is again obtained and its spectrum is contained in S.

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