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Application of Spectral Representations to the Nonrelativistic and the Relativistic Bethe-Salpeter Equation

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Abstract. The eigenvalue problem of the scalar Bethe-Salpeter equation is solved by application of the vertical Dyson representation. The method of solution is developed in complete analogy to the solution of Schrödinger's equation by a Stieltjes representation in the case of a Yukawa potential. The eigenvalues are zeros of a characteristic determinant, which can be understood as a generalization of the nonrelativistic Jost function.

I. Introduction

Recently Coester [1] has proposed relativistic particle quantum mechanics as a possible alternative of quantum field theory. From the mathematical point of view Coester's approach has the virtue of being based on the firm ground of functional analysis, but physically it suffers from serious shortcomings. It does not offer physical arguments for the choice of interaction operators, nor does it seem possible to include the principle of causality in a simple way. As a consequence of causality matrix elements should satisfy dispersion relations as in field theory.

The opposite situation is encountered in field theory. We consider as an example the formulation of the relativistic two-body system in terms of the Bethe-Salpeter equation. Here the principle of causality is included from the outset and possible approximations for the interaction can be taken from perturbation theory. On the other hand the mathematical structure of the eigenvalue problem is obscure. It is the purpose of this paper to shed some light on this question.

To avoid kinematical and renormalization difficulties we consider the B-S equation for an S-wave bound state in a super-renormalizable theory of three scalar fields with trilinear interaction. Our approach to the solution of the eigenvalue problem is based on a suitable adaption of Jost's method to the relativistic situation. We briefly review the solution of Schrödinger's equation in momentum space for an S-wave bound state in a Yukawa potential in Sec. II. A Stieltjes transformation leads to an inhomogeneous integral equation for the spectral function [2], [3], which can be solved by iteration. The relationship of this method to Jost's approach to a solution of the differential equation in configuration space is investigated. In Sec. III we summarize the general properties of the relativistic B-S amplitude and decompose it into a one-particle singular and a regular part. This device is crucial for a successful adaption of the nonrelativistic method to the B-S equation, which is the subject of Sec. IV. We use the vertical Dyson representation as a possible substitute of the Stieltjes representation and transform the B-S equation into an inhomogeneous integral equation for the spectral function by splitting off the one-particle singular part. This is done for the ladder approximation, but the method applies equally well to the complete B-S equation. The integral equation can again be solved by iteration, while the boundary conditions are expressed in terms of two coupled integral equations for the absorptive parts of the two vertex functions with one particle off the mass shell. The Fredholm determinant of this system is the generalization of the nonrelativistic Jost function.

II. The nonrelativistic amplitude

In analogy to the relativistic formulation we describe a bound state of two particles in the nonrelativistic theory by the two-point amplitude

$$\chi(x_1, x_2) = (2\pi)^{3/2} \langle 0 | T(\psi_1(x_1)\, \psi_2(x_2)) | P \rangle \tag{2.1}$$

where T means Wick's chronological operator and x stands for $x = \{\mathbf{x}, x_0 = t\}$. The field operators $\psi_1(x_1)$ and $\psi_2(x_2)$ are related to spinless particles with masses m_1 and m_2. They satisfy the commutation relations

$$[\psi_i(\mathbf{x}, x_0), \psi_i^\dagger(\mathbf{x}', x_0)] = \delta(\mathbf{x} - \mathbf{x}'), \quad i = 1, 2\,. \tag{2.2}$$

All other commutators vanish. The theory is supposed to be invariant under Galilean transformations. The invariant "mass shell" conditions read ($\hbar = 1$):

$$\frac{\mathbf{p}_1^2}{2m_1} - p_{10} = 0; \frac{\mathbf{p}_2^2}{2m_2} - p_{20} = 0 \tag{2.3}$$

for the basic particles and

$$\frac{\mathbf{P}^2}{2(m_1 + m_2)} - P_0 = \varepsilon_B > 0\,. \tag{2.4}$$

for the bound state with binding energy ε_B.

The amplitude (2.1) can be written in the form

$$\chi(x_1, x_2) = \varphi(x_1 - x_2) \exp i \left(\mathbf{P} \cdot \frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - P_0 \frac{x_{10} + x_{20}}{2}\right), \tag{2.5}$$

where

$$\varphi(y) = (2\pi)^{3/2} \left\langle 0 \middle| T\left(\psi_1\left(\frac{y}{2}\right) \psi_2\left(-\frac{y}{2}\right)\right) \middle| P \right\rangle. \tag{2.6}$$

According to the definition of the T-operator we have

$$\varphi(y) = (2\pi)^{3/2}\left\langle 0\left|\psi_1\left(\frac{y}{2}\right)\psi_2\left(-\frac{y}{2}\right)\right|P\right\rangle\theta(y_0) + \\ + (2\pi)^{3/2}\left\langle 0\left|\psi_2\left(-\frac{y}{2}\right)\psi_1\left(\frac{y}{2}\right)\right|P\right\rangle\theta(-y_0)\,. \tag{2.7}$$

The factors of the θ-functions in (2.7) can be represented by Fourier integrals:

$$(2\pi)^{3/2}\left\langle 0\left|\psi_1\left(\frac{y}{2}\right)\psi_2\left(-\frac{y}{2}\right)\right|P\right\rangle = \int d^4q\, e^{i(\mathbf{q}\cdot\mathbf{y}-q_0y_0)} \times \\ \times\,\delta\left(\frac{\left(\mathbf{q}+\frac{\mathbf{P}}{2}\right)^2}{2m_1} - \left(q_0+\frac{P_0}{2}\right)\right)\left\langle q+\frac{P}{2}\right|\psi_2(0)\left|P\right\rangle \\ (2\pi)^{3/2}\left\langle 0\left|\psi_2\left(-\frac{y}{2}\right)\psi_1\left(\frac{y}{2}\right)\right|P\right\rangle = \int d^4q\, e^{i(\mathbf{q}\cdot\mathbf{y}-q_0y_0)} \times \\ \times\,\delta\left(\frac{\left(\frac{\mathbf{P}}{2}-\mathbf{q}\right)^2}{2m_2} - \left(\frac{P_0}{2}-q_0\right)\right)\left\langle -q+\frac{P}{2}\right|\psi_1(0)\left|P\right\rangle, \tag{2.8}$$

where we have chosen the normalization:

$$(2\pi)^{3/2}\,\langle 0|\psi_i(0)|\mathbf{p}_i\rangle = 1\,,\quad i = 1, 2\,. \tag{2.9}$$

The restriction of the support of the Fourier transforms in (2.8) to the mass shell parabolas

$$\frac{\left(\frac{\mathbf{P}}{2}+\mathbf{q}\right)^2}{2m_1} - \left(\frac{P_0}{2}+q_0\right) = 0;\ \frac{\left(\frac{\mathbf{P}}{2}-\mathbf{q}\right)^2}{2m_2} - \left(\frac{P_0}{2}-q_0\right) = 0 \tag{2.10}$$

is of course enforced by the conservation of particle numbers. By substitution of (2.8) into (2.7) and Fourier transformation we obtain the following representation of the two-particle amplitude in momentum space:

$$f(q) = \frac{1}{(2\pi)^4}\int d^4y\, e^{-i(\mathbf{q}\cdot\mathbf{y}-q_0y_0)}\,\varphi(y) \\ = -\frac{i}{(2\pi)^4}\left\{\frac{(2\pi)^3\left\langle\frac{P}{2}+q|\psi_1(0)|P\right\rangle}{\frac{\left(\frac{\mathbf{P}}{2}+\mathbf{q}\right)^2}{2m_1} - \left(\frac{P_0}{2}+q_0\right) - i\varepsilon} + \right. \\ \left. + \frac{(2\pi)^3\left\langle\frac{P}{2}-q|\psi_2(0)|P\right\rangle}{\frac{\left(\frac{\mathbf{P}}{2}-\mathbf{q}\right)^2}{2m_2} - \left(\frac{P_0}{2}-\mathbf{q}_0\right) - i\varepsilon}\right\}. \tag{2.11}$$

Let us assume that the bound state has zero angular momentum. $f(q)$ can then be considered as a function of the two Galilean invariants

$$s_1 = \frac{\left(\frac{\mathbf{P}}{2} + \mathbf{q}\right)^2}{2m_1} - \left(\frac{P_0}{2} + q_0\right), \qquad s_2 = \frac{\left(\frac{\mathbf{P}}{2} - \mathbf{q}\right)^2}{2m_2} - \left(\frac{P_0}{2} - q_0\right). \tag{2.12}$$

The sum $s_1 + s_2$ is independent of q_0 and related to the energy of the relative motion:

$$s_1 + s_2 = \frac{\mathbf{k}^2}{2M} + \varepsilon_B; \ \mathbf{k} = \frac{m_2\left(\frac{\mathbf{P}}{2} + \mathbf{q}\right) - m_1\left(\frac{\mathbf{P}}{2} - \mathbf{q}\right)}{m_1 + m_2}; \ \frac{1}{M} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{2.13}$$

The matrix elements in (2.11) are the Schrödinger wave-functions of the relative motion. They do not depend on q_0, because q_0 is fixed by the corresponding mass shell relation. Hence they depend on $s_1 + s_2$ only. Finally the vanishing of the equal time commutator together with (2.8) tells us that both matrixelements can be represented by the same function

$$(2\pi)^3 \left\langle \frac{P}{2} + q \,|\psi_1(0)|\, P \right\rangle = F(s_1 + s_2) = (2\pi)^3 \left\langle \frac{P}{2} - q\,|\psi_2(0)|\, P\right\rangle. \tag{2.14}$$

These statements enable us to write the amplitude in a form that takes into account the restrictions imposed by Galilean invariance, spectral conditions, and equal time commutation relations:

$$\begin{aligned} f(q) = \tilde{f}(s_1, s_2) &= -\frac{i}{(2\pi)^4}\left\{\frac{1}{s_1 - i\varepsilon} + \frac{1}{s_2 - i\varepsilon}\right\} F(s_1 + s_2) \\ &= -\frac{i}{(2\pi)^4} \frac{1}{(s_1 - i\varepsilon)(s_2 - i\varepsilon)} \Gamma(s_1 + s_2). \end{aligned} \tag{2.15}$$

The function

$$\Gamma(s_1 + s_2) = (s_1 + s_2)\, F(s_1 + s_2) \tag{2.16}$$

is the nonrelativistic vertex function.

The properties of the vertex function depend on the dynamics of the system under discussion. For comparison with the relativistic case it is convenient to set up the dynamical problem in terms of a Bethe-Salpeter (B-S) equation. The B-S equation with a local two-particle potential $V(\mathbf{x}_1 - \mathbf{x}_2)$ reads ($\partial_{10} = \partial/\partial x_{10}$ etc.)

$$\begin{aligned} &\left(i\partial_{10} + \frac{1}{2m_1}\Delta_1\right)\left(i\partial_{20} + \frac{1}{2m_2}\Delta_2\right)\chi(x_1, x_2) \\ &\qquad = i\, V(\mathbf{x}_1 - \mathbf{x}_2)\, \delta(x_{10} - x_{20})\, \chi(x_1, x_2). \end{aligned} \tag{2.17}$$

Separation of the center of mass (s. (2.5)) leads to

$$\begin{aligned} &\left\{\frac{P_0}{2} + i\partial_0 + \frac{1}{2m_1}\left(i\frac{\mathbf{P}}{2} + \nabla_{\mathbf{y}}\right)^2\right\}\left\{\frac{P_0}{2} - i\partial_0 + \frac{1}{2m_2}\left(i\frac{\mathbf{P}}{2} - \nabla_{\mathbf{y}}\right)^2\right\} \times \\ &\qquad \times \varphi(y) = i\, V(\mathbf{y})\, \delta(y_0)\, \varphi(y). \end{aligned} \tag{2.18}$$

The Fourier transform of (2.18)

$$f(q) = \frac{i}{(2\pi)^4} \times$$

$$\times \frac{1}{\left(\left(\frac{\mathbf{P}}{2} + \mathbf{q}\right)^2 \Big/ 2m_1 - \left(\frac{P_0}{2} + q_0\right) - i\varepsilon\right)\left(\left(\frac{\mathbf{P}}{2} - \mathbf{q}\right)^2 \Big/ 2m_2 - \left(\frac{P_0}{2} - q_0\right) - i\varepsilon\right)} \times$$

$$\times (2\pi)^3 \int d^4q' \, V(\mathbf{q} - \mathbf{q}') \, f(q') \qquad (2.19)$$

has the structure required by (2.15). If we introduce the „Ansatz" (2.15) into (2.19), we can perform the q_0 integration to obtain SCHRÖDINGER's equation in momentum space for the wavefunction $F(s_1 + s_2) \to F(k^2)$:

$$F(\mathbf{k}^2) = -\frac{1}{\frac{\mathbf{k}^2}{2M} + \varepsilon_B} \int d\tau(\mathbf{k}') \, V(\mathbf{k} - \mathbf{k}') \, F(\mathbf{k}'^2) \,. \qquad (2.20)$$

The potential in closest agreement with the relativistic B-S equation, we shall study later on, is the Yukawa potential

$$V(\mathbf{y}) = -\frac{\lambda}{4\pi} \frac{e^{-\mu|\mathbf{y}|}}{|\mathbf{y}|}\,; \quad V(\mathbf{k}) = -\frac{\lambda}{(2\pi)^3} \frac{1}{\mathbf{k}^2 + \mu^2}\,, \qquad (2.21)$$

where λ and μ are parameters. The equation

$$F(\mathbf{k}^2) = \frac{\lambda^1}{k^2 + \alpha^2} \int d\tau(\mathbf{k}') \frac{1}{(\mathbf{k} - \mathbf{k}')^2 + \mu^2} F(\mathbf{k}'^2),$$

$$\lambda' = \frac{\lambda}{(2\pi)^3} 2M \qquad \alpha^2 = 2M\varepsilon_B \qquad (2.22)$$

has been solved by WANDERS [2] and BLANKENBECLER and COOK [3] by means of a Stieltjes transformation that displays the analytic properties of the wave function $F(k^2)$. For comparison with the solution of the relativistic problem we briefly outline this method in a form convenient for our purposes.

We consider (2.22) as an integral equation for the nonrelativistic vertex function

$$\Gamma(k^2) = (k^2 + \alpha^2) \, F(k^2) \,. \qquad (2.23)$$

Introducing the spectral representation (Stieltjes transform)

$$\Gamma(k^2) = \int^{\infty} ds' \frac{\varrho(s')}{s' + k^2} \qquad (2.24)$$

we obtain

$$\int^{\infty} ds \frac{\varrho(s)}{s + k^2} = \lambda' \int^{\infty} ds' \frac{\varrho(s')}{s' - \alpha^2} \int d\tau(\mathbf{k}') \frac{1}{(\mathbf{k} - \mathbf{k}')^2 + \mu^2} \times$$

$$\times \left(\frac{1}{k'^2 + \alpha^2} - \frac{1}{k'^2 + s'}\right). \qquad (2.25)$$

The integrals over k space occurring in (2.25) are of the same type and can be simplified by symmetrical integration, e.g.

$$I = \int d\tau(\mathbf{k}') \frac{1}{(\mathbf{k}-\mathbf{k}')^2 + \mu^2} \frac{1}{k'^2+\alpha^2} = 2\pi \int_0^\infty dk' \int_0^1 dz \times \times \frac{1}{k'^2 + z(1-z)k^2 + z\mu^2 + (1-z)\alpha^2} . \tag{2.26}$$

The integral over z can be dispersed with the standard formula (s. [4]):

$$\int_0^1 dz \frac{1}{z(1-z)k^2 + z(\mu^2+k'^2) + (1-z)(\alpha^2+k'^2)} = 2 \int_{(\sqrt{\mu^2+k'^2}+\sqrt{\alpha^2+k'^2})^2}^\infty \frac{ds'}{\sqrt{\lambda(s',\mu^2+k'^2,\alpha^2+k'^2)}} \frac{1}{s'+k^2}, \tag{2.27}$$

where

$$\lambda(a,b,c) = a^2+b^2+c^2-2ab-2ac-2bc . \tag{2.28}$$

Finally we substitute (2.27) into (2.26), interchange the order of integrations, and evaluate the k' integral. This yields the result of BLANKENBECLER and NAMBU [5]

$$I = \pi^2 \int_{(\mu+\alpha)^2}^\infty \frac{ds'}{\sqrt{s'}} \frac{1}{s'+k^2} . \tag{2.29}$$

Using (2.29) we may derive from (2.25) an integral equation for $\varrho(s)$,

$$\varrho(s) = \lambda' \Gamma(-\alpha^2)\, \varrho^{(1)}(s)\, \theta(s-(\mu+\alpha)^2) - \lambda' \varrho^{(1)}(s) \times \times \int_{(\mu+\alpha)^2}^\infty ds' \frac{\varrho(s')}{s'-\alpha^2} \theta(s-(\mu+\sqrt{s'})^2) \tag{2.30}$$

with

$$\varrho^{(1)}(s) = \frac{\pi^2}{\sqrt{s}} .$$

Equation (2.30) may be considered as an inhomogeneous integral equation for the function $\varrho(s)/(s-\alpha^2)$, which can be solved by inversion

$$\frac{\varrho(s)}{s-\alpha^2} = \lambda' \Gamma(-\alpha^2) \frac{\varrho^{(1)}(s)}{s-\alpha^2} \theta(s-(\mu+\alpha)^2) + + \lambda' \int ds' R(s,s')\, \lambda' \Gamma(-\alpha^2) \frac{\varrho^{(1)}(s')}{s'-\alpha^2} \theta(s'-(\mu+\alpha)^2) . \tag{2.31}$$

It is shown in Appendix A that the resolvent $R(s,s')$ is bounded by

$$|R(s,s')| < \frac{\varrho^{(1)}(s)}{s-\alpha^2} \exp\left\{ |\lambda'| \int_{(\mu+\sqrt{s'})^2}^\infty dx \frac{\varrho^{(1)}(x)}{x-\alpha^2} \right\} . \tag{2.32}$$

It is, therefore, an entire function of λ' and can be expressed by the Neumann series

$$R(s, s') = \sum_{n=1}^{\infty} \lambda'^{n-1} K^{(n)}(s', s) \quad \text{where} \quad K(s, s') = - \\ - \frac{\varrho^{(1)}(s)}{s - \alpha^2} \theta(s - (\mu + \sqrt{s'})^2) . \tag{2.33}$$

Observing that

$$\frac{\varrho^{(1)}(s)}{s - \alpha^2} \theta(s - (\mu + \alpha)^2) = -K(s, -\alpha^2) \tag{2.34}$$

we can write the solution (2.31) in a more compact form

$$\frac{\varrho(s)}{s - \alpha^2} = -\lambda' \Gamma(-\alpha^2) R(s, -\alpha^2) . \tag{2.35}$$

Integration of (2.35) yields

$$\Gamma(-\alpha^2) (1 + \lambda' \int^{\infty} ds \, R(s, -\alpha^2)) = 0 , \tag{2.36}$$

because of (2.24).

It is important to realize that the function

$$f_0(-i\alpha) = 1 + \lambda' \int^{\infty} ds \, R(s, -\alpha^2) \tag{2.37}$$

is the Jost function $f_0(k)$ with argument $k = -i\alpha$. As is well known (s. e.g. [6]), $f_0(k)$ is analytic in the complex k plane cut along the positive imaginary axis from $k = \frac{1}{2} i \mu$ to infinity in the case of a Yukawa potential (2.21). This is just what the representation (2.37) says. Hence (2.36) is identical with the statement that a bound state is a zero of the Jost function on the negative imaginary axis. As is seen from (2.36) and (2.33) zeros can occur only for an attractive potential ($\lambda' > 0$), because $f_0(-i\alpha) > 0$ if $\lambda' < 0$. According to the general properties of the Jost function the number of zeros is finite [6].

III. General properties of the relativistic amplitude

The simplest matrixelement arising in the field-theoretic description of a bound state due to the interaction of two fields is the two-point amplitude

$$\chi(x_1, x_2) = (2\pi)^{3/2} \langle 0 | T(A_1(x_1) A_2(x_2)) | P \rangle , \tag{3.1}$$

where T is again WICK's chronological operator and x is the four-vector $(x^0, \mathbf{x})$. For the sake of simplicity we assume that the field operators $A_1(x_1)$ and $A_2(x_2)$ are asymptotically related to neutral scalar particles with masses m_1 and m_2, and transform like scalars under the inhomogeneous Lorentz group. They are supposed to commute for spacelike distances

$$[A_i(x), A_k(x')] = 0, \quad (x - x')^2 < 0, \quad i, k = 1, 2 . \tag{3.2}$$

The mass shell conditions are

$$p_1^2 = p_{10}^2 - \mathbf{p}_1^2 = m_1^2; \quad p_2^2 = p_{20}^2 - \mathbf{p}_2^2 = m_2^2 \tag{3.3}$$

for the basic particles and

$$P^2 = P_0^2 - \mathbf{P}^2 = M^2, \quad M = m_1 + m_2 - \varepsilon_B \tag{3.4}$$

for the bound state with binding energy ε_B and spin zero.

By translation invariance we have

$$\chi(x_1, x_2) = \varphi(x_1 - x_2)\, e^{-iP\cdot\frac{x_1+x_2}{2}}, \tag{3.5}$$

where

$$\varphi(y) = (2\pi)^{3/2}\left\langle 0 \left| T\left(A_1\left(\frac{y}{2}\right) A_2\left(-\frac{y}{2}\right)\right)\right| P\right\rangle. \tag{3.6}$$

We decompose $\varphi(y)$ according to the definition of the T-operator

$$\begin{aligned}\varphi(y) = {} & (2\pi)^{3/2}\left\langle 0 \left| A_1\left(\frac{y}{2}\right) A_2\left(-\frac{y}{2}\right)\right| P\right\rangle \theta(y_0) + \\ & + (2\pi)^{3/2}\left\langle 0 \left| A_2\left(-\frac{y}{2}\right) A_1\left(\frac{y}{2}\right)\right| P\right\rangle \theta(-y_0).\end{aligned} \tag{3.7}$$

To introduce the spectral conditions we expand the matrixelements in (3.7) in a complete set of physical states:

$$\begin{aligned}(2\pi)^{3/2}\left\langle 0 \left| A_1\left(\frac{y}{2}\right) A_2\left(-\frac{y}{2}\right)\right| P\right\rangle = {} & \int d^4q\, e^{-iqy}\Big\{\delta(q_+^2 + m_1^2)\,\theta(q_{+0}) \times \\ & \times \langle q_+|A_2(0)|P\rangle + \int_{C_1} ds\, \delta(q_+^2 + s)\,\theta(q_{+0})\,\langle 0|A_1(0)|q_+\rangle \times \\ & (2\pi)^{3/2}\,\langle q_+|A_2(0)|P\rangle\Big\} \\ (2\pi)^{3/2}\left\langle 0 \left| A_2\left(-\frac{y}{2}\right) A_1\left(\frac{y}{2}\right)\right| P\right\rangle = {} & \int d^4q\, e^{-iqy}\Big\{\delta(q_-^2 + m_2^2)\,\theta(q_{-0}) \times \\ & \times \langle q_-|A_1(0)|P\rangle + \int_{C_2} dt\,\delta(q_-^2 + t)\,\theta(q_{-0})\,\langle 0|A_2(0)|q_-\rangle \times \\ & \times (2\pi)^{3/2}\,\langle q_-|A_1(0)|P\rangle\Big\},\end{aligned} \tag{3.8}$$

where

$$q_\pm = \frac{P}{2} \pm q. \tag{3.9}$$

We have used the normalization

$$(2\pi)^{3/2}\langle 0|A_i(0)|p_i\rangle = 1, \quad i = 1, 2 \tag{3.10}$$

and have given the one particle contributions with masses m_1, m_2 and the continuum contributions explicitly for comparison with the non-relativistic expansion (2.8).

The Lorentz invariance of the amplitude

$$\varphi(y) = \frac{1}{2}(2\pi)^{3/2}\left\langle 0\left|\left[A_1\left(\frac{y}{2}\right), A_2\left(-\frac{y}{2}\right)\right]_+\right|P\right\rangle + \\ + \frac{1}{2}\varepsilon(y_0)\,(2\pi)^{3/2}\left\langle 0\left|\left[A_1\left(\frac{y}{2}\right), A_2\left(-\frac{y}{2}\right)\right]\right|P\right\rangle \tag{3.11}$$

depends critically on the condition of locality (3.2). To secure the causal structure of the amplitude we use the Jost-Lehmann-Dyson representation in DYSON's volume form [7] for the matrix element of the comutator

$$(2\pi)^{3/2}\left\langle 0\left|\left[A_1\left(\frac{y}{2}\right), A_2\left(-\frac{y}{2}\right)\right]\right|P\right\rangle \\ = \int d^4q\, e^{-iqy}\int_0^\infty d\lambda^2\int d^4u\,\varepsilon(q_0 - u_0)\,\delta((q-u)^2 - \lambda^2)\,\sigma(\lambda^2, u)\,. \tag{3.12}$$

The support of $\sigma(\lambda^2, u)$ is restricted to the region

$$u_+ \in V_+,\ u_- \in V_+\ \ \lambda^2 \geqq \operatorname{Max}\{0, m_1 - \\ - \sqrt{u_+^2}, m_2 - \sqrt{u_-^2}\}\ \text{ where }\ u_\pm = \frac{P}{2} \pm u, \tag{3.13}$$

and V_+ is the forward light cone. (We assume $M > |m_1 - m_2|$). Observing (3.13) we can decompose (3.12) into its positive and negative frequency part.

$$(2\pi)^{3/2}\left\langle 0\left|A_1\left(\frac{y}{2}\right)A_2\left(-\frac{y}{2}\right)\right|P\right\rangle \\ = \int d^4q\, e^{-iqy}\int_0^\infty d\lambda^2\int d^4u\,\theta(q_0 - u_0)\,\delta((q-u)^2 - \lambda^2)\,\sigma(\lambda^2, u) \\ (2\pi)^{3/2}\left\langle 0\left|A_2\left(-\frac{y}{2}\right)A_1\left(\frac{y}{2}\right)\right|P\right\rangle \\ = \int d^4q\, e^{-iqy}\int_0^\infty d\lambda^2\int d^4u\,\theta(u - q_0)\,\delta((q-u)^2 - \lambda^2)\,\sigma(\lambda^2, u)\,. \tag{3.14}$$

Ignoring the question of subtractions required by the possibly singular behaviour of the matrix element (3.12) at $y = 0$ we obtain the following representation of $\varphi(y)$

$$\varphi(y) = \int d^4q\, e^{-iqy}\int_0^\infty d\lambda^2\int d^4u\,\frac{\sigma(\lambda^2, u)}{(q-u) - \lambda^2 + i\varepsilon}\,. \tag{3.15}$$

By Lorentz invariance the Fourier transform

$$\hat{\varphi}(q) = \frac{1}{(2\pi)^4}\int d^4y\, e^{iqy}\varphi(y) = \int_0^\infty d\lambda^2\int d^4u\,\frac{\sigma(\lambda^2, u)}{(q-u)^2 - \lambda^2 + i\varepsilon} \\ = f(q_+^2, q_-^2) \tag{3.16}$$

depends only on the scalar invariants q_+^2, q_-^2 and P^2, where the latter is to be considered as fixed.

A more refined representation exhibiting the one particle singularities of $f(q_+^2, q_-^2)$ is obtained from the Dyson representation of the vertex function $\tilde{\Gamma}$ [1]:

$$\frac{i}{(2\pi)^4}\tilde{\Gamma}(q_+^2, q_-^2) = (q_+^2 - m_1^2)(q_-^2 - m_2^2) f(q_+^2, q_-^2)$$
$$= P(q_+^2, q_-^2) + \int_0^\infty d\lambda^2 \int d^4u \frac{\varrho(\lambda^2, u)}{(q-u)^2 - \lambda^2 + i\varepsilon}, \tag{3.17}$$

where

$$P(q_+^2, q_-^2) = \frac{1}{(2\pi)^4}\int d^4y\, e^{iqy}\,\delta(y_0)\,(2\pi)^{3/2}\left\langle 0\left|\left[\partial_0 A_1\left(\frac{y}{2}\right), j_2\left(-\frac{y}{2}\right)\right]\right|P\right\rangle,$$

$$\int_0^\infty d\lambda^2 \int d^4u \frac{\varrho(\lambda^2, u)}{(q-u)^2 - \lambda^2 + i\varepsilon} \tag{3.18}$$
$$= \frac{1}{(2\pi)^4}\int d^4y\, e^{iqy}\,(2\pi)^{3/2}\left\langle 0\left| T\left(j_1\left(\frac{y}{2}\right) j_2\left(-\frac{y}{2}\right)\right)\right|P\right\rangle,$$

and

$$j_i(x) = (\square + m_i^2)\,A_i(x); \quad i = 1, 2\,. \tag{3.19}$$

Hence the support of $\varrho(\lambda^2, u)$ is given by (3.13) with m_1 and m_2 replaced by the least masses of the continua C_1 and C_2. The structure of the polynomial $P(q_+^2, q_-^2)$ depends on the dynamics. (We have assumed that fields and currents commute for equal times.)

Next we introduce the quantities

$$\Gamma_0 = \tilde{\Gamma}(m_1, m_2) = \Gamma(m_1^2, m_2^2)$$
$$F_1(q_+^2) = \frac{\tilde{\Gamma}(q_+^2, m_2^2) - \tilde{\Gamma}(m_1^2, m_2^2)}{q_+^2 - m_1^2}, \quad F_2(q_-^2) = \frac{\tilde{\Gamma}(m_1^2, q_-^2) - \tilde{\Gamma}(m_1, m_2^2)}{q_-^2 - m_2^2} \tag{3.20}$$

and decompose $f(q_+^2, q_-^2)$ into a one-particle singular part and a regular part $f_R(q_+^2, q_-^2)$

$$f(q_+^2, q_-^2) = \frac{i}{(2\pi)^4}\left\{\frac{\Gamma_0}{(q_+^2 - m_1^2)(q_-^2 - m_2^2)} + \frac{F_1(q_+^2)}{q_-^2 - m_2^2} + \frac{F_2(q_-^2)}{q_+^2 - m_1^2}\right\} + f_R(q_+^2, q_-^2)$$

$$f_R(q_+^2, q_-^2) = \frac{i}{(2\pi)^4}\frac{1}{(q_+^2 - m_1^2)(q_-^2 - m_2^2)} \times \tag{3.21}$$
$$\times \{\tilde{\Gamma}(q_+^2, q_-^2) - \tilde{\Gamma}(m_1^2, q_-^2) - \tilde{\Gamma}(q_+^2, m_2^2) + \tilde{\Gamma}(m_1^2, m_2^2)\}\,.$$

[1] $\tilde{\Gamma}$ differs from the field-theoretic vertex function Γ by a factor

$$\Delta'_{1F}(q_+^2)\,\Delta'_{2F}(q_-^2)\,\Delta^{-1}_{1F}(q_+^2)\,\Delta^{-1}_{2F}(q_-^2)\,,$$

where Δ'_F and Δ_F are Feynman's Green's functions for interacting and free fields respectively.

The residues of the singular part are related to physical matrix elements

$$\frac{\Gamma_0}{q_+^2 - m_1^2} + F_1(q_+^2) = \begin{cases} (2\pi)^3 \langle q_- | A_1(0) | P \rangle & q_{-0} > 0 \\ (2\pi)^3 \langle 0 | A_1(0) | P, -q_- \rangle & q_{-0} < 0 \end{cases}$$

$$\frac{\Gamma_0}{q_-^2 - m_2^2} + F_2(q_-^2) = \begin{cases} (2\pi)^3 \langle q_+ | A_2(0) | P \rangle & q_{+0} > 0 \\ (2\pi)^3 \langle 0 | A_2(0) | P, -q_+ \rangle & q_{+0} < 0 \end{cases} \tag{3.22}$$

$$\Gamma_0 = (2\pi)^3 \langle q_- | j_1(0) | P \rangle |_{q_+^2 = m_1^2} = (2\pi)^3 \langle q_+ | j_2(0) | P \rangle |_{q_-^2 = m_2^2} .$$

In cases where single variable dispersion relations can be proved for the vertices with two particles on the mass shell:

$$(2\pi)^3 \langle q_- | j_1(0) | P \rangle = \int_{a_1}^{\infty} ds \frac{\varrho_1(s)}{s - q_+^2} ; \quad (2\pi)^3 \langle q_+ | j_2(0) | P \rangle = \int_{a_2}^{\infty} dt \frac{\varrho_2(t)}{t - q_-^2} , \tag{3.23}$$

the one-particle singular part of $f(q_+^2, q_-^2)$ is local, i.e. the corresponding contribution to the matrixelement of the commutator vanishes for spacelike distances. The singular part may then be considered as a one-particle approximation of $f(q_+^2, q_-^2)$ that is in agreement with locality. As we shall show in the next section, the regular part is completely determined by the singular part in the simple model of the B-S equation in ladder approximation.

IV. The ladder approximation

We now turn to the properties of the two-point amplitude in the ladder approximation. Let us assume that the fields $A_1(x)$ and $A_2(x)$ interact with a neutral scalar field $C(x)$ of mass μ with the property

$$\langle 0 | C(x) | P \rangle = 0 . \tag{4.1}$$

Then the ladder approprimation of the B-S equation reads in configuration space

$$\chi(x_1, x_2) = -g^2 \int d^4x_1' \int d^4x_2' \frac{1}{2} \Delta_F(x_1 - x_1', m_1^2) \times \\ \times \frac{1}{2} \Delta_F(x_2 - x_2'; m_2^2) \frac{1}{2} \Delta_F(x_1' - x_2', \mu^2) \chi(x_1', x_2') , \tag{4.2}$$

or in momentum space

$$f(q_+^2, q_-^2) = \frac{1}{(q_+^2 - m_1^2)(q_-^2 - m_2^2)} \frac{ig^2}{(2\pi)^4} \int d^4q' \frac{f(q_+'^2, q_-'^2)}{(q - q')^2 - \mu^2} , \tag{4.3}$$

where g is the coupling constant, and all masses have small negative imaginary parts. The corresponding equation for the vertex function ($\Gamma = \tilde{\Gamma}$ within the ladder approximation) is

$$\Gamma(q_+^2, q_-^2) = \frac{ig^2}{(2\pi)^4} \int d^4q' \frac{1}{(q - q')^2 - \mu^2} \frac{1}{q_+'^2 - m_1^2} \frac{1}{q_-'^2 - m_2^2} \times \\ \times \Gamma(q_+'^2, q_-'^2) . \tag{4.4}$$

The polynomial $P(q_+^2, q_-^2)$ in (3.17) vanishes due to (4.1). Equation (4.4) is then in agreement with locality, if the integral operator on the r.h.s. of (4.4) reproduces the general form of the Dyson representation

$$\Gamma(q_+^2, q_-^2) = \int_0^\infty d\zeta \int d^4u \frac{\varrho(\zeta, u)}{\zeta + i\varepsilon - (q-u)^2} \cdot \tag{4.5}$$

Actually even the vertical Dyson representation

$$\Gamma(q_+^2, q_-^2) = \int_0^\infty d\zeta \int_{-1}^{+1} dz \frac{\varrho(\zeta, z)}{\zeta + i\varepsilon - \left(q + z\frac{P}{2}\right)^2} \tag{4.6}$$

is compatible with (4.4). It is well known [8] that the vertical representation as proposed by DESER, GILBERT, and SUDARSHAN [9] is not a completely general one, i.e. it does not follow from the general postulates of local field theory alone. Its validity depends on the interaction. In this respect we have a similar situation as in the nonrelativistic case, where the existence of the Stieltjes transform (2.24) is due to the analytic properties of the Yukawa potential.

Because the representation (4.6) seems to be generally valid in perturbation theory [10], it is natural to use it also for the solutions of the B-S equation as has first been suggested by WANDERS [2]. It remains, however, an open question, whether every solution can be represented by (4.6). (But see in this connection the work of IDA and MAKI [11]). We shall not discuss the problem of uniqueness here, but restrict ourselves to solutions of the form (4.6).

Formally (4.5) and (4.6) are related by

$$\tilde{\varrho}(\zeta, u) = \int_{-1}^{+1} dz\, \varrho(\zeta, z)\, \delta\left(u + z\frac{P}{2}\right) . \tag{4.7}$$

While the more general form

$$\tilde{\varrho}(\zeta, u) = \int_{-\infty}^{\infty} dz\, \varrho(\zeta, z)\, \delta\left(u + z\frac{P}{2}\right) \tag{4.8}$$

of DYSON's spectral function follows from the assumption that a Fourier-Bessel transform exists with respect to the variables y^2 and $y \cdot P$ of the commutator matrixelement of the currents [12],

$$(2\pi)^{3/2} \left\langle 0 \left| \left[j_1\left(\frac{y}{2}\right), j_2\left(-\frac{y}{2}\right)\right] \right| P \right\rangle = (2\pi)^4 \int_0^\infty d\zeta \int_{-\infty}^{\infty} dz\, \varrho(\zeta, z)\, \Delta(y^2, \zeta)\, e^{-iz\left(P\cdot\frac{y}{2}\right)}, \tag{4.9}$$

one cannot conclude from (4.8) via the spectral conditions (3.13) that

$$(1+z)\frac{P}{2}\in V_+,\quad (1-z)\frac{P}{2}\in V_+$$
$$\zeta \geqq \mathrm{Max}\left\{0,\ m_1+\mu-(1-z)\frac{M}{2},\ m_2+\mu-(1+z)\frac{M}{2}\right\}, \tag{4.10}$$

and (4.7) holds, because DYSON's volume form (4.5) is not unique. Nevertheless, we shall see that the solutions of (4.4) satisfy the support conditions (4.10).

We now introduce the representation (4.6) into (4.4) and, similarly as in the nonrelativistic case (2.25), split off the contribution from the double pole term of $f(q_+^2, q_-^2)$ (3.21):

$$\int_{-1}^{+1} dz \int d\zeta \frac{\varrho(\zeta,z)}{\zeta-\left(q+z\frac{P}{2}\right)^2}$$
$$= i\lambda \int_{-1}^{+1} dz' \int d\zeta' \frac{\varrho(\zeta',z')}{\zeta'-M^2(z')} \int d^4q' \frac{1}{(q-q')^2-\mu^2}\times \tag{4.11}$$
$$\times\left\{\frac{1}{(q_+'^2-m_1^2)(q_-'^2-m_2^2)}+\left(\frac{\frac{1+z'}{2}}{q_-'^2-m_2^2}+\frac{\frac{1-z'}{2}}{q_+'^2-m_1^2}\right)\frac{1}{\zeta'-\left(q'+z'\frac{P}{2}\right)^2}\right\},$$

where

$$\lambda=\frac{g^2}{(2\pi)^4};\quad M^2(z)=m_1^2\frac{1+z}{2}+m_2^2\frac{1-z}{2}-(1-z^2)\frac{P^2}{4}$$
$$= M^2(m_1^2, m_2^2, z)\,. \tag{4.12}$$

By symmetrical integration we find

$$I = i\int d^4q' \frac{1}{(q-q')^2-\mu^2}\frac{1}{q_+'^2-m_1^2}\frac{1}{q_-'^2-m_2^2}$$
$$=\frac{\pi^2}{2}\int_{-1}^{+1} dz\int_0^1 d\alpha \frac{1-\alpha}{\alpha\mu^2+(1-\alpha)M^2(z)-\alpha(1-\alpha)\left(q+z\frac{P}{2}\right)^2} \tag{4.13}$$

The integral over α can be dispersed as (2.27):

$$I=\frac{\pi^2}{2}\int_{-1}^{+1} dz \int_{(\mu+M(z))^2}^{\infty} d\zeta \frac{\zeta+\mu^2-M^2(z)}{\zeta\sqrt{\lambda(\zeta,\mu^2,M^2(z))}}\frac{1}{\zeta-\left(q+z\frac{P}{2}\right)^2}\,. \tag{4.14}$$

The second term in (4.11) is of the same type, because

$$i\int d^4q' \frac{1}{(q-q')^2-\mu^2}\left(\frac{\frac{1+z'}{2}}{q_-'^2-m_2^2}+\frac{\frac{1-z'}{2}}{q_+'^2-m_1^2}\right)\frac{1}{\zeta'-\left(q'+z'\frac{P}{2}\right)^2}$$

$$=\frac{\pi^2}{2}\int_{-1}^{+1}dz\int_0^1 d\alpha\frac{1-\alpha}{\alpha\mu^2+(1-\alpha)\tilde{M}^2(\zeta',z',z)-\alpha(1-\alpha)\left(q+z\frac{P}{2}\right)} \tag{4.15}$$

where

$$\tilde{M}^2(\zeta',z',z)=\begin{cases} M_+^2(\zeta',z',z)=m_1^2\left(\frac{1+z}{2}\right)+ \\ +\frac{\zeta'+(1-z'^2)\frac{P^2}{4}-m_1^2\left(\frac{1+z'}{2}\right)}{\frac{1-z'}{2}}\left(\frac{1-z}{2}\right)-(1-z^2)\frac{P^2}{4}, \quad z>z' \\ M_-^2(\zeta',z',z)=\frac{\zeta'+(1-z'^2)\frac{P^2}{2}-m_2^2\left(\frac{1-z'}{2}\right)}{\frac{1+z'}{2}}\left(\frac{1+z}{2}\right)+ \\ +m_2^2\left(\frac{1-z}{2}\right)-(1-z^2)\frac{P^2}{4}, \quad z<z'. \end{cases} \tag{4.16}$$

Using (4.14) we finally obtain the following equation for the spectral function $\varrho(\zeta, z)$:

$$\varrho(\zeta,z)=\lambda\Gamma(m_1^2,m_2^2)\,\varrho^{(1)}(\zeta,z)\,\theta\big(\zeta-(\mu+M(z))^2\big)-$$
$$-\lambda\int_{-1}^{+1}dz'\int d\zeta'\frac{\varrho(\zeta',z')}{\zeta'-M^2(z')}\varrho^{(1)}(\zeta,\tilde{M}^2(\zeta',z',z))\,\theta\big(\zeta-(\mu+\tilde{M}(\zeta',z',z))^2\big), \tag{4.17}$$

where

$$\varrho^{(1)}(\zeta,M^2)=\frac{\pi^2}{2}\frac{\zeta+\mu^2-M^2}{\zeta\sqrt{\lambda(\zeta,\mu^2,M^2)}}. \tag{4.18}$$

(4.17) can be considered as an inhomogeneous integral equation for $\varrho(\zeta, z)$, but in contrast to the nonrelativistic equation it is not of the Volterra type. To see this, we determine the region, where the iterated terms are different from zero. The support of the inhomogeneous term is bounded by the curve

$$\zeta=W_0(z)=(\mu+M(z))^2 \tag{4.19}$$

and for the n-th term we have

$$W_n(z)=\mathrm{Min}\begin{Bmatrix}\mathrm{Min}_{-1\leq z'\leq z}\big(\mu+M_+(W_{n-1}(z'),z',z)\big)^2, \\ \mathrm{Min}_{z\leq z'\leq 1}\big(\mu+M_-(W_{n-1}(z'),z',z)\big)^2\end{Bmatrix} n=1,2,\dots. \tag{4.20}$$

The details of this minimization problem are given in Appendix B, the result is sketched in Fig. 1. As long as anomalous thresholds exist, the support decreases with increasing order of iteration. This is in complete analogy to the nonrelativistic case, where only anomalous thresholds occur. But in the relativistic case the anomalous thresholds pass the normal thresholds after a finite number N of iterations, and the support of all higher order terms is bounded by the same curve $W_N(z)$, equ. (B 7), (B 8) of Appendix B.

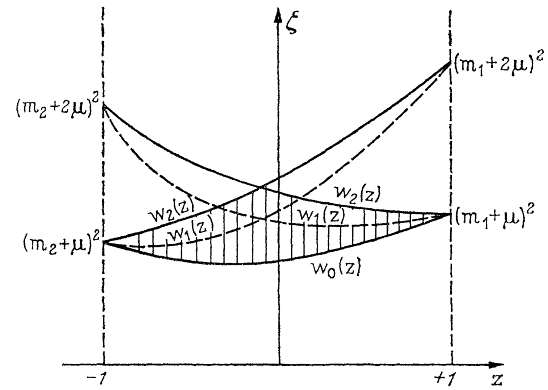

Fig. 1. Support of iterated terms of the integral equation (4.17) ($N = 2$)

The preceding discussion makes clear, in which sense the singularities of the vertex function $\Gamma(q_+^2, q_-^2)$ in the ladder approcimation are "majorized" by the lowest order term. From (4.6) and (4.17) we have

$$\Gamma^{(1)}(q_+^2, q_-^2) = \lambda \Gamma(m_1^2, m_2^2) \int_{-1}^{+1} dz \int_{(\mu + M(z))^2}^{\infty} d\zeta \times \tag{4.21}$$
$$\times \frac{\varrho^{(1)}(\zeta, M^2(z))}{\zeta - q_+^2\left(\frac{1+z}{2}\right) - q_-^2\left(\frac{1-z}{2}\right) + (1-z^2)\frac{P^2}{4}}.$$

The singularities of $\Gamma^{(1)}(q_+^2, q_-^2)$, considered as a function of two complex variables q_+^2, q_-^2, are easily derived from (4.21). The normal thresholds

$$q_+^2 = (\mu + m_1)^2, \quad q_-^2 = (\mu + m_2)^2 \tag{4.22}$$

result from endpoint singularities of the z-integration at $z = \pm 1$ respectively, while the singular manifold [13]

$$-\Phi(q_+^2, q_-^2, P^2) = \begin{vmatrix} 2\mu^2 & \mu^2 + m_1^2 - q_+^2 & \mu^2 + m_2^2 - q_-^2 \\ \mu^2 + m_1^2 - q_+^2 & 2m_1^2 & m_1^2 + m_2^2 - P^2 \\ \mu + m_2^2 - q_-^2 & m_1^2 + m_2^2 - P^2 & 2m_2^2 \end{vmatrix} = 0 \tag{4.23}$$

is due to a double root of

$$\begin{aligned}(\mu + M(z))^2 - q_+^2 \left(\frac{1+z}{2}\right) - q_-^2 \left(\frac{1-z}{2}\right) + (1-z^2)\frac{P^2}{4} = \\ = \mu^2 + 2\mu M(z) - (q_+^2 - m_1^2)\left(\frac{1+z}{2}\right) - (q_-^2 - m_2^2)\left(\frac{1-z}{2}\right) = 0\,.\end{aligned} \tag{4.24}$$

If one of the variables q_+^2, q_-^2 is on the mass shell, (4.23) gives the anomalous thresholds. This agrees with eq. (B 2) of appendix B, because $\Phi(q_+^2, q_-^2, P^2)$ is proportional to the discriminant of the quadratic equation equivalent with (4.24).

A more consistent approach to the solution of the B-S equation (4.4) is suggested by the general structure (3.21) of the amplitude $f(q_+^2, q_-^2)$. If we split the righthand side of (4.4) into the contributions from the complete one-particle singular part and the regular part of $f(q_+^2, q_-^2)$, instead of separating off only the double pole singularity as in (4.11), we are led to the following integral equation for the spectral function $\varrho(\zeta, z)$:

$$\begin{aligned}\varrho(\zeta, z) &= \lambda\Gamma(m_1^2, m_2^2)\,\varrho^{(1)}(\zeta, M^2(z))\,\theta\big(\zeta - (\mu + M(z))^2\big) - \\ &- \lambda\int_{-1}^{+1} dz' \int d\zeta' \frac{\varrho(\zeta', z')}{\zeta' - M^2(z')}\,\varrho^{(1)}(\zeta, M_+^2(\zeta', z', z)) \times \\ &\qquad\qquad \times \theta\big(\zeta - (\mu + M_+(\zeta', z', z))^2\big) - \\ &- \lambda\int_{-1}^{+1} dz' \int d\zeta' \frac{\varrho(\zeta', z')}{\zeta' - M^2(z')}\,\varrho^{(1)}(\zeta, M_-^2(\zeta', z', z)) \times \\ &\qquad\qquad \times \theta\big(\zeta - (\mu + M_-(\zeta', z', z))^2\big) + \\ &+ \lambda\int_{-1}^{+1} dz' \int d\zeta' \frac{\varrho(\zeta', z')}{\zeta' - M^2(z')}\,\varrho^{(1)}(\zeta, \overline{M}^2(\zeta', z', z)) \times \\ &\qquad\qquad \times \theta\big(\zeta - (\mu + \overline{M}(\zeta', z', z))^2\big)\,,\end{aligned} \tag{4.25}$$

where

$$\overline{M}^2(\zeta', z', z) = \begin{cases} M_+^2(\zeta', z', z) & z' > z \\ M_-^2(\zeta', z', z) & z' < z\,, \end{cases} \tag{4.26}$$

in contrast to the definition (4.16) of $\tilde{M}^2$. The first three terms result from the one-particle singular part, while the last term is due to the regular part of $f(q_+^2, q_-^2)$. (4.25) can also be derived from (4.17) by adding and subtracting the last term of (4.25). Eq. (4.25) assumes a more transparent form, if we use

$$\begin{aligned} s_1 &= \frac{\zeta' + (1 - z'^2)\dfrac{P^2}{4} - m_2^2\left(\dfrac{1-z'}{2}\right)}{\dfrac{1+z'}{2}}\,, \\ s_2 &= \frac{\zeta' + (1 - z'^2)\dfrac{P^2}{4} - m_1^2\left(\dfrac{1+z'}{2}\right)}{\dfrac{1-z'}{2}} \end{aligned} \tag{4.27}$$

as integration variables for the second and the third term respectively and introduce the quantities (see (4.12))

$$\varrho_1(s_1) = \int_{-1}^{+1} dz\, \varrho(M^2(s_1, m_2^2, z), z), \quad \varrho_2(s_2) = \int_{-1}^{+1} dz\, \varrho(M^2(m_1^2, s_2, z), z). \quad (4.28)$$

This yields

$$\begin{aligned} \varrho(\zeta, z) = {} & \lambda \Gamma(m_1^2, m_2^2)\, \varrho^{(1)}(\zeta, M^2(m_1^2, m_2^2, z))\, \theta\big(\zeta - (\mu + M(m_1^2, m_2^2, z))^2\big) - \\ & - \lambda \int_{a_2^{(1)}}^{\infty} ds_1 \frac{\varrho_1(s_1)}{s_1 - m_1^2}\, \varrho^{(1)}(\zeta, M^2(s_1, m_2^2, z))\, \theta\big(\zeta - (\mu + M(s_1, m_2^2, z))^2\big) - \\ & - \lambda \int_{a_2^{(1)}}^{\infty} ds_2 \frac{\varrho_2(s_2)}{s_2 - m_2^2}\, \varrho^{(1)}(\zeta, M^2(m_1^2, s_2, z))\, \theta\big(\zeta - (\mu + M(m_1^2, s_2, z))^2\big) + \quad (4.29) \\ & + \lambda \int_{-1}^{+1} dz' \int d\zeta' \frac{\varrho(\zeta', z')}{\zeta' - M^2(z')}\, \varrho^{(1)}(\zeta, \overline{M}^2(\zeta', z', z)) \times \\ & \times \theta\big(\zeta - (\mu + \overline{M}(\zeta', z', z))^2\big). \end{aligned}$$

It is shown in Appendix B that the minima of s_1, s_2 are tantamount to the anomalous thresholds $a_1^{(1)}$, $a_1^{(2)}$ respectively. The meaning of $\varrho_1(s_1)$ and $\varrho_2(s_2)$ becomes clear from the representation (4.5) with one of the variables q_+^2, q_-^2 on the mass shell,

$$\Gamma(q_+^2, m_2^2) = \int_{a_1^{(1)}}^{\infty} ds_1 \frac{\varrho_1(s_1)}{s_1 - q_+^2}\,; \quad \Gamma(m_1^2, q_-^2) = \int_{a_2^{(1)}}^{\infty} ds_2 \frac{\varrho_2(s_2)}{s_2 - q_-^2}\,. \quad (4.30)$$

Hence, the coupling constant $\Gamma(m_1^2, m_2^2)$ of the bound particle can also be expressed in terms of $\varrho_1(s_1)$ or $\varrho_2(s_2)$.

Eq. (4.29) should be considered as the relativistic analogue of (2.30). It is, in fact, an integral equation of Volterra type, and the resolvent (solving for $\varrho(\zeta, z)/(\zeta - M^2(z))$

$$\begin{aligned} R(\zeta, z|\zeta', z') &= \sum_{n=1}^{\infty} \lambda^{n-1} K^{(n)}(\zeta, z|\zeta', z') \\ K(\zeta, z|\zeta', z') &= \frac{\varrho^{(1)}(\zeta, \overline{M}^2(\zeta', z', z))}{\zeta - M^2(z)}\, \theta\big(\zeta - (\mu + \overline{M}(\zeta', z', z))^2\big) \end{aligned} \quad (4.31)$$

is an entire function of λ. Fig. 2 shows the relevant domains of dependence for the kernels of (4.17) and (4.29). The convergence of the Neumann series (4.31) is proved in Appendix C.

As in the nonrelativistic equation ((2.31), (2.34)) we can express the inhomogeneous terms of (4.29) by the kernel (4.31),

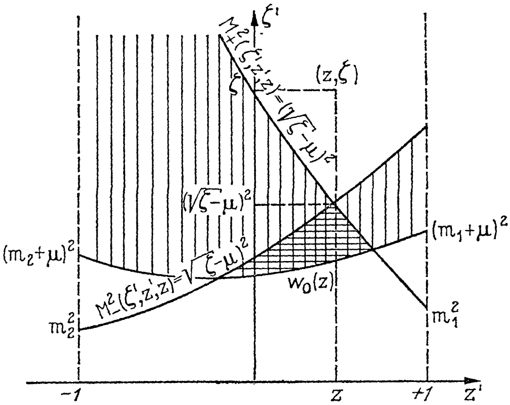

Fig. 2. Domain of dependence for the kernels of the integral equations (4.17) (hatched) and (4.29) (cross-hatched)

$$\begin{aligned}
&\frac{\varrho^{(1)}(\zeta, M^2(m_1^2, m_2^2, z))}{\zeta - M^2(z)}\,\theta\big(\zeta - (\mu + M(m_1^2, m_2^2, z))^2\big) \\
&\quad = K(\zeta, z\,|\,M^2(m_1^2, m_2^2, z'), z')|_{z'=\pm 1} \\
&\frac{\varrho^{(1)}(\zeta, M^2(s_1, m_2^2, z))}{\zeta - M^2(z)}\,\theta\big(\zeta - (\mu + M(s_1, m_2^2, z))^2\big) \\
&\quad = K(\zeta, z\,|\,M^2(s_1, m_2^2, z'), z')|_{z'=-1} \\
&\frac{\varrho^{(1)}(\zeta, M^2(m_1^2, s_2, z))}{\zeta - M^2(z)}\,\theta\big(\zeta - (\mu + M(m_1^2, s_2, z))^2\big) \\
&\quad = K(\zeta, z\,|\,M^2(m_1^2, s_2, z'), z')|_{z'=+1}\,,
\end{aligned} \tag{4.32}$$

where the kernel in the first line is actually independent of z'. Hence

$$\begin{aligned}
\varrho(\zeta, z) = {} & \lambda\Gamma(m_1^2, m_2^2)\, R(\zeta, z\,|\,M^2(m_1^2, m_2^2, z'), z')|_{z'=\pm 1} - \\
& - \lambda \int_{a_1^{(1)}}^{\infty} ds_1 \frac{\varrho_1(s_1)}{s_1 - m_1^2}\, R(\zeta, z\,|\,M^2(s_1, m_2^2, z'), z')|_{z'=-1} - \\
& - \lambda \int_{a_1^{(2)}}^{\infty} ds_2 \frac{\varrho_2(s_2)}{s_2 - m_2^2}\, R(\zeta, z\,|\,M^2(m_1^2, s_2, z'), z')|_{z'=+1}\,.
\end{aligned} \tag{4.33}$$

Referring to (4.28) we get a system of coupled integral equations for $\varrho_1(s_1)$ and $\varrho_2(s_2)$ from (4.33):

$$\begin{aligned}
\frac{\varrho_1(s_1)}{s_1 - m_1^2} = {} & \lambda\Gamma(m_1^2, m_2^2)\, K_{11}(s_1, m_2^2) - \lambda \int ds_1'\, K_{11}(s_1, s_1') \frac{\varrho_1(s_1')}{s_1' - m_1^2} - \\
& - \lambda \int ds_2'\, K_{12}(s_1, s_2') \frac{\varrho_2(s_2')}{s_2' - m_2^2}
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
\frac{\varrho_2(s_2)}{s_2 - m_2^2} = {} & \lambda\Gamma(m_1^2, m_2^2)\, K_{22}(s_2, m_2^2) - \lambda \int ds_1'\, K_{21}(s_2, s_1') \frac{\varrho_1(s_1')}{s_1' - m_1^2} - \\
& - \lambda \int ds_2'\, K_{22}(s_2, s_2') \frac{\varrho_2(s_2')}{s_2' - m_2^2}\,,
\end{aligned} \tag{4.35}$$

where e.g.

$$K_{11}(s_1, s_1') = \int_{-1}^{+1} dz \frac{1+z}{2} R(M^2(s_1, m_2^2, z), z \,|\, M^2(s_1', m_2^2, z'), z')|_{z'=-1} \quad (4.36)$$

etc. A similar system of equations for the absorptive parts $\varrho_1(s_1)$ and $\varrho_2(s_2)$ has been obtained by NAKANISHI [14] in the unphysical case $P^2 \leqq 0$. His approach is based on a double spectral representation of $f(q_+^2, q_-^2)$ that is valid only, if $P^2 \leqq 0$. The more general vertical Dyson representation leads to corresponding results in the physical region of eigenvalues ($P^2 > 0$).

The system (4.35) of coupled Fredholm integral equations is the relativistic generalization of (2.36). The much more complicated structure of these equations is of course due to the fact that in configuration space the B-S equation is a fourth order partial differential equation, whereas the nonrelativistic Schrödinger equation is only a second order ordinary differential equation (for fixed angular momentum). Further light is shed on this point by an analysis of the case $P = 0$, where the B-S equation reduces to a fourth order ordinary differential equation. From (4.6) we have with $P = 0$

$$\Gamma(q^2) = \int_0^\infty d\zeta \frac{\varrho(\zeta)}{\zeta - q^2 + i\varepsilon} \,. \quad (4.37)$$

Again we split the amplitude $f(q^2)$ into a one-particle singular part and a regular part $f_R(q^2)$

$$f(q^2) = \frac{1}{q^2 - m_1^2} \frac{\Gamma(m_1^2)}{m_1^2 - m_2^2} + \frac{1}{q_-^2 - m_2^2} \frac{\Gamma(m_2^2)}{m_2^2 - m_1^2} + f_R(q^2) \,. \quad (4.38)$$

We then obtain the following integral equation for the spectral function $\varrho(\zeta)$ from the B-S equation (4.4) with $P = 0$:

$$\begin{aligned}\varrho(\zeta) = &-2\lambda \frac{\Gamma(m_1^2)}{m_1^2 - m_2^2} \int_{m_1^2}^{(\sqrt{\zeta}-\mu)^2} ds_1\, \varrho^{(1)}(\zeta, s_1)\, \theta(\zeta - (\mu + m_1)^2) - \\ &-2\lambda \frac{\Gamma(m_2^2)}{m_2^2 - m_1^2} \int_{m_2^2}^{(\sqrt{\zeta}-\mu)^2} ds_2\, \varrho^{(1)}(\zeta, s_2)\, \theta(\zeta - (\mu + m_2)^2) + \\ &+ 2\lambda \int d\zeta' \frac{\varrho(\zeta')}{(\zeta' - m^2)(\zeta' - m_2^2)} \int_{\zeta'}^{(\sqrt{\zeta}-\mu)^2} ds\, \varrho^{(1)}(\zeta, s)\, \theta(\zeta - (\mu + \sqrt{\zeta'})^2) \,.\end{aligned} \quad (4.39)$$

This equation is again of Volterra type and can be solved by iteration. Introducing the solution into the representation (4.37) for $q^2 = m_1^2$ and $q^2 = m_2^2$ we are led to a system of two linear equations for the constants $\Gamma(m_1^2)$, $\Gamma(m_2^2)$:

$$\begin{aligned}\Gamma(m_1^2) &= a_{11}\Gamma(m_1^2) + a_{12}\Gamma(m_2^2) \\ \Gamma(m_2^2) &= a_{21}\Gamma(m_1^2) + a_{22}\Gamma(m_2^2) \,,\end{aligned} \quad (4.40)$$

which is the analogue of (4.35). Each of the elements a_{ik} is given as a power series in λ. The eigenvalues λ must satisfy

$$\Delta_0(\lambda) = \begin{vmatrix} a_{11}(\lambda) - 1 & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) - 1 \end{vmatrix} = 0 . \tag{4.41}$$

The characteristic determinant $\Delta_0(\lambda)$ is the generalization of the Jost function to a fourth order differential equation. In the general case $P^2 > 0$ finally, the Jost function blows up to the Fredholm determinant $\Delta(\lambda, P^2)$ of the system (4.35) and the eigenvalues of the B-S equation (4.4) are obtained by solving

$$\Delta(\lambda, P^2) = 0 \tag{4.42}$$

for λ or P^2. Every element of $\Delta(\lambda, P^2)$ is expanded into powers of λ as the nonrelativistic Jost function $f_0(\lambda, k)$.

Our approach does not answer the question whether the eigenvalues in λ or P^2 are real. As in the nonrelativistic case, operator analysis offers more powerful tools for an attack on such questions. We know from the pioneering work of Wick [15] that the eigenvalue problem of the B-S equation (4.4) is equivalent to that of a completely continuous Hermitian integral operator in the equal mass case. This is also true for $P = 0$ in the unequal mass case, but for $P^2 > 0$ the type of the λ-spectrum is still an open question.

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Appendix A. Bound for the nonrelativistic resolvent

We use the following bounds for the iterated kernels of lowest order:

$$\begin{aligned}
|K^{(1)}(s, s')| &= \frac{\varrho^{(1)}(s)}{s-\alpha^2}\,\theta(s-(\mu+\sqrt{s'})^2) \\
|K^{(2)}(s, s')| &= \frac{\varrho^{(1)}(s)}{s-\alpha^2}\int\limits_{(\mu+\sqrt{s'})^2}^{\infty} dx_1\,\frac{\varrho^{(1)}(x_1)}{x_1-\alpha^2}\,\theta(s-(\mu+\sqrt{x_1})^2) \leqq \\
&\leqq \frac{\varrho^{(1)}(s)}{s-\alpha^2}\int\limits_{(\mu+\sqrt{s'})^2}^{\infty} dx_1\,\frac{\varrho^{(1)}(x_1)}{x_1-\alpha^2}\,\theta(s-(\mu+\sqrt{s'})^2) \\
|K^{(3)}(s, s')| &= \frac{\varrho^{(1)}(s)}{s-\alpha^2}\int\limits_{(\mu+\sqrt{s'})^2}^{\infty} dx_1\,\frac{\varrho^{(1)}(x_1)}{x_1-\alpha^2}\int\limits_{(\mu+\sqrt{x_1})^2}^{\infty} dx_2\,\frac{\varrho^{(1)}(x_2)}{x_2-\alpha^2}\,\theta(s-(\mu+\sqrt{x_2})^2) < \\
&< \frac{\varrho^{(1)}(s)}{s-\alpha^2}\int\limits_{(\mu+\sqrt{s'})^2}^{\infty} dx_1\,\frac{\varrho^{(1)}(x_1)}{x_1-\alpha^2}\int\limits_{x_1}^{\infty} dx_2\,\frac{\varrho^{(1)}(x_2)}{x_2-\alpha^2}\,\theta(s-(\mu+\sqrt{s'})^2) \\
&= \frac{\varrho^{(1)}(s)}{s-\alpha^2}\,\frac{1}{2!}\left(\int\limits_{(\mu+\sqrt{s'})^2}^{\infty} dx\,\frac{\varrho^{(1)}(x)}{x-\alpha^2}\right)^2\theta(s-(\mu+\sqrt{s'})^2) .
\end{aligned} \tag{A.1}$$

The generalization to $|K^{(n)}(s, s')|$ is obvious. Hence

$$R(s, s') \leqq \sum_{n=1}^{\infty} |\lambda'|^{n-1} |K^{(n)}(s, s')| < \frac{\varrho^{(1)}(s)}{s-\alpha^2} \exp\left\{|\lambda'| \int\limits_{(\mu+\sqrt{s'})^2}^{\infty} dx \frac{\varrho^{(1)}(x)}{x-\alpha^2}\right\} \times \theta(s-(\mu+\sqrt{s'})^2) . \qquad (A.2)$$

Appendix B. Support of iterated terms

From (4.16) we have

$$M_+^2 (W_0(z'), z', z) = m_1^2 \left(\frac{1+z}{2}\right) + \frac{W_0(z') + (1-z'^2)\frac{P^2}{4} - m_1^2\left(\frac{1+z'}{2}\right)}{\frac{1-z'}{2}} \left(\frac{1-z}{2}\right) - (1-z^2)\frac{P^2}{4}$$

$$M_-^2 (W_0(z'), z', z) = \frac{W_0(z') + (1-z'^2)\frac{P^2}{4} - m_2^2\left(\frac{1-z'}{2}\right)}{\frac{1+z'}{2}} \left(\frac{1+z}{2}\right) + m_2^2\left(\frac{1-z}{2}\right) - (1-z^2)\frac{P^2}{4} . \qquad (B.1)$$

We first minimize the expressions

$$\underset{-1\leqq z'\leqq 1}{\text{Min}} \left(\frac{W_0(z') + (1-z'^2)\frac{P^2}{4} - m_2^2\left(\frac{1-z'}{2}\right)}{\frac{1+z'}{2}}\right) = \underset{-1\leqq z'\leqq 1}{\text{Min}} \left(m_1^2 + \frac{\mu^2 + 2\mu M(z')}{\frac{1+z'}{2}}\right) = a_1^{(1)}$$

$$\underset{-1\leqq z'\leqq 1}{\text{Min}} \left(\frac{W_0(z') + (1-z'^2)\frac{P^2}{4} - m_1^2\left(\frac{1+z'}{2}\right)}{\frac{1-z'}{2}}\right) = \underset{-1\leqq z'\leqq 1}{\text{Min}} \left(m_2^2 + \frac{\mu^2 + 2\mu M(z')}{\frac{1-z'}{2}}\right) = a_2^{(1)} . \qquad (B.2)$$

If absolute minima occur at points $-1 \leqq z_1^{(1)}, z_2^{(1)} \leqq 1$, these are the anomalous thresholds with respect to the variables q_+^2, q_-^2. Otherwise the minima are given by the boundary values at $z = \pm 1$ respectively, i.e. $a_1^{(1)} = (m_1 + \mu)^2$, $a_2^{(1)} = (m_2 + \mu)^2$.

Next we minimize (B.1):

$$\min_{-1\leqq z'\leqq z} M_+^2 (W_0(z'), z', z) = (M_+^{(1)}(z))^2$$

$$= \begin{cases} m_1^2 \left(\frac{1+z}{2}\right) + a_2^{(1)} \left(\frac{1-z}{2}\right) - (1-z^2)\frac{P^2}{4} & z_2^{(1)} \leqq z \leqq 1 \\ (\mu + M(z))^2 & -1 \leqq z \leqq z_2^{(1)} \end{cases}$$

$$\min_{z\leqq z'\leqq 1} M_-^2 (W_0(z'), z', z) = (M_-^{(1)}(z))^2 \tag{B.3}$$

$$= \begin{cases} (\mu + M(z))^2 & z_1^{(1)} \leqq z \leqq 1 \\ a_1^{(1)} \left(\frac{1+z}{2}\right) + m_2^2 \left(\frac{1-z}{2}\right) - (1-z^2)\frac{P^2}{4} & -1 \leqq z \leqq z_1^{(1)} \end{cases}$$

and obtain

$$W_1(z) = \mathrm{Min}\{(\mu + M_+^{(1)}(z))^2, (\mu + M_-^{(1)}(z))^2\}. \tag{B.4}$$

Repeating the procedure we find a sequence of anomalous thresholds

$$\min_{-1\leqq z'\leqq 1} \left(\frac{W_{k-1}(z') + (1-z'^2)\frac{P^2}{4} - m_2^2\left(\frac{1-z'}{2}\right)}{\frac{1+z'}{2}} \right) = a_1^{(k)} > a_1^{(k-1)} > \cdots > a_1^{(1)}$$

$$\min_{-1\leqq z'\leqq 1} \left(\frac{W_{k-1}(z') + (1-z'^2)\frac{P^2}{4} - m_1^2\left(\frac{1+z'}{2}\right)}{\frac{1-z'}{2}} \right) = a_2^{(k)} > a_2^{(k-1)} > \cdots > a_2^{(1)} \tag{B.5}$$

and after a finite number of steps we arrive at

$$a_1^{(N)} = (m_1 + \mu)^2, \quad a_2^{(N)} = (m_2 + \mu)^2. \tag{B.6}$$

This yields

$$\left.\begin{aligned} (M_+^{(N)}(z))^2 &= m_1^2\left(\frac{1+z}{2}\right) + (m_2+\mu)^2\left(\frac{1-z}{2}\right) - (1-z^2)\frac{P^2}{4} \\ (M_-^{(N)}(z))^2 &= (m_1+\mu)^2\left(\frac{1+z}{2}\right) + m_2^2\left(\frac{1-z}{2}\right) - (1-z^2)\frac{P^2}{4} \end{aligned}\right\} -1 \leqq z \leqq 1 \tag{B.7}$$

and

$$W_N(z) = \mathrm{Min}\{(\mu + M_+^{(N)}(z))^2, (\mu + M_-^{(N)}(z))^2\}. \tag{B.8}$$

Hence the support decreases with each step as long as $k \leqq N$ and is bounded by $W_N(z)$ for all iterated terms of order $k > N$ (Fig. 1). It should be noted that anomalous thresholds related to higher normal thresholds, e.g. $(m_1 + 2\mu)^2$, can appear below the normal thresholds (B.6). But they also disappear from the physical sheet after a finite number of iterations.

Appendix C. Bound for the relativistic resolvent

The proof of convergence for the Neumann series of the relativistic resolvent is slightly more involved than in the nonrelativistic case, because the kernel (4.31)

$$K(\zeta, z|\zeta', z') = \frac{\varrho^{(1)}(\zeta, \overline{M}^2(\zeta', z', z))}{\zeta - M^2(z)}\,\theta\big(\zeta - (\mu + \overline{M}(\zeta', z', z))^2\big) \tag{C.1}$$

is unbounded (s. (4.18)). But the second iterated kernel

$$K^{(2)}(\zeta, z|\zeta', z') = \int_{-1}^{+1} dz_1 \int_{(\mu+\overline{M}(\zeta', z', z_1))^2}^{\infty} d\zeta_1 \frac{\varrho^{(1)}(\zeta, \overline{M}^2(\zeta_1, z_1, z))}{\zeta - M^2(z)} \cdot \frac{\varrho^{(1)}(\zeta_1, \overline{M}^2(\zeta', z', z_1))}{\zeta_1 - M^2(z_1)}\,\theta\big(\zeta - (\mu + \overline{M}(\zeta_1, z_1, z))^2\big) \tag{C.2}$$

is bounded. The singularities of the integrand are due to the zeros of (s. (2.28))

$$\begin{aligned} \lambda(\zeta_1, \mu^2, \overline{M}^2(\zeta', z', z_1)) &= (\zeta_1 - (\mu + \overline{M})^2)\,(\zeta_1 - (\mu - \overline{M})^2) \\ \lambda(\zeta, \mu^2, \overline{M}^2(\zeta_1, z_1, z)) &= ((\sqrt{\zeta} + \mu)^2 - \overline{M}^2)\,((\sqrt{\zeta} - \mu)^2 - \overline{M}^2)\,. \end{aligned} \tag{C.3}$$

Since

$$\begin{aligned} &\frac{\zeta_1 + \mu^2 - \overline{M}^2(\zeta', z', z_1)}{\zeta_1 - M^2(z_1)} = 1 + \frac{\mu^2 - (\overline{M}^2 - M^2)}{\zeta_1 - M^2} < 1 + \frac{\mu^2}{\mu^2 + 2\mu\overline{M}} < 2\,, \\ &\frac{\zeta + \mu^2 - \overline{M}^2(\zeta_1, z_1, z)}{\zeta - M^2(z)} < 2\,, \\ &\frac{1}{(\zeta_1 - (\mu - \overline{M})^2)^{1/2}} \leqq \frac{1}{\sqrt{4\mu\overline{M}}}\,;\; \frac{1}{((\sqrt{\zeta} + \mu^2) - \overline{M}^2)^{1/2}} \leqq \frac{1}{\sqrt{4\mu\sqrt{\zeta}}} \end{aligned} \tag{C.4}$$

inside the domain of integration, we may write

$$|K^{(2)}| < \left(\frac{\pi^2}{2}\right)^2 \frac{1}{\zeta} \frac{1}{\sqrt{\mu\sqrt{\zeta}}} \int_{-1}^{+1} dz_1 \frac{1}{\sqrt{\mu\overline{M}(\zeta', z', z_1)}} \times \tag{C.5}$$

$$\times \int_{(\mu+\overline{M}(\zeta', z', z_1))^2}^{\overline{M}^2(\zeta_1, z_1, z) = (\sqrt{\zeta}-\mu)^2} d\zeta_1 \frac{1}{\zeta_1(\zeta_1 - (\mu + \overline{M}(\zeta', z', z_1))^2)^{1/2}} \frac{1}{(-\overline{M}^2(\zeta_1, z_1, z) + (\sqrt{\zeta} - \mu)^2)^{1/2}}\,.$$

Next we evaluate the ζ_1-integral

$$\int_a^b dx \frac{1}{x} \frac{1}{\sqrt{b - x}} \frac{1}{\sqrt{x - a}} = \frac{\pi}{\sqrt{ab}} \tag{C.6}$$

and determine the minimum value of $M(\zeta', z', z_1)$ from (4.26) and (4.16):

$$\operatorname*{Min}_{-1 \leqq z_1 < 1} \overline{M}(\zeta', z', z_1) = \sqrt{\zeta'}\,, \tag{C.7}$$

for large enough values of ζ'. This yields

$$|K^{(2)}| < \left(\frac{\pi^2}{2}\right) \frac{1}{\zeta} \frac{1}{\sqrt{\mu\sqrt{\zeta}}} \frac{1}{\sqrt{\mu\sqrt{\zeta'}}} \frac{1}{(\mu + \sqrt{\zeta'})(\sqrt{\zeta} - \mu)}\,. \tag{C.8}$$

Hence $K^{(2)}$ is a square-integrable kernel.

The third iterated kernel,

$$K^{(3)}(\zeta, z\,|\,\zeta', z') = \int\limits_{-1}^{+1} dz_1 \int\limits_{(\mu+\overline{M}(\zeta',z',z_1))^2}^{\infty} d\zeta_1 \frac{\varrho^{(1)}(\zeta_1, \overline{M}^2(\zeta', z', z_1))}{\zeta_1 - M^2(z_1)} \times \\ \times \int\limits_{-1}^{+1} dz_2 \int\limits_{(\mu+\overline{M}(\zeta_1,z_1,z_2))^2}^{\infty} d\zeta_2 \frac{\varrho^{(1)}(\zeta_2, \overline{M}^2(\zeta_1, z_1, z_2))}{\zeta_2 - M^2(z_2)} \times \\ \times \frac{\varrho^{(1)}(\zeta, \overline{M}^2(\zeta_2, z_2, z))}{\zeta - M^2(z)}\, \theta\big(\zeta - (\mu + \overline{M}(\zeta_2, z_2, z))^2\big) \tag{C.9}$$

can be estimated in a similar way. With

$$\int\limits_a^{\infty} dx \frac{1}{x} \frac{1}{\sqrt{x-a}} = \frac{\pi}{\sqrt{a}}, \tag{C.10}$$

(C.4) and (C.7) we find

$$\left| \int\limits_{-1}^{+1} dz_1 \int\limits_{(\mu+\overline{M}(\zeta',z',z_1))^2}^{\infty} d\zeta_1 \frac{\varrho^{(1)}(\zeta_1, \overline{M}^2(\zeta', z', z_1))}{\zeta_1 - M^2(z_1)} \right| < \frac{\pi^2}{2} \frac{1}{\sqrt{\mu}\sqrt{\zeta'}} \frac{2\pi}{\mu + \sqrt{\zeta'}}. \tag{C.11}$$

The second integral is bounded by (C.8) with $\sqrt{\zeta'}$ replaced by $\sqrt{\zeta'} + \mu$, because

$$\min_{-1 \leqq z_1 \leqq 1;\, \zeta_1} \overline{M}(\zeta_1, z_1, z_2) = \mu + \sqrt{\zeta'}\,. \tag{C.12}$$

Hence

$$|K^{(3)}| < |K^{(2)}|\, \varphi(\mu + \sqrt{\zeta'})\,, \tag{C.13}$$

where

$$\varphi(x) = \frac{\pi^2}{2} \frac{1}{\sqrt{\mu x}} \frac{2\pi}{\mu + x}\,. \tag{C.14}$$

The generalization

$$|K^{(n)}| < |K^{(n-1)}|\, \varphi((n-2)\,\mu + \sqrt{\zeta'}), \quad n = 3, 4, \ldots \tag{C.15}$$

is obvious and proves the convergence of the Neumann series.

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