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Proof of the Bogoliubov-Parasiuk Theorem on Renormalization

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Abstract. A new proof is given that the subtraction rules of BOGOLIUBOV and PARASIUK lead to well-defined renormalized Green's distributions. A collection of the counter-terms in "trees" removes the difficulties with overlapping divergences and allows fairly simple estimates and closed expressions for renormalized Feynman integrals. The renormalization procedure, which also applies to conventionally non-renormalizable theories, is illustrated in the φ^4-theory.

1. Introduction

Renormalization in Lagrangian quantum field theory is in the interpretation of BOGOLIUBOV and PARASIUK [1], [2] the extension of certain linear functionals, defined on a subspace of $\mathcal{S}(R^{4n})$, to tempered distributions in $\mathcal{S}'(R^{4n})$ [3]. For instance, the Gell-Mann Low perturbation expansion of the truncated time-ordered distributions has the form

$$\langle T\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle^T = \sum_{n=m}^{\infty}\frac{(-i)^{n-m}}{(n-m)!}\int dx_{m+1}\ldots dx_n \times \\ \times \langle T\varphi_1^{\mathrm{I}}(x_1)\ldots\varphi_m^{\mathrm{I}}(x_m)\mathcal{H}^{\mathrm{I}}(x_{m+1})\ldots\mathcal{H}^{\mathrm{I}}(x_n)\rangle^T . \tag{1.1}$$

Here the truncated vacuum expectation values $\langle\varphi_1^{\mathrm{I}}(x_1)\ldots\mathcal{H}^{\mathrm{I}}(x_n)\rangle^T$ are well-defined for $\mathcal{H}^{\mathrm{I}}(x)$, which are WICK polynomials of the free fields $\varphi_i^{\mathrm{I}}(x)$. On the other hand the straightforward construction of $\langle T\varphi_1^{\mathrm{I}}(x_1)\ldots\mathcal{H}^{\mathrm{I}}(x_n)\rangle^T$ by WICK's theorem [4] leads to a product of distributions

$$\prod_{l\in\mathcal{L}} \Delta_l^F(x_{i_l}-x_{f_l}) . \tag{1.2}$$

Formula (1.2) is in general not meaningful as one sees from the definition of Δ_l^F in p-space

$$\tilde{\Delta}_l^F(p) = \lim_{\varepsilon\downarrow 0} i\,P_l(p)\,(p^2-m_l^2+i\varepsilon)^{-1} , \tag{1.3}$$

where $P_l(p)$ is a polynomial and where $m_l > 0$ is always assumed. Then the convolutions in p-space corresponding to (1.2) can lead to "ultraviolet divergences".

Nevertheless the product (1.2) taken with regularized [5] propagators is a good starting point for the definition of $\langle T\varphi_1^{\mathrm{I}}(x_1)\ldots\mathcal{H}^{\mathrm{I}}(x_n)\rangle^T$.

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The choice of regulators will turn out to be rather arbitrary, but it is desirable to maintain Lorentz covariance. Using

$$\tilde{\Delta}_l^F(p) = \lim_{\varepsilon\downarrow 0} P_l(p) \int_0^\infty d\alpha \exp[i\alpha(p^2 - m_l^2 + i\varepsilon)] \tag{1.4}$$

we define for $\varepsilon > 0$, $r > 0$

$$\tilde{\Delta}_l^{r,\varepsilon}(p) = P_l(p) \int_r^\infty d\alpha \exp[i\alpha(p^2 - m_l^2 + i\varepsilon)]\,. \tag{1.5}$$

$\tilde{\Delta}_l^{r,\varepsilon}$ belongs to $\mathcal{O}_M(R^4) \cap \mathcal{O}'_C(R^4)$ ([3], vol. 2, p. 101) and Fourier transforms and convolutions of several $\tilde{\Delta}_l^{r,\varepsilon}$ can be obtained (in the sense of distribution theory) by interchange with the α-integration. Evidently $\tilde{\Delta}_l^F(p) - \lim_{\varepsilon\downarrow 0} \tilde{\Delta}_l^{r,\varepsilon}(p)$ converges to zero in $\mathcal{O}_M(R^4)$ for $r \downarrow 0$. One has

$$\Delta_l^{r,\varepsilon}(x) = -\frac{i}{4} P_l\left(i\frac{\partial}{\partial x}\right) \cdot \int_r^\infty \frac{d\alpha}{\alpha^2} \exp\left[-i\alpha(m_l^2 - i\varepsilon) - i\frac{x^2}{4\alpha}\right] \tag{1.6}$$

in $\mathcal{O}_M(R^4) \cap \mathcal{O}'_C(R^4)$ and thus the product $\prod_{l\in\mathcal{L}} \Delta_l^{r,\varepsilon}(x_{i_l} - x_{f_l})$ is well-defined. Furthermore it can be shown ([6]; see sec. 4) that

$$\lim_{\varepsilon\downarrow 0} \lim_{r\downarrow 0} \prod_{l\in\mathcal{L}} \Delta_l^{r,\varepsilon}(x_{i_l} - x_{f_l})$$

is a continuous linear functional on the subspace $\mathcal{S}_N(R^{4n})$ of those test functions $\varphi \in \mathcal{S}(R^{4n})$, which vanish of sufficiently high order N whenever two arguments x_i, x_j, $1 \leqq i < j \leqq n$, coincide.

The renormalization theory of DYSON [7] is in this framework a constructive form of the Hahn-Banach theorem: one subtracts from $\prod \Delta_l^{r,\varepsilon}(x_{i_l} - x_{f_l})$ counter terms which vanish on $\mathcal{S}_N(R^{4n})$, such that the remainder has a limit in $\mathcal{S}'(R^{4n})$ for $r \downarrow 0$ and $\varepsilon \downarrow 0$. The fact that these subtractions can be implemented by formal counter terms in $\mathcal{H}^{\mathrm{I}}(x)$ is an inherently beautiful feature of Lagrangian quantum field theory.

Feynman graphs efficiently organize the combinatorics of the counter terms. We map the n arguments $x_1, \ldots x_n$ of $\prod_{\nu=1}^{L} \Delta_{l_\nu}^{r,\varepsilon}(x_{i_\nu} - x_{f_\nu})$ onto n points $V_1, \ldots V_n$ in a plane, called vertices, and each propagator $\Delta_{l_\nu}^{r,\varepsilon}(x_{i_\nu} - x_{f_\nu})$ on an oriented line l_ν from V_{i_ν} to V_{f_ν}. This gives (up to topological equivalence) the graph $G(V_1, \ldots V_n, \mathcal{L})$ for $\prod \Delta_l^{\varepsilon,r}$, where $\mathcal{L} = \{l_1, \ldots l_L\}$.

Def.: A subset $U = \{V'_1, \ldots V'_m\} \subset \{V_1, \ldots V_n\}$ is called a generalized vertex of $G(V_1, \ldots V_n, \mathcal{L})$.

Def.: Let $U_i = \{V_{i1}, \ldots V_{ir(i)}\}$, $1 \leqq i \leqq m$, be pairwise disjoint generalized vertices of $G(V_1, \ldots V_n, \mathcal{L})$ and $\mathcal{M} \subset \mathcal{L}$. Then the graph $G(U_1, \ldots U_m, \mathcal{M})$ is obtained by representing the sets $U_1, \ldots U_m$ by m points in a plane (again denoted by U_i) and by connecting them by those $l \in \mathscr{M}$, which run between different U_i (i.e. between $V_{i_l} \in U_i$, $V_{f_l} \in U_f$, $U_i \neq U_f$). Sets $\{V_i\}$ are also denoted by V_i.

Example: Let $\mathscr{L} = \{l_1, \ldots l_6\}$ and $G(V_1, \ldots V_4, \mathscr{L})$ be as in Fig. 1. If $U_1 = \{V_1, V_2\}$ and $\mathscr{M} = \{l_1, l_4, l_5, l_6\}$, then $G(U_1, V_3, \mathscr{M})$ becomes Fig. 1.

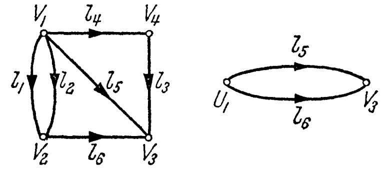

Fig. 1. Graph and generalized subgraph

Def.: $G(U_1, \ldots U_m, \mathscr{M})$ is connected, if $\{U_1, \ldots U_m\}$ is connected by the lines in $\mathscr{M}$. $G(U_1, \ldots U_m, \mathscr{M})$ is 1-particle irreducible (IPI), if for all $l \in \mathscr{M}$ and $\mathscr{M}/l = \mathscr{M} - \{l\}$, $G(U_1, \ldots U_m, \mathscr{M}/l\}$ is connected. Otherwise $G(U_1, \ldots U_m, \mathscr{M})$ is called 1-particle reducible (IPR).

Def.: The superficial divergence $\nu(V_1', \ldots V_m')$ of $G(V_1', \ldots V_m', \mathscr{L})$, $\{V_1', \ldots V_m'\} \subset \{V_1, \ldots V_n\}$, is defined by

$$\nu(V_1', \ldots V_m') = \sum_{\text{conn}} (r_l + 2) + 4(m - 1)\,, \tag{1.7}$$

where $\sum_{\text{conn}}$ extends over all $l \in \mathscr{L}$, which connect two vertices from $\{V_1', \ldots V_m'\}$ and r_l is the degree of the polynomial P_l in (1.3). In the above example one has for a scalar theory ($r_l = 0$):

$$\nu(V_1, V_2) = 0\,,\ \nu(V_1, V_2, V_3) = 0\,,\ \nu(V_1, V_2, V_3, V_4) = 0\,.$$

To each generalized vertex $\{V_1', \ldots V_m'\}$ we define a vertex part $\mathscr{X}_{\mathscr{L}}^{r,\varepsilon}(V_1', \ldots V_m')$ as the following distribution with support in $\{x_1' = \cdots = x_m'\}$:

Def.: Let $\{V_1', \ldots V_m'\} \subset \{V_1, \ldots V_n\}$. Then

$$\mathscr{X}_{\mathscr{L}}^{r,\varepsilon}(V_1', \ldots V_m') = \begin{cases} 1 & , \text{ if } m = 1 \\ 0 & , \text{ for IPR } G(V_1', \ldots V_m', \mathscr{L}) \\ -M\overline{\mathscr{R}}_{\mathscr{L}}^{r,\varepsilon}(V_1', \ldots V_m'), & \text{otherwise}\,, \end{cases} \tag{1.8}$$

$$\overline{\mathscr{R}}_{\mathscr{L}}^{r,\varepsilon}(V_1', \ldots V_m') = \sum_P{}' \prod_{j=1}^{k(P)} \mathscr{X}_{\mathscr{L}}^{r,\varepsilon}(V_{j1}^P, \ldots V_{jr(j)}^P) \prod_{\text{conn}} \Delta^{r,\varepsilon}\,. \tag{1.9}$$

Here $\sum_P{}'$ extends over all partitions $\{\{V_{j1}^P, \ldots V_{jr(j)}^P\},\ 1 \leqq j \leqq k(P)\}$, of $\{V_1', \ldots V_m'\}$ into $1 < k(P) \leqq m$ sets and $\prod_{\text{conn}}$ is taken over all $l \in \mathscr{L}$ which connect different sets of the partition. The operation M maps $\overline{\mathscr{R}}_{\mathscr{L}}^{r,\varepsilon}(V_1', \ldots V_m')$, being in p-space of the form $\delta(p_1' + \cdots + p_m') \times$ $\times F(p_1', \ldots p_m')$ with $F \in \mathscr{O}_M(R^{4m})$, into $\delta(p_1' + \cdots + p_m')\, T(p_1', \ldots p_m')$, where the polynomial T is the Taylor series of F around $p_1' = \cdots p_m' = 0$

up to the order $\nu(V'_1, \ldots V'_m)$, $T = 0$ for $\nu < 0$. One verifies by induction that for $\varepsilon > 0$, $r > 0$ the above definition makes sense. Finally we set

$$\mathcal{R}^{r,\varepsilon}_{\mathcal{L}}(V'_1, \ldots V'_m) = \overline{\mathcal{R}}^{r,\varepsilon}_{\mathcal{L}}(V'_1, \ldots V'_m) + \mathcal{X}^{r,\varepsilon}_{\mathcal{L}}(V'_1, \ldots V'_m)\,. \qquad (1.10)$$

Example: If we denote graphically each $(-M)$-operation by a dotted encircling of the corresponding subgraph, then we obtain in a typical case Fig. 2. We have only listed the non-vanishing terms. One sees the

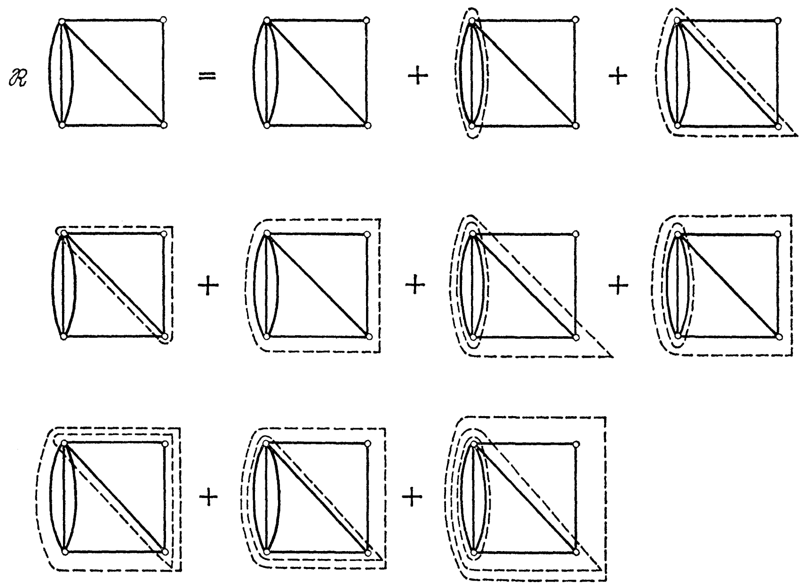

Fig. 2. Counterterms

emergence of subtraction rules, which are similar but not identical with those proposed by SALAM [8].

The main theorem of BOGOLIUBOV and PARASIUK is that for an arbitrary product of Feynman propagators (1.2) $\lim\limits_{\varepsilon\downarrow 0}\lim\limits_{r\downarrow 0}\mathcal{R}^{r,\varepsilon}_{\mathcal{L}}(V_1, \ldots V_n)$ exists as a tempered distribution and defines a Lorentz covariant continuation of (1.2) which can be implemented by (formal) counter terms in $\mathcal{H}^{\mathrm{I}}(x)$.

Unfortunately the papers of BOGOLIUBOV and PARASIUK [1], [2] come close to not satisfying SALAM's criterion: it is hard to find two theoreticians whose understanding of the essential steps of the proof is isomorphic. This is particularly regrettable, since the very ingenious and elaborate treatment of the authors is the most general discussion of renormalization in Lagrangian quantum field theory. Our aim is to give a new and possibly clearer account of the fundamental operations and to circumvent a number of errors in the original proof of the main theorem. We shall give a new treatment of the combinatorial structure of the $\mathscr{R}$-operation (Lemma 2.4), which deals efficiently with the overlapping divergences and leads to a direct majorization of the counter terms in partial sums of tree structure (Lemma 3.2). Each tree gives a renormalized Feynman integral in the limit $r \downarrow 0$, which has the usual analyticity properties and is a tempered distribution for $\varepsilon \downarrow 0$. In an example we shall discuss the renormalized perturbation series for the φ^4-theory [9].

2. Tree structure of the $\mathscr{R}$-Operation

In momentum space each term in $\mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n)$ has the following structure

$$\delta(\sum p_i) \int_r^\infty \ldots \int_r^\infty \prod_{l=1}^{L} d\alpha_l \, R(\alpha, p) \exp\left[i \sum_1^n A_{ij} p_i p_j - i \sum_1^L \alpha_l (m_l^2 - i\varepsilon)\right] \quad (2.1)$$

The $A_{ij} = A_{ij}(\alpha)$ are rational and homogeneous of degree $+1$ and $R(\alpha, p)$ rational and in general not locally α-integrable for $\alpha \downarrow 0$. Thus the "ultraviolet divergences" become manifest and the counter terms are introduced to enforce local integrability when $r \downarrow 0$. The greatest difficulty in proving the existence of $\lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n)$ arises from "overlapping divergences", i.e. from the necessity of subtractions for generalized vertices $U_i = \{V_{i1}, \ldots V_{ik}\}$, $U_j = \{V_{j1}, \ldots V_{jl}\}$ with $U_i \cap U_j \neq \emptyset$ and neither $U_i \supset U_j$ nor $U_j \supset U_i$. Although each term in $\mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n)$ diverges in general for $r \downarrow 0$, one easily sees in simple examples that certain partial sums of counter terms converge individually in regions of the type $\alpha_{l_1} \geqq \cdots \geqq \alpha_{l_L} \geqq 0$.

This leads to the following combinatorial problem: Given an ordering $l_1 > \cdots > l_L$ of the lines $\mathscr{L}$. Can one decompose $\mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n)$ into partial sums involving only non-overlapping subtractions, while keeping definite order relations between the lines $\mathscr{L}$ such that "the right subtractions arise at the right places"? The answer is affirmative, and the problem of keeping track of all counter terms has found a surprisingly simple solution by Lemma 2.2 due to Bogoliubov and Parasiuk [1], [2]. In this section we shall not denote explicitly the $r > 0$ and $\varepsilon > 0$ dependence (by writing e.g. Δ_l for $\Delta_l^{r,\varepsilon}$).

We first introduce a generalized $\mathscr{R}$-operation for a set $U_1, \ldots U_s$ of pairwise disjoint generalized vertices $U_i = \{V_{i1}, \ldots V_{ir(i)}\}$ with vertex parts $\mathscr{X}(U_i)$ (i.e. distributions with support in $\{x_{i1} = \cdots = x_{ir(i)}\}$) and for a subset $\mathscr{M} \subset \mathscr{L}$. For any union $U_1' \cup \cdots \cup U_t'$, $U_i' \in \{U_1, \ldots U_s\}$, we define a vertex part as in (1.8, 9, 10) by

$$\mathscr{X}_{\mathscr{M}}(U_1', \ldots U_t') = \begin{cases} \mathscr{X}(U_1') & \text{if} \quad t = 1 \\ 0 \quad \text{if} \quad G(U_1', \ldots U_t', \mathscr{M}) \text{ is IPR} \\ -M\overline{\mathscr{R}}_{\mathscr{M}}(U_1', \ldots U_t') & \text{otherwise}, \end{cases} \tag{2.2}$$

$$\overline{\mathscr{R}}_{\mathscr{M}}(U_1', \ldots U_t') = \sum_P{}' \prod_{j=1}^{k(P)} \mathscr{X}_{\mathscr{M}}(U_{j1}^P, \ldots U_{jr(j)}^P) \prod_{\text{conn}} \Delta_l . \tag{2.3}$$

$\sum_P{}'$ runs over all partitions P of $\{U_1', \ldots U_t'\}$ into $1 < k(P) \leqq t$ sets and $\prod_{\text{conn}}$ over all lines $l \in \mathscr{L}$, for which V_{i_l} and V_{f_l} lie in different sets of the partition. M is defined in p-space as Taylor expansion of the coefficient of $\delta\left(\sum_{i,j} p_{ij}\right)$ up to the order $\nu(V_{11}, \ldots V_{tk(t)})$ around the origin. Finally

$$\mathscr{R}_{\mathscr{M}}(U_1', \ldots U_t') = \overline{\mathscr{R}}_{\mathscr{M}}(U_1', \ldots U_t') + \mathscr{X}_{\mathscr{M}}(U_1', \ldots U_t') . \tag{2.4}$$

It is important to observe that $\mathscr{M}$ only accounts for IPI in (2.2), while $\prod_{\text{conn}}$ extends over all connecting lines $l \in \mathscr{L}$ and $\mathscr{L}$ determines $\nu(V_{11}, \ldots V_{tk(t)})$ by (1.7). Thus by varying $\mathscr{M}$ one changes the number of counter terms in (2.4) without affecting the analytical form of the remaining terms.

Lemma 2.1. If $G(U_1, \ldots U_s, \mathscr{M})$ is IPR, then there exists a unique partition of $\{U_1, \ldots U_s\}$ into $\{U_{01}\}, \ldots \{U_{0s(0)}\}$, $\{U_{11}, \ldots U_{1s(1)}\}, \ldots \ldots \{U_{r1}, \ldots U_{rs(r)}\}$, $s(1), \ldots s(r) > 1$, such that $G(U_{i1}, \ldots U_{is(i)}, \mathscr{M})$ are IPI and

$$\begin{aligned} \mathscr{R}_{\mathscr{M}}(U_1, \ldots U_s) &= \overline{\mathscr{R}}_{\mathscr{M}}(U_1, \ldots U_s) \\ &= \prod_{i=1}^{s(0)} \mathscr{X}(U_{0i}) \prod_{j=1}^{r} \mathscr{R}_{\mathscr{M}}(U_{j1}, \ldots U_{js(j)}) \prod_{\text{conn}} \Delta_l , \end{aligned} \tag{2.5}$$

where $\prod_{\text{conn}}$ extends over all $l \in \mathscr{L}$ connecting generalized vertices from different sets of the partition.

Proof: One obtains trivially the decomposition of $G(U_1, \ldots U_s, \mathscr{M})$ into IPI components by looking at the corresponding graph. These components are some generalized vertices U_{0i} and some IPI subgraphs $G(U_{j1}, \ldots U_{js(j)}, \mathscr{M})$. Since the definition of $\mathscr{R}_{\mathscr{M}}(U_1, \ldots U_s)$ excludes all partitions of $\{U_1, \ldots U_s\}$, which are not finer than $\{U_{01}\}, \ldots \ldots \{U_{r1}, \ldots U_{rs(r)}\}$ (because otherwise there would appear at least one IPR subgraph) one obtains (2.5).

Since the connecting lines between the IPI components of any $G(V_1, \ldots V_n, \mathscr{L})$ do not form closed loops, we can in the sequel restrict ourselves to the renormalization of IPI graphs. The following lemma shows how to regroup the counter terms in $\mathscr{X}_{\mathscr{M}}(U_1, \ldots U_m)$, if $\mathscr{M}$ is replaced by $\mathscr{M}/l$:

Lemma 2.2: Given $\mathscr{M} \subset \mathscr{L}$ and a disjoint set of generalized vertices $U_1, \ldots U_m$ with vertex parts $\mathscr{X}(U_i)$. If $l \in \mathscr{M}$ connects different U_i, then

$$\mathscr{X}_{\mathscr{M}}(U_1, \ldots U_m) = \mathscr{X}_{\mathscr{M}/l}(U_1, \ldots U_m) + \sum_j \mathscr{X}_{\mathscr{M}/l}(U_{j1} \cup \cdots \cup U_{jr(j)}, U_{j(r(j)+1)}, \ldots U_{jm}) \qquad (2.6)$$

Here $\sum_j$ extends over all IPI $G(U_{j1}, \ldots U_{jr(j)}, \mathscr{M})$, $1 < r(j) \leqq m$, which become IPR without l. $\mathscr{X}_{\mathscr{M}/l}(U_{j1} \cup \cdots \cup U_{jr(j)}, \ldots U_{jm})$ is defined by (2.2) starting from $\mathscr{X}(U_{jk})$, $k > r(j)$, and from new vertex parts for $U_{j1} \cup \cdots \cup U_{jr(j)}$:

$$\mathscr{X}_{\mathscr{M}/l}(U_{j1} \cup \cdots \cup U_{jr(j)}) = -M\overline{\mathscr{R}}_{\mathscr{M}/l}(U_{j1}, \ldots U_{jr(j)}) \,. \qquad (2.7)$$

Proof[1] by induction: The lemma is true for $m = 1$. We assume that Lemma 2.2 holds for all proper subsets $\{U'_1, \ldots U'_k\}$ of $\{U_1, \ldots U_m\}$, where $k < m$ and $l \in \mathscr{M} \subset \mathscr{L}$ is fixed.

If $G(U_1, \ldots U_m, \mathscr{M})$ is IPR, then also $G(U_1, \ldots U_m, \mathscr{M}/l)$. Since any IPI $G(U_{j1}, \ldots U_{jr(j)}, \mathscr{M})$ must lie in a IPI component of $G(U_1, \ldots U_m, \mathscr{M})$, all the graphs $G(U_{j1} \cup \cdots \cup U_{jr(j)}, \ldots U_{jm}, \mathscr{M}/l)$ are IPR. Therefore both sides of (2.6) vanish.

If $G(U_1, \ldots U_m, \mathscr{M})$ is IPI and $m > 1$, then we use (2.2) and (2.3). For each of the $\mathscr{X}_{\mathscr{M}}$ on the right hand side of (2.3) the induction hypothesis applies. If l does not connect two vertices from $U^P_{j1}, \ldots U^P_{jr(j)}$, then obviously $\mathscr{X}_{\mathscr{M}}(U^P_{j1}, \ldots U^P_{jr(j)}) = \mathscr{X}_{\mathscr{M}/l}(U^P_{j1}, \ldots U^P_{jr(j)})$. The other alternative occurs at most once in every product in (2.3), say for $j = 1$. By the induction hypothesis (2.3) becomes

$$\begin{aligned} \mathscr{X}_{\mathscr{M}}(U_1, \ldots U_m) = {} & -M\overline{\mathscr{R}}_{\mathscr{M}/l}(U_1, \ldots U_m) - \\ & - M \sum_P{}'' \sum_{a=1}^{A} \mathscr{X}_{\mathscr{M}/l}(U^P_{1a1} \cup \cdots \cup U^P_{1as(a)}, \ldots U^P_{1ar(1)}) \times \\ & \times \prod_{j=2}^{k(P)} \mathscr{X}_{\mathscr{M}/l}(U^P_{j1}, \ldots U^P_{jr(j)}) \prod_{\text{conn}} \Delta_l \,. \end{aligned} \qquad (2.8)$$

Here $\sum_P''$ extends over those proper partitions P of $\{U_1, \ldots U_m\}$, where l connects two vertices from $U^P_{11}, \ldots U^P_{1r(1)}$, and $\sum_{a=1}^{A}$ over all IPI subgraphs $G(U^P_{1a1}, \ldots U^P_{1as(a)}, \mathscr{M})$, $1 < s(a) \leqq r(1)$, which become IPR without l.

It is easy to see that the right hand side of (2.6) and (2.8) coincide:

$$-M\overline{\mathscr{R}}_{\mathscr{M}/l}(U_1, \ldots U_m)$$

becomes $\mathscr{X}_{\mathscr{M}/l}(U_1, \ldots U_m)$, if $G(U_1, \ldots U_m, \mathscr{M}/l)$ is IPI and otherwise

[1] The proof in [1], [2] is incorrect. Unfortunately the erroneous identity $R(G_{a_1} \ldots G_{a_k} : \Gamma') = \Delta(\Gamma') R(G_{a_1} \ldots G_{a_k})$ has been used repeatedly in discussing the analytical properties of the R-operation (e.g. [1], (4.4)ff; [1], Satz 4).

$\mathscr{X}_{\mathscr{M}/l}(U_1 \cup \cdots \cup U_m)$. By inverting the order of the summations $\sum\limits_P''$ and $\sum\limits_a$ the second term in (2.8) is identified with the sum over all IPI $G(U_{j1}, \ldots U_{jr(j)}, \mathscr{M})$ which are IPR without l and over all proper partitions of $\{U_{j1} \cup \cdots \cup U_{jr(j)}, U_{j(r(j)+1)}, \ldots U_{jm}\}$.

This proves Lemma 2.2 and under the same assumptions:

Lemma 2.3:

$$\overline{\mathscr{R}}_{\mathscr{M}}(U_1, \ldots U_m) = \overline{\mathscr{R}}_{\mathscr{M}/l}(U_1, \ldots U_m) + \\ + \sum_j{}' \overline{\mathscr{R}}_{\mathscr{M}/l}(U_{j1} \cup \cdots \cup U_{jr(j)}, U_{j(r(j)+1)}, \ldots U_{jm}), \tag{2.9}$$

where $\sum\limits_j{}'$ extends over all IPI $G(U_{j1}, \ldots U_{jr(j)}, \mathscr{M})$, $1 < r(j) < m$, which become IPR without l.

If we collect sums of iterated M-operations appearing in $\mathscr{R}_{\mathscr{L}}(V_1, \ldots V_n)$ into non-overlapping sequences of M- and $(1 - \mathrm{M})$-operations, the following definition arises:

Def.: Let $G(V_1, \ldots V_n, \mathscr{L})$ be IPI. A tree $T = (\mathfrak{U}, \mathscr{M}, \sigma)$ is a set $\mathfrak{U}$ of generalized vertices, a subset $\mathscr{M} \subset \mathscr{L}$ and a mapping $\sigma : \mathfrak{U} \to \{-1, 0, +1\}$ satisfying (A), (B) and (C):

(A) $\{V_1, \ldots V_n\} \in \mathfrak{U}$; $V_i \in \mathfrak{U}$, $1 \leqq i \leqq n$; if $U_1, U_2 \in \mathfrak{U}$, then either $U_1 \cap U_2 = \emptyset$ or $U_1 \subset U_2$ or $U_2 \subset U_1$.

Remark: The $U \in \mathfrak{U}$ can be uniquely labeled by their position in a chain of elements in $\mathfrak{U}$:

$$U(i_0, \ldots i_k) \subset U(i_0, \ldots i_{k-1}) \subset \cdots \subset U(i_0) = \{V_1, \ldots V_n\}, \tag{2.10}$$

where $i_0 = 1$ and the $i_r \geqq 1$ are integers. We set $I = (i_0, \ldots i_k)$ and $(I, i) = i_0, \ldots i_k, i)$ and thus $U(I, i) \subset U(I)$. $\mathscr{I} = \{I : U(I) \in \mathfrak{U}\}$.

(B) Either $U(I) = V_i$ for some $1 \leqq i \leqq n$ or $\mathscr{M}$ connects the $\{U(I, i) : (I, i) \in \mathscr{I}\}$ either IPI or linearly without closed loops.

Example: Fig. 3.

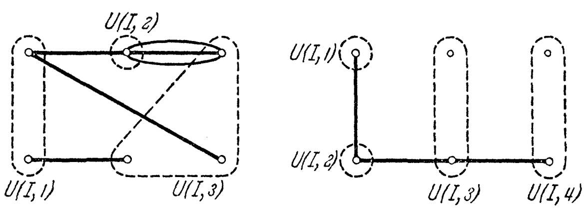

Fig. 3. Connection of the $\{U(I, i)\} \subset U(I)$ by $\mathscr{M}$.

(C) $\sigma(U(i_0)) = +1$; $\sigma(U(I)) = 0$, iff $U(I) = V_i$ for some $1 \leqq i \leqq n$. If $\sigma(U(I)) = -1$, then $G(\{U(I, i)\}, \mathscr{M})$ is IPR. If $G(\{U(I, i)\}, \mathscr{M})$ is IPI, then $\sigma(U(I)) = +1$ and $\sigma(U(I, i)) \leqq 0$ for all $(I, i) \in \mathscr{I}$.

Def.: Let $T = (\mathfrak{U}, \mathcal{M}, \sigma)$ be a tree for a IPI $G(V_1, \ldots V_n, \mathcal{L})$. Then the Feynman amplitude $\mathcal{F}_T(V_1, \ldots V_n) = \mathcal{F}_{\mathcal{M}}(U(i_0))$ is defined recursively from the $\mathcal{F}_{\mathcal{M}}(U(I))$ of the "branches" $U(I) \in \mathfrak{U}$ by:

(a) If $U(I) = V_i$, then $\mathcal{F}_{\mathcal{M}}(U(I)) = 1$ and $U(I)$ is called a "twig".

(b) If $\sigma(U(I)) = -1$, then $U(I)$ is a "twig" and

$$\mathcal{F}_{\mathcal{M}}(U(I)) = -M\Big\{\prod_{(I,i)\in\mathcal{I}} \mathcal{F}_{\mathcal{M}}(U(I,i)) \prod_{\text{conn}} \Delta_l\Big\}. \tag{2.11}$$

(c) If $\sigma(U(I)) = +1$ and $G(\{U(I,i)\}, \mathcal{M})$ is IPR, then $U(I)$ is called a "bough" and

$$\mathcal{F}_{\mathcal{M}}(U(I)) = (1-M)\Big\{\prod_{(I,i)\in\mathcal{I}} \mathcal{F}_{\mathcal{M}}(U(I,i)) \prod_{\text{conn}} \Delta_l\Big\}. \tag{2.12}$$

(d) If $\sigma(U(I)) = +1$ and $G(\{U(I,i)\}, \mathcal{M})$ is IPI, then $U(I)$ is called a "bud" and

$$\mathcal{F}_{\mathcal{M}}(U(I)) = (1-M)\,\overline{\mathcal{R}}_{\mathcal{M}}(\{U(I,i)\}) \tag{2.13}$$

is defined by (2.2), (2.3) starting from the $\mathcal{F}_{\mathcal{M}}(U(I,i))$ [$= 1$ or (2.11)] as vertex parts $\mathcal{X}(U(I,i))$.

In (2.11), (2.12) $\prod_{\text{conn}} \Delta_l$ extends over all $l \in \mathcal{L}$, which connect different $U(I,i) \subset U(I)$.

Example: If $G(V_1, \ldots V_n, \mathcal{L})$ is IPI, then $\mathcal{R}_{\mathcal{L}}(V_1, \ldots V_n) = (1-M)\,\overline{\mathcal{R}}_{\mathcal{L}}(V_1, \ldots V_n)$ is a tree with the bud $\{V_1, \ldots V_n\}$ and the twigs V_i, $1 \leqq i \leqq n$. A repeated application of Lemma 2.3 will again lead to sums of trees (see Lemma 2.4).

Def.: The order of $U(I)$, $I = (i_0, \ldots i_k)$, is the length k of the chain (2.10). Let $\sigma(U(I)) = 1$. Then the *b*-order of $U(I)$ is the length of the subchain of (2.10) consisting only of buds and boughs.

Def.: l is contained in $U(I)$, if $V_{i_l}, V_{f_l} \in U(I)$.

For each $l \in \mathcal{L}$ there exists an $U(I) \in \mathfrak{U}$ of maximal order which contains l: if $V_{i_l} = U(I')$, $V_{f_l} = U(I'')$, then $i'_\varrho = i''_\varrho$ for $0 \leqq \varrho \leqq r$ and $i'_{r+1} \neq i''_{r+1}$ for some r; then $I = (i'_0, \ldots, i'_r)$. If $\sigma(U(I)) = -1$, then the bud or bough of maximal order containing l contains $U(I)$ properly. The following lemma motivates our arborological language:

Lemma 2.4.: Let $G(V_1, \ldots V_n, \mathcal{L})$ be IPI and $l_1 > \cdots > l_L$ be any ordering of the lines $\mathcal{L}$. Then there exists a finite set of trees $T = (\mathfrak{U}, \mathcal{M}, \sigma)$ such that

(a) $\mathcal{R}_{\mathcal{L}}(V_1, \ldots V_n) = \sum_T \mathfrak{F}_T(V_1, \ldots V_n)$.

(b) Each $\mathcal{M}$ contains exactly $n-1$ lines.

(c) If $l \in \mathcal{L} - \mathcal{M}$ and if $U(i_0, \ldots i_s)$ is the branch and $U(i_0, \ldots i_r)$ the bough of maximal order containing l, then $l > l'$ for all $l' \in \mathcal{M}_1(i_0, \ldots i_\varrho)$, $r \leqq \varrho \leqq s$, where $\mathcal{M}_1(I)$ is the set of all $l' \in \mathcal{M}$, which are contained in $U(I)$ but in no twig $U(I') \subset U(I)$, $U(I') \neq U(I)$.

Proof: Using the fact that $\mathscr{R}_{\mathscr{L}}(V_1, \ldots V_n)$ is a tree and repeatedly the reduction formula (2.9) we shall prove Lemma 2.4 by complete induction with respect to the number of lines in $\mathscr{L} - \mathscr{M}$.

We assume that after m steps, $0 \leqq m < L + 1 - n$, $\mathscr{R}_{\mathscr{L}}(V_1, \ldots V_n)$ can be represented as a finite sum $\sum \mathscr{F}_T(V_1, \ldots V_n)$ of trees $T = (\mathfrak{U}, \mathscr{M}, \sigma)$. In each tree every bough is of b-order $\leqq k$ and every bud of b-order $\geqq k$ for some $k \geqq 0$. $\mathscr{M}$ consists of exactly $L - m$ lines. If $l \in \mathscr{L} - \mathscr{M}$ and $U(i_0, \ldots i_s)$ is the branch and $U(i_0, \ldots i_r)$ the bud or bough of maximal order containing l, then $l > l'$ for all $l' \in \mathscr{M}_1(i_0, \ldots i_\varrho)$, $r \leqq \varrho \leqq s$.

Obviously $\mathscr{R}_{\mathscr{L}}(V_1, \ldots V_n)$ satisfies all these properties for $m = 0$. We shall show that the induction hypothesis remains valid, if in any tree T we reduce one of the buds $U(I)$ of smallest b-order by applying Lemma 2.3 to its Feynman amplitude (2.13) and by reducing with respect to the largest line l in $\mathscr{M}_1(I)$. By definition all branches $U(I, i) \subset U(I)$ are twigs. Therefore one obtains

$$\begin{aligned}\mathscr{F}_{\mathscr{M}}(U(I)) = (1 - M)\, \overline{\mathscr{R}}_{\mathscr{M}/l}(U(I, 1), \ldots U(I, s)) + \\ + \sum_j (1 - M)\, \overline{\mathscr{R}}_{\mathscr{M}/l}(U(I, j_1) \cup \cdots \cup U(I, j_{r(j)}), \ldots U(I, j_s)) .\end{aligned} \tag{2.14}$$

$\sum_j$ extends over all IPI $G(U(I, j_1), \ldots U(I, j_{r(j)}), \mathscr{M})$, which are IPR without l. Let $\{U(I, h_i)\}$, $1 \leqq i \leqq a$, and $\{U(I, h_{ij}), 1 \leqq j \leqq c(i)\}$, $1 \leqq i \leqq b$, be the partition of $\{U(I, j_1), \ldots U(I, j_{r(j)})\}$, which characterizes the IPI components of $G(U(I, j_1), \ldots U(I, j_{r(j)}), \mathscr{M}/l)$. Then the vertex part for $U(I, j_1) \cup \cdots \cup U(I, j_{r(j)})$ factorizes by Lemma 2.1 into

$$\begin{aligned}&\mathscr{X}_{\mathscr{M}/l}(U(I, j_1) \cup \cdots \cup U(I, j_{r(j)})) \\ &= - M \overline{\mathscr{R}}_{\mathscr{M}/l}(U(I, j_1), \ldots U(I, j_{r(j)})) \\ &= - M \left\{ \prod_{i=1}^{a} \mathscr{F}_{\mathscr{M}}(U(I, h_i)) \times \right. \\ &\left. \times \prod_{i=1}^{b} [(1 - M)\, \overline{\mathscr{R}}_{\mathscr{M}/l}(U(I, h_{i1}), \ldots U(I, h_{i\,c(i)}))] \prod_{\text{conn}} \Delta_l \right\} .\end{aligned} \tag{2.15}$$

Similarly, if $G(U(I, 1), \ldots U(I, s), \mathscr{M}/l)$ is IPR, the first term on the right hand side of (2.14) can be decomposed:

$$\begin{aligned}&(1 - M)\, \overline{\mathscr{R}}_{\mathscr{M}/l}(U(I, 1), \ldots U(I, s)) \\ &= (1 - M) \left\{ \prod_{i=1}^{d} \mathscr{F}_{\mathscr{M}}(U(I, k_i)) \times \right. \\ &\left. \times \prod_{i=1}^{e} [(1 - M)\, \overline{\mathscr{R}}_{\mathscr{M}/l}(U(I, k_{i1}), \ldots U(I, k_{i\,f(i)}))] \prod_{\text{conn}} \Delta_l \right\} .\end{aligned} \tag{2.16}$$

On the other hand all $G(U(I, j_1) \cup \cdots \cup U(I, j_{r(j)}), \ldots U(I, j_s), \mathscr{M}/l)$ are IPI in a bud $U(I)$, and the corresponding Feynman amplitude does not factorize beyond (2.14).

It is easy to see that only in $\mathscr{F}_{\mathscr{M}}(U(I))$ the reduction of $\mathscr{M}$ to $\mathscr{M}/l$ has the effect of introducing new branches corresponding to a rearrangement of counter terms. By definition $\mathscr{M}$ connects linearly the $\{U(I', i)\}$ in a bough or twig $U(I')$, while in a bud $G(\{U(I', i)\}, \mathscr{M})$ is IPI and $\mathscr{M}$ determines the counter terms in $(1 - M)\, \overline{\mathscr{R}}_{\mathscr{M}}(\{U(I', i)\})$ by (2.2).

Now, the bud $U(I)$ in (2.14) was chosen to lie in no other bud. Therefore all branches containing $U(I)$ are boughs or twigs, whose sets of connecting lines in $\mathscr{M}$ is unaffected by omitting any $l \in \mathscr{M}$ contained in $U(I)$. For the same reason $\mathscr{M}$ and $\mathscr{M}/l$ are equivalent in branches $U(I') \cap U(I) = \emptyset$. Lastly every $U(I') \subset U(I)$ is contained in some $U(I, i)$, and we can also here replace $\mathscr{M}$ by $\mathscr{M}/l$, since l was chosen not to lie in any $U(I, i)$.

Thus if we insert (2.14) into the recursive definition of $\mathscr{F}_T(V_1, \dots V_n)$, the contribution of each summand defines a new tree $T' = (\mathfrak{U}', \mathscr{M}', \sigma')$, $\mathscr{M}' = \mathscr{M}/l$, if one separates the twigs and buds in (2.15), (2.16). Since the bud $U(I)$ was of minimal b-order, any bough arising from (2.14), if $G(\{U(I, i)\}, \mathscr{M}/l)$ is IPR, lies outside of all buds, and every bud in T' is of b-order $\geqq k'$, $k' = k$ or $k + 1$.

It remains to check the order relations in $(\mathfrak{U}', \mathscr{M}', \sigma')$ between the lines in $\mathscr{M}' = \mathscr{M}/l$ and $\mathscr{L} - \mathscr{M}'$. Consider the line l. The bough or bud U of maximal order containing l is either $U(I)$ or one of the new buds $U(I, h_{i1}) \cup \cdots \cup U(I, h_{i\,c(i)})$ in (2.15) or $U(I, k_{i1}) \cup \cdots \cup U(I, k_{i\,f(i)})$ in (2.16), and l does not lie in any twig contained in U, since each of these twigs is contained in one of the $U(I, i)$. Therefore the lines in $\mathscr{M}/l$, which lie in U, but in no twig in U, are a subset of $\mathscr{M}_1(I)$, from which l was chosen as the largest line.

Consider now an $l' \in \mathscr{L} - \mathscr{M}$, which is not contained in $U(I)$. Since in all $(\mathfrak{U}', \mathscr{M}', \sigma')$ the structure of the branches outside of $U(I)$ is the same as in $(\mathfrak{U}, \mathscr{M}, \sigma)$ and since the set of $l'' \in \mathscr{M}'$, which lie in $U(I)$ but in no twigs in $U(I)$, is a subset of $\mathscr{M}_1(I)$, the order relations hold for a fortiori for l'.

Suppose finally that $l' \in \mathscr{L} - \mathscr{M}$ is contained in $U(I)$. Then the bud or bough U of maximal order containing l' satisfies one of the following alternatives:

a) U is one of the buds of $(\mathfrak{U}, \mathscr{M}, \sigma)$ contained in some $U(I, i)$. As the reduction did not change the structure inside of the $U(I, i)$, the order relations are preserved.

b) U is one of the newly created buds in (2.15) or (2.16). Then the twigs in U are contained in the $U(I, i)$. Thus $l'' \in \mathscr{M}_1(I)$ for all $l'' \in \mathscr{M}'$, which lie in U but in no twig contained in U. If l' lies outside of all twigs in U, then by the order relations in $(\mathfrak{U}, \mathscr{M}, \sigma)$ $l' > l''$ for all $l'' \in \mathscr{M}_1(I)$. If the branch U' of maximal order containing l' is different from U, then the chain of twigs between U' and U is contained in some

$U(I, i): U' \subset U'' \subset \cdots \subset U(I, i) \subset U$. In the chain $U' \subset U'' \subset \cdots \subset U(I, i)$ the order relations of $(\mathfrak{U}, \mathcal{M}, \sigma)$ continue to hold, and as before l' satisfies the order relations in U.

c) $U = U(I)$. Then it is possible that the twig U' of maximal order containing l' generates a chain of twigs $U' \subset \cdots \subset U(I, i) \subset U'' \subset U(I)$, where U'' is one of the twigs $U(I, j_1) \cup \cdots \cup U(I, j_{r(j)})$ in (2.14). Again in $U' \subset \cdots \subset U(I, i)$ the order relations are not affected by going from $\mathcal{M}$ to $\mathcal{M}/l$, while in U'' or $U(I)$ the critical lines belong to $\mathcal{M}_1(I)$.

This completes the induction. After $L + 1 - n$ reductions one has eliminated all buds in all trees, and one arrives at the statement of Lemma 2.4.

Example: Consider the graph $G(V_1, \ldots V_4, \mathcal{L})$ (Fig. 4).

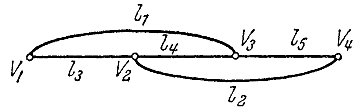

Fig. 4. $G(V_1, \ldots V_4, \mathcal{L})$

For $l_1 > \cdots > l_5$ $\mathcal{R}_{\mathcal{L}}(V_1, \ldots V_4)$ leads to the trees (Fig. 5). Here twigs are denoted by a dotted, boughs by a continuous encircling. The lines in $\mathcal{M}$ are solidly marked. In the case $l_4 > l_3 > l_5 > l_1 > l_2$ one obtains Fig. 6.

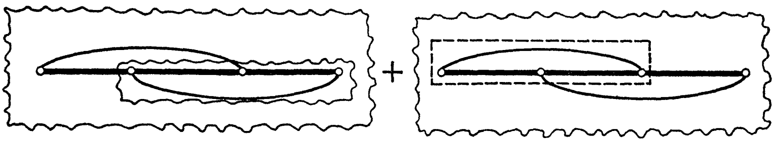

Fig. 5. Trees in the sector $l_1 > \cdots > l_5$

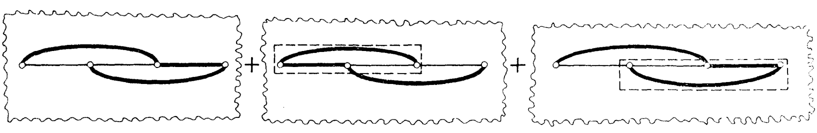

Fig. 6. Trees in the sector $l_4 > l_3 > l_5 > l_1 > l_2$

3. Analytical properties of the $\mathcal{R}$-operation

In this section we shall determine the analytical structure of the $\mathcal{R}$-operation in p-space. For $\varepsilon > 0$, $r > 0$ we first carry out all convolution integrals in p-space and study the remaining α-integrands. The essential simplification occurs after reducing in every sector

$$\alpha_{l_1} \geqq \cdots \geqq \alpha_{l_L} \geqq r \tag{3.1}$$

the α-integrands of $\mathcal{R}_{\mathcal{L}}(V_1, \ldots V_n)$ into a sum of trees with respect to the order relation $l_1 > \cdots > l_L$.

For a tree $(\mathfrak{U}, \mathscr{M}, \sigma)$ we define:

$$\begin{aligned}
&\mathscr{V}_0(I) = \{V_i \in U(I)\}\\
&\mathscr{L}_0(I) = \{l \in \mathscr{L} : V_{i_l}, V_{f_l} \in U(I)\}\\
&\mathscr{L}(I) = \{l \in \mathscr{L}\text{: connecting different } U(I, i) \subset U(I)\}\\
&\mathscr{M}(I) = \{l \in \mathscr{M}\text{: connecting different } U(I, i) \subset U(I)\}\\
&\mathscr{M}_0(I) = \{l \in \mathscr{M} : V_{i_l}, V_{f_l} \in U(I)\} \qquad (3.2)\\
&\mathscr{M}_1(I) = \{l \in \mathscr{M}\text{: contained in } U(I) \text{ but in no twig } U(I') \subset U(I),\\
&\qquad U(I') \neq U(I)\}\\
&\mathfrak{U}_1(I) = \{U(I') \subset U(I)\text{: not contained in any twig } U(I'') \subset U(I),\\
&\qquad U(I'') \neq U(I)\}\,.
\end{aligned}$$

The number of elements in any of these sets $\mathscr{N}$ is denoted by $|\mathscr{N}|$, e.g., $|\mathscr{L}_0(i_0)| = L$. $I \geqq I'$, if $U(I) \supset U(I')$, and $I > I'$, if in addition $U(I) \neq U(I')$. The superficial divergence (1.7) of $G(\mathscr{V}_0(I), \mathscr{L})$ becomes

$$\nu(I) = \sum_{l \in \mathscr{L}_0(I)} (r_l + 2) - 4(|\mathscr{V}_0(I)| - 1)\,. \qquad (3.3)$$

Lemma 3.1: Let $T = (\mathfrak{U}, \mathscr{M}, \sigma)$ be a tree in the decomposition of $\mathscr{R}_{\mathscr{L}}(V_1, \ldots V_n)$ in (3.1). Then the Feynman amplitude $\mathscr{F}_{\mathscr{M}}(U(I))$ of any bough $U(I) \in \mathfrak{U}$ is for fixed α a finite sum of terms of the form

$$\begin{aligned}
&\delta\Big(\sum_{V_i \in \mathscr{V}_0(I)} p_i\Big) \int_0^1 \cdots \int_0^1 \Big[\prod_{\substack{I' \leqq I\\ \sigma(I') = 1}} d\tau(I')\Big] P^{\mathrm{I}}(p)\, Q^{\mathrm{I}}(\alpha, \tau) \times\\
&\qquad \times \Big[\prod_{I' \leqq I} D^{\mathrm{I}'} R^{\mathrm{I}'}(A^{\mathrm{I}'})\, S^{\mathrm{I}'}(B^{\mathrm{I}'})\Big] \times \qquad (3.4)\\
&\times \exp\Big[i \sum_{V_i, V_j \in \mathscr{V}_0(I)} \tau(I)^2 A^{\mathrm{I}}_{ij} p_i p_j - i \sum_{l \in \mathscr{L}_0(I)} \alpha_l (m_l^2 - i\varepsilon)\Big]\,.
\end{aligned}$$

If $\sigma(U(I)) = -1$, then $\mathscr{F}_{\mathscr{M}}(U(I))$ has the same structure (3.4) except for $\sum \tau(I)^2 A^{\mathrm{I}}_{ij} p_i p_j$ being replaced by zero. The integrand in (3.4) satisfies:

(1) $P^{\mathrm{I}}(p)$ is a monomial in p_i, $V_i \in \mathscr{V}_0(I)$, of degree $2x(I) + z(I)$.

(2) $Q^{\mathrm{I}}(\alpha, \tau)$ is a rational function in α_l, $l \in \mathscr{L}_0(I)$ and $\tau(I')$, $I' \leqq I$, homogeneous of degree 0 in α and uniformly bounded in (3.1) for $r \geqq 0$ and all $0 \leqq \tau(I') \leqq 1$.

(3) $D^{\mathrm{I}'} = \prod_{l \in \mathscr{L}(I') - \mathscr{M}} D_l^{-2}$, where $D_l = D_l(\alpha, \tau)$ is rational in α, τ, homogeneous of degree $+1$ in α and $D_l \geqq \alpha_l$ for $r \geqq 0$.

(4) $A^{\mathrm{I}'} = (A^{\mathrm{I}'}_{ij})$ is a positive semidefinite quadratic form. The $A^{\mathrm{I}'}_{ij}(\alpha, \tau)$ are rational in α, τ, homogeneous of degree $+1$ in α and satisfy uniformly for $r \geqq 0$

$$|A^{\mathrm{I}'}_{ij}| \leqq c(I') \max\{\alpha_l : l \in \mathscr{M}_1(I')\},\ c(I') < \infty\,. \qquad (3.5)$$

(5) $R^{\mathrm{I}'}(A^{\mathrm{I}'})$ is a monomial of degree $x(I')$ in $A^{\mathrm{I}'}_{ij}$, V_i, $V_j \in \mathscr{V}_0(I')$.

(6) $B^{\mathrm{I}'}_{ij}(\alpha, \tau)$ is rational in α, τ, homogeneous of degree -1 in α, and satisfies uniformly in (3.1) for $r \geqq 0$

$$|B^{\mathrm{I}'}_{ij}| < d(I') \max\{\alpha_l^{-1}, l \in \mathscr{L}(I') - \mathscr{M}\},\ d(I') < \infty\,. \qquad (3.6)$$

(7) $S^{I'}(B^{I'})$ is a monomial of degree $y(I')$ in the $B^{I'}_{ij}$, $1 \leqq i$, $j \leqq |\mathscr{V}_0(I')| + |\mathscr{L}(I')| - |\mathscr{M}(I')|$.

(8) $x(I')$, $y(I')$, $z(I') \geqq 0$ are integers satisfying:

$$x(I') = y(I') = z(I') = 0, \quad \text{for} \quad \sigma(U(I')) = 0$$

$$x(I') \geqq \left[\left[\frac{v(I') + 1 - z(I')}{2}\right]\right], \quad \text{for} \quad \sigma(U(I')) = 1 \tag{3.7}$$

$$x(I') \leqq \frac{v(I') - z(I')}{2} \quad \text{for} \quad \sigma(U(I')) = -1 \tag{3.8}$$

$$2y(I') + z(I') \leqq \sum_{l \in \mathscr{L}(I)} r_l + \sum_{(I',i) \in \mathscr{I}} [2x(I',i) + z(I',i)], \tag{3.9}$$

where $[[k]]$ is the smallest integer $\geqq k$.

Remark: This theorem sharpens the statements in [1], Satz 4. The representation of the renormalized amplitudes as sums of terms of the form (3.4), each of which being locally α-integrable in the whole range $0 \leqq \alpha_l < \infty$, $1 \leqq l \leqq L$, as stated in [2], (8.3), (8.4), has not been proved.

Proof: Let $s(I)$ be the length of the longest chain of branches $U(I) \supset U(I_1) \supset \cdots \supset U(I_{s(I)})$ (with proper inclusion) contained in $U(I)$. We shall prove Lemma 3.1 by induction on $s(I)$, starting from an original vertex with $s(I) = 0$. As induction assumption in the case $s(I) > 0$ we take Lemma 3.1 for granted for all $U(I')$ with $s(I') < s(I)$ and thus for all $U(I') \subset U(I)$.

We compute $\mathscr{F}_{\mathscr{M}}(U(I))$ in p-space using (2.11) or (2.12). Then the p-dependent part of $\mathscr{F}_{\mathscr{M}}(U(I,1))\,\mathscr{F}_{\mathscr{M}}(U(I,2))\,\Delta_{l_{12}}$ becomes for $\sigma(U(I,1)) = \sigma(U(I,2)) = +1$ a sum of terms

$$\begin{aligned} &\int dk\, \delta\Big(\sum_{V_i \in \mathscr{V}_0(I,1)} (p_i + e_i k)\Big)\, \delta\Big(\sum_{V_j \in \mathscr{V}_0(I,2)} (p_j + e_j k)\Big)\, P_{l_{12}}(k) \\ &\qquad P^{(I,1)}(\{p_i + e_i k\})\, P^{(I,2)}(\{p_j + e_j k\}) \\ &\qquad \exp\Big[i\alpha_{l_{12}} k^2 + i\tau(I,1)^2 \sum_{i,i'} A^{(I,1)}_{ii'}(p_i + e_i k)(p_{i'} + e_{i'} k) + \\ &\qquad\qquad + i\tau(I,2)^2 \sum_{j,j'} A^{(I,2)}_{jj'}(p_j + e_j k)(p_{j'} + e_{j'} k)\Big], \end{aligned} \tag{3.10}$$

where $e_i = +1$ for $i = i_{l_{12}}$, $e_i = -1$ for $i = f_{l_{12}}$ and 0 otherwise. If $U(I,1)$ and $U(I,2)$ are not both boughs, one obtains similar terms. After carrying out the k-integration with the help of one of the δ-functions (3.10) becomes a sum of terms

$$\delta\Big(\sum_{V_i \in \mathscr{V}_0(I,1) \cup \mathscr{V}_0(I,2)} p_i\Big)\, P'(p) \exp[i \textstyle\sum A'_{ij} p_i p_j], \tag{3.11}$$

where $P'(p)$ is a monomial in p_i, $V_i \in \mathscr{F}_0(I,1) \cup \mathscr{F}_0(I,2)$, of degree $\leqq r_{l_{12}} + \sum_{i=1}^{2} [2x(I,i) + z(I,i)]$ and A' is a positive semidefinite quadratic form, whose coefficients A'_{ij} are linear combinations of the $A^{(I,1)}_{ii'}\tau(I,1)^2$, $A^{(I,2)}_{jj'}\tau(I,2)^2$ and $\alpha_{l_{12}}$. By a similar argument one obtains for

$$\prod_{(I,i) \in \mathscr{I}} \mathscr{F}_{\mathscr{M}}(U(I,i)) \prod_{l \in \mathscr{M}(I)} \Delta_l \tag{3.12}$$

in p-space a sum of terms

$$\delta\Big(\sum_{V_i\in\mathscr{V}(I)} p_i\Big)\int_0^1\ldots\int_0^1 \prod_{I'<I} d\tau(I')\prod_{I'<I} D^{I'}R^{I'}(A^{I'})\,S^{I'}(B^{I'})\times \\ \times \prod_{(I,i)\in\mathscr{J}} Q^{(I,i)}(\alpha,\tau)\,P^{(r)}(p)\exp\big[i\sum A_{ij}^{(r)}p_ip_j - i\sum \alpha_l(m_l^2 - i\varepsilon)\big]\,, \tag{3.13}$$

where $P^{(r)}(p)$ is a monomial in the p_i, $V_i\in\mathscr{F}_0(I)$, of degree $\leqq \sum_{l\in\mathscr{M}(I)} r_l + \sum_{(I,i)\in\mathscr{J}} [2x(I,i)+z(I,i)]$ and $A_{ij}^{(r)}$ is a linear combination of the α_l, $l\in\mathscr{M}(I)$, and the $A_{ij}^{(I,i)}\tau(I,i)^2$, $\sigma(I,i)=1$. Following [1] we use $p = -i\dfrac{\partial}{\partial r}\exp[ipr]|_{r=0}$ and obtain

$$P^{(r)}(p) = P^{(r)}(-i\nabla)\exp\Big[i\sum_{V_j\in\mathscr{V}_0(I)} p_jr_j\Big]\Big|_{r=0}\,. \tag{3.14}$$

Let us assume that after having incorporated a set $\mathscr{N}$ of n lines, $r\leqq n<\mathscr{L}(I)$, we hav in p-space instead of

$$P^{(r)}(p)\exp\big[i\sum A_{ij}^{(r)}p_ip_j - i\sum\alpha_l(m_l^2 - i\varepsilon)\big] \tag{3.15}$$

in (3.13) a sum of terms

$$\prod_{l\in\mathscr{N}-\mathscr{M}} D_l^{-2}p^{(n)}(-i\nabla)\exp\Big[i\sum_{V_i,V_j\in\mathscr{V}_0(I)} A_{ij}^{(n)}p_ip_j + i\sum_{k,l=1}^{n} B_{kl}^{(n)}r_kr_l + \\ + i\sum_{V_i\in\mathscr{V}_0(I)}\sum_{k=1}^{n} C_{ik}^{(n)}p_ir_k - i\sum_{l\in\mathscr{N}}\alpha_l(m_l^2 - i\varepsilon)\Big]\Big|_{r=0} \tag{3.16}$$

where D_l satisfies (3), $P^{(n)}(-i\nabla)$ is a monomial in the $\dfrac{\partial}{\partial r_i}$, $1\leqq i\leqq n$, of degree

$$\leqq \sum_{(I,i)\in\mathscr{J}} [2x(I,i)+z(I,i)] + \sum_{l\in\mathscr{N}} r_l\,, \tag{3.17}$$

$A^{(n)}$ satisfies (5), $B^{(n)}$ satisfies (7) and $C_{ik}^{(n)}(\alpha,\tau)$ is rational in the α,τ, homogeneous of degree 0 in the α and uniformly bounded in (3.1) for $r\geqq 0$.

We now incorporate another $\Delta_{l'}(x_{i_{l'}} - x_{f_{l'}})$, where $l'\in\mathscr{L}(I)-\mathscr{N}$. Let $e_i=+1$ for $i=i_{l'}$, $e_i=-1$ for $i=f_{l'}$ and $e_i=0$ otherwise. For the convolution in p-space we have to integrate over $k\in R^4$

$$P_{l'}\Big(-i\frac{\partial}{\partial r_{n+1}}\Big)\exp[i\alpha_{l'}(k^2 - m_{l'}^2 + i\varepsilon) + ir_{n+1}k]_{r_{n+1}} = 0 \tag{3.18}$$

multiplied with an expression similar to (3.16) but with p_i replaced by $p_i + e_ik$. Then (3.16) times $P_{l'}\Big(-i\dfrac{\partial}{\partial r_{n+1}}\Big)\exp[-i\alpha_{l'}(m_{l'}^2 - i\varepsilon)]$ operates on

$$\int dk\exp\big[i(\alpha_{l'}+\sum A_{ij}^{(n)}e_ie_j)\,k^2 + i(2\sum A_{ij}^{(n)}p_ie_j + \\ + \sum C_{ik}^{(n)}e_ir_k + r_{n+1})k\big]_{r=0}\,. \tag{3.19}$$

The Gaussian integral (3.19) can be evaluated by using the identity for

$a > 0$:

$$\lim_{\eta \downarrow 0} \int dk \exp[iak^2 + ipk - \eta(k_0^2 + \underset{\sim}{k}^2)] = \frac{\pi^2}{ia^2} \exp\left[-i \frac{p^2}{4a}\right]. \quad (3.20)$$

Then one obtains again an expression of the form (3.16) with $\prod_{\mathcal{N}-\mathcal{M}} D_l^{-2}$ multiplied by $D_{l'}^{-2}$, where

$$D_{l'} = (\alpha_{l'} + \sum A_{ij}^{(n)} e_i e_j). \quad (3.21)$$

Furthermore

$$P^{(n+1)}(-i\nabla) = P^{(n)}(-i\nabla) P_{l'}\left(-i\frac{\partial}{\partial r_{n+1}}\right) \quad (3.22)$$

$$A_{ii'}^{(n+1)} = A_{ii'}^{(n)} - D_{l'}^{-1}\left[\sum_{jj'} A_{ij}^{(n)} e_j A_{i'j'}^{(n)} e_{j'}\right] \quad (3.23)$$

$$B_{kk'}^{(n+1)} = B_{kk'}^{(n)} - \frac{1}{4} D_{l'}^{-1}\left[\sum_{i,i'} C_{ik}^{(n)} e_i C_{i'k'}^{(n)} e_{i'}\right], \quad 1 \leqq k, k' \leqq n,$$

$$B_{n+1,k}^{(n+1)} = B_{k,n+1}^{(n+1)} = \frac{1}{4} D_{l'}^{-1}\left[\sum_i C_{ik}^{(n)} e_i\right] \quad (3.24)$$

$$B_{n+1,n+1}^{(n+1)} = -\frac{1}{4} D_{l'}^{-1}$$

$$C_{ik}^{(n+1)} = C_{ik}^{(n)} - D_{l'}^{-1}\left[\sum_{j,j'} A_{ij}^{(n)} e_j C_{j'k}^{(n)} e_{j'}\right], \quad 1 \leqq k \leqq n,$$

$$C_{i,n+1}^{(n+1)} = -D_{l'}^{-1} \sum_j A_{ij}^{(n)} e_j. \quad (3.25)$$

Since $A^{(n)}$ is positive semidefinite, $D_{l'} \geqq \alpha_{l'}$ satisfies again (3). Since $l' \in \mathcal{L}(I) - \mathcal{M}$, $\alpha_{l'} \geqq \alpha_l$ for all $l \in \mathcal{M}_1(I)$ by Lemma 2.4. As $A_{ij}^{(n)}$ is majorized by (3.5) for some $c_n < \infty$ and $D_{l'} \geqq \alpha_{l'}$, $D_{l'}^{-1} A_{ij}^{(n)}$ is uniformly bounded in (3.1) for $r \geqq 0$ and $A_{ij}^{(n+1)}$ satisfies (5). The positive semidefiniteness of $A^{(n+1)}$ follows from the inequality

$$\sum_{i,j} A_{ij}^{(n+1)} x_i x_j \geqq D_{l'}^{-1}\left[\sum_{i,i',j,j'} A_{ij}^{(n)} x_i x_j A_{i'j'}^{(n)} e_{i'} e_{j'} - \left(\sum_{i,j} A_{ij}^{(n)} x_i e_j\right)^2\right] \geqq 0. \quad (3.26)$$

Similarly one checks that $B_{kl}^{(n+1)}$ satisfies (7) and that the $C_{ik}^{(n+1)}$ are uniformly bounded in (3.1) for $r \geqq 0$, rational in α, τ, and homogeneous of degree 0 in α. Therefore $\prod_{(I,i)\in\mathcal{S}} \mathcal{F}_{\mathcal{M}}(U(I,i)) \prod_{\mathcal{L}(I)} \Delta_l$ has in p-space the form (3.13) with (3.16) and $\mathcal{N} = \mathcal{L}(I) - \mathcal{M}$, $n = |\mathcal{L}(I)|$ inserted instead of $P^{(r)} \exp[\ldots]$. We set $A_{ij}^{\mathrm{I}} = A_{ij}^{(n)}$, $B_{ik}^{\mathrm{I}} = B_{ik}^{(n)}$, $D^{\mathrm{I}} = \prod_{\mathcal{L}(I)-\mathcal{M}} D_l^{-2}$.

Finally we carry out the r-differentiation. The only terms which survive after setting all $r_k = 0$, have in (3.13) $P^{(n)}(-i\nabla)$ replaced by a sum of terms

$$Q(\alpha, \tau) P(p) S^{\mathrm{I}}(B^{\mathrm{I}}), \quad (3.27)$$

where Q is a polynomial in the $C_{ik}^{(n)}$, $P(p)$ a monomial in the p_i, $V_i \in \mathcal{V}_0(I)$, of degree $z(I)$ and $S^{\mathrm{I}}(B^{\mathrm{I}})$ a polynomial in the B_{kl}^{I} of degree $y(I)$ with

$$2y(I) + z(I) \leqq \sum_{(I,i)\in\mathcal{S}} [2x(I,i) + y(I,i)] + \sum_{l\in\mathcal{L}(I)} r_l. \quad (3.28)$$

The inequality (3.28) holds, since every r-differentiation of

$$\exp[i \sum B_{kl} r_k r_l + i \sum C_{ik} p_i r_k]$$

brings down factors like $B_{kl} r_k$ and $C_{ik} p_i$. Each $B_{kl} r_k$ has to be differentiated again in order to survive for $r_k \to 0$. Thus $\deg P(p) + 2 \deg S^{\mathrm{I}}(B^{\mathrm{I}}) = z(I) + 2y(I) \leqq \deg P^{(n)}$.

Let $\sigma(U(I)) = -1$. If we apply the operation $(-M)$, we obtain instead of $Q(\alpha, \tau)\, P(p)\, S^{\mathrm{I}}(B^{\mathrm{I}}) \exp[i \sum A^{\mathrm{I}}_{ij} p_i p_j]$ a sum of terms of the form

$$Q^{\mathrm{I}}(\alpha, \tau)\, P^{\mathrm{I}}(p)\, R^{\mathrm{I}}(A^{\mathrm{I}})\, S^{\mathrm{I}}(B^{\mathrm{I}})\,. \tag{3.29}$$

Here $P^{\mathrm{I}}(p)$ satisfies (1), and is zero for $z(I) > \nu(I)$ and otherwise of degree

$$2x(I) + z(I) \leqq \nu(I)\,. \tag{3.30}$$

Furthermore Q^{I} satisfies (2); R^{I}, (4); and S^{I}, (6). In the case $\sigma(U(I)) = +1$ we use for the remainder of the Taylor series of $f(x_1, \ldots x_m)$ around $(0, \ldots 0)$ up to order n the expression

$$\frac{1}{n!} \int_0^1 d\tau (1-\tau)^n \frac{\partial^{n+1}}{\partial \tau^{n+1}} f(\tau x_1, \ldots \tau x_m)\,. \tag{3.31}$$

This leads to (3.4) with $x(I) \geqq \left[\left[\frac{\nu(I) - z(I) + 1}{2}\right]\right]$. Q.E.D.

Lemma 3.2: Let $\mathscr{R}_{\mathscr{L}}(V_1, \ldots V_n) = \sum_T \mathscr{F}_T(V_1, \ldots V_n)$ hold as above in (3.1). Then the α-integrand of every $\mathscr{F}_T(V_1, \ldots V_n)$ is, together with all p-derivatives, absolutely integrable for $r \downarrow 0$.

Proof: Let $\lambda = (2L)^{-1}$. If we use Lemma 3.1 for $\mathscr{F}_{\mathscr{M}}(U(I))$, then

$$\prod_{l \in \mathscr{L}_0(I) - \mathscr{M}} D_l^{\lambda - 1} \prod_{I' \leqq I} \prod_{l \in \mathscr{M}(I')} \alpha_l^{\lambda - 1} \tag{3.22}$$

is locally integrable in (3.1) for $r \downarrow 0$. As $\exp[-i \sum \alpha_l (m_l^2 - i\varepsilon)]$ is strongly decreasing at infinity for $\varepsilon > 0$, we have only to show that in every bough $U(I)$

$$\prod_{I' \leqq I} \left[\prod_{l \in \mathscr{L}(I') - \mathscr{M}} D_l^{-(1+\lambda)}\right] \left[\prod_{l \in \mathscr{M}(I')} \alpha_l^{(1-\lambda)}\right] R^{\mathrm{I}'}(A^{\mathrm{I}'})\, S^{\mathrm{I}'}(B^{\mathrm{I}'}) = T^{\mathrm{I}} \tag{3.33}$$

remains continuous and bounded for $r \downarrow 0$.

We use induction with respect to the length $s(I)$ of the longest chain of branches contained properly in $U(I)$ and make the following assumption for any $U(I)$, $0 \leqq s(I) < s$:

(1) If $\sigma(U(I)) = 0$, then $T^{\mathrm{I}} = 1$.

(2) If $\sigma(U(I)) = +1$, then (3.33) is of the form

$$F^{\mathrm{I}}(\alpha, \tau)\, G^{\mathrm{I}}(\alpha, \tau)\,, \tag{3.34}$$

where F^{I} and G^{I} are tempered continuous functions in (3.1) for $r \downarrow 0$. F^{I} is homogeneous in $A^{\mathrm{I}'}_{ij}$, $U(I') \in \mathfrak{U}_1(I)$ and α_l, $l \in \mathscr{M}_1(I)$ of degree $\frac{1}{2}[2x(I) + z(I) - \nu(I)] - \lambda |\mathscr{L}_0(I)|]$, which is $\geqq 0$ by (3.7).

(3) If $\sigma(U(I))=-1$, then (3.33) multiplied with any $H^I(\alpha,\tau)$ is tempered and continuous in (3.1) for $r\downarrow 0$. Here $H^I(\alpha,\tau)$ is homogeneous of degree $\frac{1}{2}[\nu(I)-2x(I)-z(I)]+\lambda|\mathscr{L}_0(I)|$ ($\geqq 0$ by (3.8)) in the variables $A^{\tilde{I}}_{ij}$, $U(\tilde{I})\in\mathfrak{U}_1(I)\cup\mathfrak{U}_1(I')\ldots\cup\mathfrak{U}_1(I^k)$, and α_l, $l\in\mathscr{M}_1(I)\cup\cdots\cdots\cup\mathscr{M}_1(I^k)$, where $U(I^k)$ is the bough of maximal order containing $U(I)$ and $U(I^k)\supset\cdots\supset U(I')\supset U(I)$ the chain of twigs between $U(I)$ and $U(I^k)$. These assumptions are satisfied for $s=0$. Consider a branch $U(I)$ with $s(I)=s>0$ and write (3.33) in the form

$$\prod_{l\in\mathscr{L}(I)-\mathscr{M}} D_l^{-(1+\lambda)}\prod_{l\in\mathscr{M}(I)}\alpha_l^{(1-\lambda)}R^I(A^I)\,S^I(B^I)\prod_{(I,i)\in\mathscr{I}}T^{(I,i)}(\alpha,\tau)\,. \tag{3.35}$$

By induction assumption each $T^{(I,i)}(\alpha,\tau)$ is of the form $F^{(I,i)}(\alpha,\tau)\times\times G^{(I,i)}(\alpha,\tau)$, if $\sigma(U(I))=+1$, with $\deg F^{(I,i)}=\frac{1}{2}[2x(I,i)+z(I,i)--\nu(I,i)]-\lambda|\mathscr{L}_0(I,i)|$.

Let $U(I)$ be a bough and assume that

$$\prod_{\mathscr{L}(I)-\mathscr{M}} D_l^{-1-\lambda}\prod_{\mathscr{M}(I)}\alpha_l^{(1-\lambda)}R^I(A^I)\,S^I(B^I)\prod_{\sigma(U(I,i)=1}F^{(I,i)}(\alpha,\tau) \tag{3.36}$$

is of the form

$$F^I(\alpha,\tau)\,K^I(\alpha,\tau)\prod_{\sigma(U(I,i))=-1}H^{(I,i)}(\alpha,\tau) \tag{3.37}$$

with $F^I, H^{(I,i)}$ as in (2), (3), and K^I continuous and tempered in (3.1) for $r\downarrow 0$. This is sufficient for performing the induction step, since for all twigs $U(I,i)$ the $H^{(I,i)}$ can be used by (3) to make bounded the $T^{(I,i)}$ (the bough of maximal order containing $U(I,i)$ is $U(I)$). Now

$$\prod_{\mathscr{M}(I)}\alpha_l^{(1-\lambda)}R^I(A^I)\prod_{\sigma(U(I,i))=1}F^{(I,i)}(\alpha,\tau) \tag{3.38}$$

is homogeneous in $A^{I'}_{ij}$, $U(I')\in\mathfrak{U}_1(I)$, and α_l, $l\in\mathscr{M}_1(I)$, since for $\sigma(U(I))=+1$ $\mathfrak{U}_1(I,i)\subset\mathfrak{U}_1(I)$. The degree of (3.38) is

$$\begin{aligned}&(1-\lambda)\,|\mathscr{M}(I)|+x(I)+\\ &+\sum_{\sigma(U(I,i))=1}\left[x(I,i)+\frac{z(I,i)-\nu(I,i)}{2}-\lambda\,|\mathscr{L}_0(I,i)|\right].\end{aligned} \tag{3.39}$$

By (3.7) and (3.8)

$$x(I)\geqq\frac{\nu(I)-z(I)}{2}+\frac{1}{2}\geqq\frac{\nu(I)-z(I)}{2}+\lambda\,|\mathscr{L}_0(I)|\,, \tag{3.40}$$

$$z(I)\leqq\sum_{\mathscr{L}(I)}r_l+\sum_{(I,i)\in\mathscr{I}}[2x(I,i)+z(I,i)]-2y(I)\,, \tag{3.41}$$

and (3.39) is larger than

$$\begin{aligned}&\geqq(1-\lambda)\,|\mathscr{M}(I)|+\frac{\nu(I)}{2}+\lambda|\mathscr{L}_0(I)|-\frac{1}{2}\sum_{\mathscr{L}(I)}r_l+y(I)+\\ &+\sum_{\sigma(U(I,i))=1}\left[x(I,i)+\frac{z(I,i)-\nu(I,i)}{2}-\lambda|\mathscr{L}_0(I,i)|\right]-\\ &-\sum_{(I,i)\in\mathscr{I}}\left[x(I,i)+\frac{z(I,i)}{2}\right]+\left[x(I)+\frac{z(I)-\nu(I)}{2}-\lambda|\mathscr{L}_0(I)|\right].\end{aligned} \tag{3.42}$$

We remark that $|\mathscr{V}_0(I)| - |\mathscr{M}(I)| - 1 = \sum [|\mathscr{V}_0(I, i)| - 1]$. Therefore we obtain from (3.3)

$$\nu(I) = \sum_{(I,i)\in\mathscr{I}} \nu(I, i) + \sum_{\mathscr{L}(I)} (r_l + 2) - 4|\mathscr{M}(I)| . \tag{3.43}$$

Using (3.43) and the relation $|\mathscr{L}_0(I)| - |\mathscr{L}(I)| = \sum |\mathscr{L}_0(I, i)|$ one transforms (3.42) into

$$\begin{aligned} &= y(I) + (1+\lambda)\,[|\mathscr{L}(I)| - |\mathscr{M}(I)|] + x(I) + \frac{z(I) - \nu(I)}{2} - \\ &- \lambda|\mathscr{L}_0(I)| + \sum_{\sigma(I,i)=-1} \left[\frac{\nu(I,i) - z(I,i)}{2} - x(I,i) + \lambda|\mathscr{L}_0(I,i)|\right]. \end{aligned} \tag{3.44}$$

Therefore we can split off from (3.38) a factor $F^I \cdot \prod_{\sigma(U(I,i))=-1} H^{(I,i)}$. The remainder $L^I(\alpha, \tau)$ is homogeneous of degree $\geqq y(I) + (1+\lambda)\,[|\mathscr{L}(I)| - |\mathscr{M}(I)|]$ in $A^{I'}_{ij}$, $U(I') \in \mathfrak{U}_1(I)$ and α_l, $l \in \mathscr{M}_1(I)$. By Lemma 3.1 these $A^{I'}_{ij}$ are of the order of the α_l, $l \in \mathscr{M}_1(I)$. Therefore by Lemma 2.4 L^I can be used to make bounded the divergent part of (3.36)

$$\prod_{\mathscr{L}(I)-\mathscr{M}} D_l^{-1-\lambda} S^I(B^I) \tag{3.45}$$

of degree $y(I) + (1+\lambda)\,[|\mathscr{L}(I)| - |\mathscr{M}(I)|]$.

Finally let $\sigma(U(I)) = -1$. Then it follows from the induction hypothesis that

$$\prod_{\mathscr{M}(I)} \alpha_l^{1-\lambda} R^I(A^I) \prod_{\sigma(U(I,i))=1} F^{(I,i)}(\alpha,\tau)\, H^I(\alpha,\tau) \tag{3.46}$$

is homogeneous in α_l, $l \in \mathscr{M}(I')$, and $A^{I'}_{ij}$, where $U(I') \in \mathfrak{U}_1(I) \cup \cdots \cdots \cup \mathfrak{U}_1(I^k)$. By Lemma 2.4 and 3.1 one can use (3.46) to make bounded the divergent term $\prod D_l^{-1-\lambda} S^I(B^I)$ and to produce a factor

$$\prod_{\sigma(U(I,i))=-1} H^{(I,i)}(\alpha,\tau)\,,$$

which depends on the right $A^{I'}_{ij}$, α_l for the induction step. A power counting as before shows that the degree of (3.46) is high enough for these operations. Q.E.D.

We remark that in every bough the coefficient of $\exp[\ldots]$ in (3.4) is homogeneous of degree $> -|\mathscr{L}_0(I)| + 1$ in α.

4. Renormalization

We have now accumulated enough information to discuss the limits $\varepsilon \downarrow 0$ and $r \downarrow 0$ in the $\mathscr{R}$-operation and to turn to the properties of the sum of all graph contributions in the perturbation expansion of $\langle T\varphi_1(x_1) \ldots \varphi_m(x_m)\rangle^T$.

Let us assume that $\lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n)$ exists in the topology of $\mathscr{S}'(R^{4n})$. For r, $\varepsilon > 0$ the subtractions in $\mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n)$ corresponding to partitions of $\{V_1, \ldots V_n\}$ different from $\{V_1\}, \ldots \{V_n\}$ lead to functionals

$$\varphi \in \mathscr{S}(R^{4n}) \to \left\langle \prod_{j=1}^{k(P)} \mathscr{X}^{r,\varepsilon}_{\mathscr{L}}(V^P_{j1}, \ldots V^P_{jr(j)}) \prod_{\text{conn}} \Delta^{r,\varepsilon}_l, \varphi \right\rangle, \tag{4.1}$$

which vanish identically on a certain subspace $\mathscr{S}_N(R^{4n}) \subset \mathscr{S}(R^{4n})$. Here $N \geqq 0$ is determined by the $\{\nu(V'_1, \ldots V'_m)\}$. $\mathscr{S}_N(R^{4n})$ consists of all $\varphi \in \mathscr{S}(R^{4n})$, for which $D\varphi(x_1, \ldots x_n) = 0$ whenever some $x_i = x_j$, $1 \leqq i < j \leqq n$, for all differential monomials D in the $\partial/\partial x^\mu_i$ of degree $\leqq N$. $\mathscr{S}_N(R^{4n})$ is a closed subspace of $\mathscr{S}(R^{4n})$ with the induced topology. It follows from (4.1) that

$$\lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n) = \lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \prod \Delta^{r,\varepsilon}_{\mathscr{L}} \tag{4.6}$$

holds on $\mathscr{S}_N(R^{4n})$. Therefore [6] $\prod_{\mathscr{L}} \Delta^F_l$ is a continuous linear functional on $\mathscr{S}_N(R^{4n})$ and $\lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \mathscr{R}^{\varepsilon,r}_{\mathscr{L}}(V_1, \ldots V_n)$ is its continuation to $\mathscr{S}(R^{4n})$.

By Lemma 3.1 $\mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n)$ is in p-space a sum of terms

$$\delta\left(\sum_{i=1}^{n} p_i\right) P(p)\, Q(\alpha, \tau) \exp\left[i \sum_{ij=1}^{n-1} A_{ij} p_i p_j - i \sum_{\mathscr{L}} \alpha_l (m^2_l - i\varepsilon)\right] \tag{4.3}$$

integrated over all $0 \leqq \tau(I) \leqq 1$ and over a sector (3.1). By Lemma 3.2 the $A_{ij} = A_{ij}(\alpha, \tau)$ are continuous in (3.1) for $r \downarrow 0$ and homogeneous of degree $+1$ in the α. $P(p)$ is a monomial in $p_1, \ldots p_{n-1}$; $Q(\alpha, \tau)$ is rational, homogeneous of degree $d \geqq (-L+1)$ in α, tempered and locally integrable in (3.1) for $r \downarrow 0$. Therefore $\lim_{r \downarrow 0} \mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n)$ exists in the ordinary sense and is (apart from $\delta(\sum p_i)$) a function in $\mathscr{O}_M(R^{4(n-1)})$.

We could have chosen a more general regularization for $\tilde{\Delta}^F_l(p)$

$$\tilde{\Delta}^{\varrho,\varepsilon}_l(p) = P_l(p) \int_0^\infty d\alpha\, f_\varrho(\alpha) \exp[i\alpha(p^2 - m^2_l + i\varepsilon)], \tag{4.4}$$

where $f_\varrho(\alpha)$ is continuous and vanishes of sufficiently high order for $\alpha \downarrow 0$ for all ϱ in, say, $0 < \varrho \leqq 1$, where $|f_\varrho(\alpha)| < c(1 + |\alpha|)^N$ for some c, $N < \infty$ uniformly for all $0 \leqq \varrho \leqq 1$ and $0 \leqq \alpha < \infty$ and where $f_\varrho(\alpha) \to 1$ pointwise for $\varrho \downarrow 0$. Then one obtains for an obviously defined $\tilde{\mathscr{R}}^{\varrho,\varepsilon}$-operation

$$\lim_{\varrho \downarrow 0} \tilde{\mathscr{R}}^{\varrho,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n) = \lim_{r \downarrow 0} \mathscr{R}^{r,\varepsilon}_{\mathscr{L}}(V_1, \ldots V_n) \tag{4.5}$$

by the Lebesgue dominated convergence theorem.

The limit $\varepsilon \downarrow 0$ of $\mathscr{R}^{0,\varepsilon}_{\mathscr{L}}(V_1, \dots V_n)$ in $\mathscr{S}'(R^{4n})$ has been discussed in [1], [2]. Unfortunately the argument relies on a splitting of the testing functions $\varphi \in \mathscr{S}(R^{4n})$, (see [1], (4.39)), which is in general impossible. But the limit can be expressed directly in terms of elementary distributions. For the proof we discuss a more general situation: We restrict in (4.3) the $(p_1, \dots p_{n-1})$ to some m-dimensional linear manifold, $0 \leqq m \leqq 4(n-1)$, by setting

$$p_i^\mu = \sum_{j=1}^{m} c_{j,4(i-1)+\mu} q_j, \; 1 \leqq i \leqq n-1, 0 \leqq \mu \leqq 3\,, \tag{4.6}$$

where rank $(c_{jk}) = m$. For any $\psi \in \mathscr{S}(R^m)$ we study

$$\begin{aligned} F_\varepsilon(\psi) = \int dq_1 \dots dq_m\, \psi(q_1, \dots q_m) \int_0^1 \dots \int_0^1 \Pi\, d\tau(I) \int \dots \int_{\alpha_{l_1} \geqq \cdot\, \alpha_{l_L} \geqq 0} \Pi\, d\alpha_l \times \\ \times P'(q)\, Q(\alpha, \tau) \exp\left[i \sum_{i,j=1}^{m} A'_{ij} q_i q_j - i \sum_{l=1}^{L} \alpha_l (m_l^2 - i\varepsilon)\right]. \end{aligned} \tag{4.7}$$

Here $P'(q) = P(p(q))$ and $\sum A'_{ij} q_i q_j = \sum A_{ij} p(q)_i\, p(q)_j$. If $\lim_{\varepsilon \downarrow 0} F_\varepsilon(\psi)$ exists for all $\psi \in \mathscr{S}(R^m)$, then [3] the mapping $\psi \to \lim_{\varepsilon \downarrow 0} F_\varepsilon(\psi)$ is a distribution in $\mathscr{S}'(R^m)$ and is the strong limit for $\varepsilon \downarrow 0$ of $\mathscr{R}^{0,\varepsilon}_{\mathscr{L}}(V_1, \dots V_n)$ with the coefficient of $\delta(\sum p_i)$ restricted in p-space to the manifold (4.6).

For $\varepsilon > 0$ we can freely interchange integrations in (4.7). After the coordinate transform

$$\alpha_l = \lambda \beta_l, \;\; 1 \leqq l \leqq L, \sum_{l=1}^{L} \beta_l = 1, \frac{\partial(\alpha_1, \dots \alpha_L)}{\partial(\beta_1, \dots \beta_{L-1}, \lambda)} = \lambda^{L-1} \tag{4.8}$$

we first integrate (4.7) over $0 \leqq \lambda < \infty$. This leads to the Feynman integral.

$$\begin{aligned} F_\varepsilon(\psi) = \int_0^1 \dots \int_0^1 \Pi\, d\tau(I) \int_{1 \geqq \beta_{l_1} \geqq \cdots \beta_{l_L} \geqq 0} \dots \int \Pi\, d\beta_l\, \delta(1 - \textstyle\sum \beta_l)\, Q(\beta, \tau) \times \\ \times F_\varepsilon(\psi, \beta, \tau) \end{aligned} \tag{4.9}$$

$$F_\varepsilon(\psi, \beta, \tau) = c \int dq_1 \dots dq_m \frac{P'(q)\, \psi(q_1, \dots q_m)}{[\Sigma A'_{ij}(\beta, \tau)\, q_i q_j - \Sigma \beta_l m^2 + i\varepsilon]^{d+L}}\,. \tag{4.10}$$

Since $Q(\beta, \tau)$ is locally integrable, it is sufficient to show that $\lim_{\varepsilon \downarrow 0} F_\varepsilon(\psi, \beta, \tau)$ exists and is continuous in β and τ in the compact region of integration in (4.9). Since $m_l^2 > 0$, $1 \leqq l \leqq L$, the singular support of (4.10)

$$\left\{\sum_{i,j=1}^{m} A'_{ij}(\beta, \tau)\, q_i q_j = \sum_{l=1}^{L} \beta_l m_l^2\right\} \tag{4.11}$$

is an analytic manifold in $(q_1, \dots q_m)$ for fixed β and τ. Thus $\lim_{\varepsilon \downarrow 0} F_\varepsilon(\psi, \beta, \tau)$ exists and is in local coordinates expressible in terms of derivatives of δ-functions and principal values. As the $A'_{ij}(\beta, \tau)$ are continuous in the region of integration in (4.9), the intersection of (4.11) with any compact subset of R^m varies continuously with β and τ. This proves the continuity of $\lim_{\varepsilon \downarrow 0} F_\varepsilon(\psi, \beta, \tau)$ in β and τ, since ψ is strongly decreasing at infinity, where (4.11) is well-behaved.

We now return to the Gell-Mann Low expansion (1.1) of $\langle T \varphi_1(x_1) \ldots \ldots \varphi_m(x_m)\rangle^T$, where we define $\langle T \varphi_1^{\mathrm{I}}(x_1) \ldots \mathscr{H}^{\mathrm{I}}(x_n)\rangle^T$ by a sum of $\lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \mathscr{R}_{\mathscr{L}}^{r,\varepsilon}(V_1, \ldots V_n)$. The integration over $x_{m+1}, \ldots x_n$ in (1.1) corresponds in p-space to a restriction to a $4m$-dimensional linear manifold in R^{4n}. The preceeding argument applies:

Theorem: The pertubation-theoretic Green's functions $\langle T \varphi_1(x_1) \ldots \ldots \varphi_m(x_m)\rangle^T_{(n)}$ of order n defined by the $\mathscr{R}$-operation are in p-space (up to $\delta(\sum p_i)$) Lorentz covariant boundary values in $\mathscr{S}'(R^{4(m-1)})$ of sums of Feynman integrals, which are analytic in $p_1, \ldots p_{m-1}$ without natural boundaries.

Remark: The analyticity properties in p-space follow from (4.3). Landau rules for these renormalized Feynman amplitudes can be worked out, but will not be discussed here.

The combinatorial structure of the subtractions in the $\mathscr{R}$-operation and their relation to formal counter terms in $\mathscr{H}^{\mathrm{I}}(x)$ have been lucidly treated by BOGOLIUBOV and SHIRKOV [10]. It is noteworthy that the $\mathscr{R}$-operation can also be applied to theories, which are conventionally considered as non-renormalizable. The distinction of the renormalizable theories in the restricted sense follows from the usual power counting theorem [7], [10], by which these theories have counter terms to $\mathscr{H}^{\mathrm{I}}(x)$, which involve only a finite number of WICK polynomials in the $\varphi_i^{\mathrm{I}}(x)$ uniformly for all orders n.

Example: As an illustration of the method of BOGOLIUBOV and PARASIUK we shall discuss the quartic self-interaction of a neutral scalar field. It is generally believed that the φ^4-theory [9] is in perturbation theory up to order n uniquely characterized by renormalizing

$$\tau(x_1, \ldots x_m)^T_{(n)} = \sum_{k=0}^{n} \frac{(-i)^k}{k!} \int dy_1 \ldots dy_k \langle T \varphi^{\mathrm{I}}(x_1) \ldots \mathscr{H}^{\mathrm{I}}(y_k)\rangle^T \quad (4.12)$$

using counter terms to $\mathscr{H}^{\mathrm{I}}(y) = -\frac{g}{4!} : \varphi^{\mathrm{I}}(y)^4:$ of the type

$$A : \varphi^{\mathrm{I}}(y)^2 : + B \,\Box : \varphi^{\mathrm{I}}(y)^2 : + C : \varphi^{\mathrm{I}}(y)^4 : .$$

Here g is the physical coupling constant and $\varphi^{\mathrm{I}}(y)$ is a neutral scalar free field with the physical mass m and

$$\langle T\,\varphi^{\mathrm{I}}(x_1)\,\varphi^{\mathrm{I}}(x_2)\rangle^T = i\Delta^F(m; x_1 - x_2) = \frac{i}{(2\pi)^4}\int \frac{dp\,\exp[-ip(x_1 - x_2)]}{p^2 - m^2 + i0}\,. \tag{4.13}$$

The "renormalization constants"

$$A = \sum_{k=2}^{n} A_k g^k,\ B = \sum_{k=2}^{n} B_k g^k,\ C = \sum_{k=2}^{n} C_k g^k$$

are determined by three conditions on the Green's functions $\tau(x_1, \ldots x_m)^T_{(k)}$ for $2 \leqq k \leqq n$: (A, B): The 2-point function $\tilde{\tau}(p_1, p_2)^T_{(k)} = \delta(p_1 + p_2)\,\tilde{\tau}(p_1^2)_{(k)}$ must have a pole at $p_1^2 = m^2$ with residue $(2\pi)^{-2}$

$$(2\pi)^2\,(p_1^2 - m^2)\,\tilde{\tau}(p_1^2)_{(k)} = 1 \quad \text{for} \quad p_1^2 = m^2\,. \tag{4.14}$$

(C): The 2-body scattering amplitude

$$\langle p_1, p_2^{\mathrm{out}} | -p_3, -p_4^{\mathrm{in}}\rangle_{(k)} - \langle p_1, p_2^{\mathrm{in}} | -p_3, -p_4^{\mathrm{in}}\rangle_{(k)} = i\delta(\textstyle\sum p_i)\,\tilde{T}(p_1, \ldots p_4)_{(k)} = (2\pi)^2 \displaystyle\prod_{i=1}^{4}(p_i^2 - m^2)\,\tilde{\tau}(p_1, \ldots p_4)^T_{(k)} \tag{4.15}$$

is normalized at $p_i^2 = m^2$, $(p_i + p_j)^2 = \frac{4m^2}{3}$, $i < j$, to

$$\tilde{T}(p_1, \ldots p_4)_{(k)} = \frac{g}{(2\pi)^2}\,. \tag{4.16}$$

The conditions (4.14), (4.16) are satisfied in order $n < 2$, but not for general n, if we define $\tau(x_1, \ldots x_m)^T_{(n)}$ by the $\mathscr{R}$-operation of section 1. The remedy is to use the freedom of the choice of the point around which the Taylor expansion is taken in (1.9). The results of the preceeding sections remain valid, if one defines a more general $\mathscr{R}$-operation [10] by adding recursively to any $-M\overline{\mathscr{R}}^{r,\varepsilon}_{\mathscr{L}}(V_1', \ldots V_m')$ a distribution $\tilde{\mathscr{X}}_{\mathscr{L}}(V_1' \ldots V_m')$, which is in p-space of the form $\delta(\sum p_i')\,\tilde{P}(p_1', \ldots p_m')$ with a covariant polynomial $\tilde{P}$ of degree $\leqq \nu(V_1', \ldots V_m')$ depending only on the structure of $G(V_1', \ldots V_m', \mathscr{L})$.

Theorem: In the φ^4-theory there exists a choice of the finite renormalizations consistent with (4.14) and (4.16), and any such choice leads to the same renormalized Green's distributions $\tau(x_1, \ldots x_m)^T_{(n)}$ for all m, n.

Proof: In the φ^4-theory $\nu(V_1', \ldots V_m') = 2$ or 0 for subgraphs $G(V_1', \ldots V_m', \mathscr{L})$ with two or four external lines, respectively. Otherwise no over-all subtractions are necessary.

We can always enforce (4.14) and (4.16), since the renormalized Feynman amplitudes for the self energy (SE) and vertex (V) parts are analytic around $p_1^2 = m^2$ and the Chew-Mandelstam point, respectively. Furthermore in any order $n \geqq 2$ there exist SE- and V-parts, which require a new over-all subtraction, e.g. the graphs of Fig. 7. In second

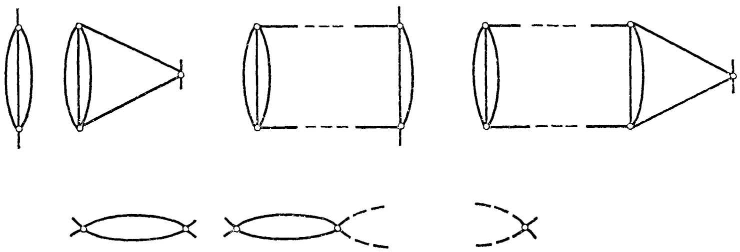

Fig. 7. *SE*- and *V*-parts

order the finite renormalizations are uniquely determined by (4.14) and (4.16), since there is only one divergent SE- and V-part. In third order the following SE-parts require an over-all subtraction:

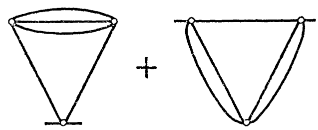

Fig. 8. Independent *SE*-parts in 3rd order

Only the sum of the counter terms, a polynomial $A_3^{\gamma,\varepsilon} + B_3^{\gamma,\varepsilon} p_1^2$, is uniquely determined. We shall see that this "renormalization gauge" leaves the renormalized Green's functions (4.12) invariant.

Consider all graphs $\mathfrak{S}_{m,n}$ of n^{th} order in the Wick expansion of (4.12). The analytical contribution of each graph is defined by the $\mathcal{R}$-operation, where the combinatorics of the counter terms is determined

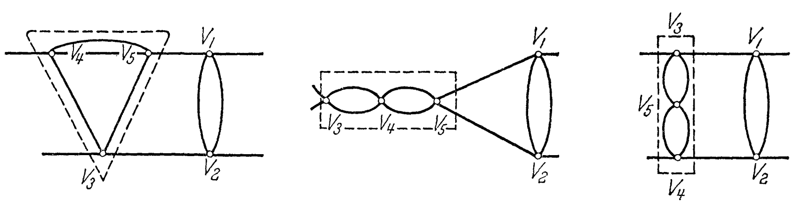

Fig. 9. Typical terms in $\mathfrak{S}_\gamma(V_1, V_2, \{V_3, V_4, V_5\})$

by the internal vertices $V_1, \ldots V_n$. For any partition

$$U_1 = \{V_{11}, \ldots V_{1r(1)}\}, \ldots U_s = \{V_{s1}, \ldots V_{sr(s)}\}$$

of $\{V_1, \ldots V_n\}$ into $s < n$ generalized vertices a certain subclass $\mathfrak{S}(U_1, \ldots U_s)$ of $\mathfrak{S}_{m,n}$ requires a subtraction. This set falls into equivalence classes $\mathfrak{G}_\gamma(U_1, \ldots U_s)$, where two graphs $G_1, G_2 \in \mathfrak{G}_\gamma(U_1, \ldots U_s)$ differ only in the structure of the internal lines and vertices in $U_1, \ldots U_s$.

Example: Graphs belonging to the same $\mathfrak{G}_\gamma(V_1, V_2, \{V_3, V_4, V_5\})$. The Wick expansion of $\langle T\varphi^I(x_1) \ldots \mathscr{H}^I(y_n)\rangle^T$ contains all possible connected contractions, where the m lines from $x_1, \ldots x_m$ and the $4n$ lines from $y_1, \ldots y_n$ connect different vertices. Thus $\mathfrak{G}_\gamma(U_1, \ldots U_s)$ is either empty or contains exactly once all possible contractions between the $4r(j)$ lines from the $r(j)$ vertices in U_j, $1 \leqq j \leqq s$, which connect different vertices and lead to IPI SE- or V-parts, respectively.

The analytical form of the sums of counter terms for all graphs in $\mathfrak{G}_\gamma(U_1, \ldots U_s)$, which arise from the subtractions associated with the generalized vertices $U_1, \ldots U_s$, is for r, $\varepsilon > 0$ in p-space of the form

$$\delta(\sum p_i) \int dk_1 \ldots dk_t \prod_{\text{conn}} \tilde{\Delta}_l^{r,\varepsilon}(q') \prod_{U_i:SE} (A^{r,\varepsilon}_{r(i)} + B^{r,\varepsilon}_{r(i)} q''^2) \prod_{U_j:V} C^{r,\varepsilon}_{r(j)}, \quad (4.17)$$

where $k_1, \ldots k_t$ are loop momenta, the q', q'' are linear combinations of the $k_1, \ldots k_t$ and the external momenta $p_1, \ldots p_m$, $\prod_{\text{conn}}$ extends over all lines which are not contained in $U_1, \ldots U_s$, $\prod_{U_i:SE}$ over all SE-parts U_i and $\prod_{U_j:V}$ over all V-parts U_j (with $r(j) > 1$) of the partition $U_1, \ldots U_s$. The essential observation is that the SE-contributions $A^{r,\varepsilon}_{r(i)} + B^{r,\varepsilon}_{r(i)} q''^2$ in (4.17) are exactly the same as the sum of all over-all subtractions in $\mathscr{R}^{r,\varepsilon}_{\mathscr{L}} \tau(x_1, x_2)^T_{(r(i))}$ from order $r(i)$ and similarly for $C^{r,\varepsilon}_{r(i)}$ in any V-part U_j.

We know that $A^{r,\varepsilon}_n$, $B^{r,\varepsilon}_n$, $C^{r,\varepsilon}_n$ are uniquely determined by (4.14), (4.16) for $n = 2$. Assume that the same holds in orders $2 \leqq k < n$, $n \geqq 3$, and consider all graphs contributing to $\langle T\varphi^I(x_1)\, \varphi^I(x_2)\, \mathscr{H}^I(y_1) \ldots$ $\ldots \mathscr{H}^I(y_n)\rangle^T$. The sums of subtractions associated with all partitions $U_1, \ldots U_s$ of $\{V_1, \ldots V_n\}$, $s > 1$, are uniquely determined by (4.17) using the induction assumption. The sum of the over-all subtractions for $\mathfrak{G}(\{V_1, \ldots V_n\})$ of the form $\delta(p_1 + p_2)\, \Delta^{r,\varepsilon}(p_1)^2 [A^{r,\varepsilon}_n + B^{r,\varepsilon}_n p_1^2]$ is then uniquely determined by (4.14), and similarly $C^{r,\varepsilon}_n$ by (4.16). Furthermore this construction is independent of the regularization (4.4).

This proves in the φ^4-theory that the "universal" subtraction procedure of Bogoliubov and Parasiuk together with the adjustment of three parameters in any order n leads to unique time-ordered distributions.

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