Commun. math. Phys. 2, 121—132 (1966)

Irreducible Tensors for the Group SU_3

L. Bányai, N. Marinescu, I. Raszillier, and V. Rittenberg*

Institute of Physics of the Academy, Bukarest, Rumania

Received October 5, 1965

Abstract. The explicit determination of the matrix elements of the SU_3 irreducible tensors is carried out by a purely algebraic method. These expressions may be used to compute the Clebsch-Gordan coefficients by orthogonalisation. For the special case of $(0, q)$ tensors simple formulas are derived.

I. Introduction

Recently compact Lie groups of rank $\geqq 2$ have found wide applications in elementary particle physics. In view of concrete physical problems, for each group the following main problems have to be solved: (a) determination of the irreducible representations (I.R.) and the matrix elements of the group generators, (b) decomposition of the direct product of two I.R. and hence the computation of the Clebsch-Gordan (C.G.) coefficients. It is well known that the groups of rank $\geqq 2$ are not multiplicity-free (the same representation may occur in the direct product more than once) so that the C.G. coefficients are not completely specified by the basis vectors. The Wigner-Eckart theorem is also modified: the number of reduced matrix elements appearing there is equal to the multiplicity of the equivalent representations.

The simplest of the above groups is SU_3. In this case the problem (a) has already been solved by a number of authors [1, 2, 3, 4, 5], while problem (b) has received until now only an incomplete solution. Moshinsky [6] has derived a compact expression for the C.G. coefficients corresponding to the product $(p, q) \otimes (p', 0)$, which is multiplicity-free, while Kurian, Lurié and Macfarlane [7] have tabulated the coefficients for the simple product $(p, q) \otimes (1, 1)$, Baird and Biedenharn [8] for the cases $(p, q) \otimes (1, 0)$, $(p, q) \otimes (0, 1)$, $(p, q) \otimes (1, 1)$ and Pandit and Mukunda [9] for the case $(p, q) \otimes (3, 0)$. We must also mention the numerical tables of SU_3 C.G. coefficients [10, 11, 12, 13] for the products of lowest representations. However, the general problem of deriving a simple analytical formula analogous to the Wigner-Racah expression for SU_2 has not yet been solved and it is doubtful if such a task is really possible.

* In partial fulfillment for the requierements of the doctoral degree at the Institute of Physics of the Academy.

In the present paper we establish an analytic expression for the matrix elements of an arbitrary irreducible tensor (I.T.). A method used by LURIÉ and MACFARLANE [15] for the (1, 1) tensor is generalised. The method consists in solving the commutation relations (C.R.) which define the I.T. We obtain equations with finite differences whose solutions contain the number of arbitrary constants corresponding to the equivalent representations which occur in the direct product. The constants are connected with the reduced matrix elements of the tensors. The C.G. coefficients are obtained by orthogonalisation.

It seems that the method used in this paper for SU_3 may be extended to other higher rank groups.

In the Sections II—V we establish the expression of the matrix elements of the I.T. in the general case while in Sec. VI, the formula for the $(0, q)$ irreducible tensors, which are multiplicity-free, is derived. These last expressions are obtained in a much easier way and are simpler than those of MOSHINSKY [6]. A brief version of the present paper has been published elsewhere [14].

II. Preliminary remarks

The irreducible tensors T^μ_ν corresponding to a representation $\mu = (p, q)$ of the SU_3 group and labelled by $\nu = (I, I_z, Y)$ are defined by their C.R. with the infinitesimal operators X:

$$[X, T^\mu_\nu] = (\mu, \nu' |X| \mu, \nu) T^\mu_{\nu'} . \tag{1}$$

The general structure of the eigenvalue diagram and hence the range of I, I_z and Y may be deduced from the paper of GINIBRE [16] and is represented in Fig. 1.

The matrix elements of the eight infinitesimal operators may be found in DE SWART's paper [17]. We shall mention only those which are used in the present paper,

$$(\mu; I', I_z + 1/2, Y + 1 |K_+| \mu; I, I_z, Y)$$
$$= C^{I, 1/2, I'}_{I_z, 1/2, I'_z} \left[\delta_{I', I+1/2} \frac{A_\mu(x)}{(2I+2)^{1/2}} - \delta_{I', I-1/2} \frac{B_\mu(y)}{(2I)^{1/2}}\right] \tag{2a}$$

$$(\mu; I', I_z + 1/2, Y - 1 |L_-| \mu; I, I_z, Y)$$
$$= C^{I, 1/2, I'}_{I_z, 1/2, I'} \left[\delta_{I', I+1/2} \frac{B_\mu(y+1)}{(2I+2)^{1/2}} + \delta_{I', I-1/2} \frac{A_\mu(x-1)}{(2I)^{1/2}}\right] \tag{2b}$$

where $C^{I', I'', I}_{I'_z, I''_z, I_z}$ are the C.G. coefficients of the SU_2 group and

$$A_\mu(x) = [(a - x)(b + x + 2)(c + x + 1)]^{1/2} \tag{3}$$

$$B_\mu(y) = [(a + y + 1)(b - y + 1)(-c + y)]^{1/2} \tag{4}$$

where

$$a = \frac{1}{3}(2p + q); \quad b = \frac{1}{3}(p + 2q); \quad c = \frac{1}{3}(p - q) \tag{5}$$

$$x = I + \frac{1}{2} Y; \quad y = I - \frac{1}{2} Y . \tag{6}$$

From Fig. 1 follows

$$-c \leqq x \leqq a; \quad c \leqq y \leqq b . \tag{7}$$

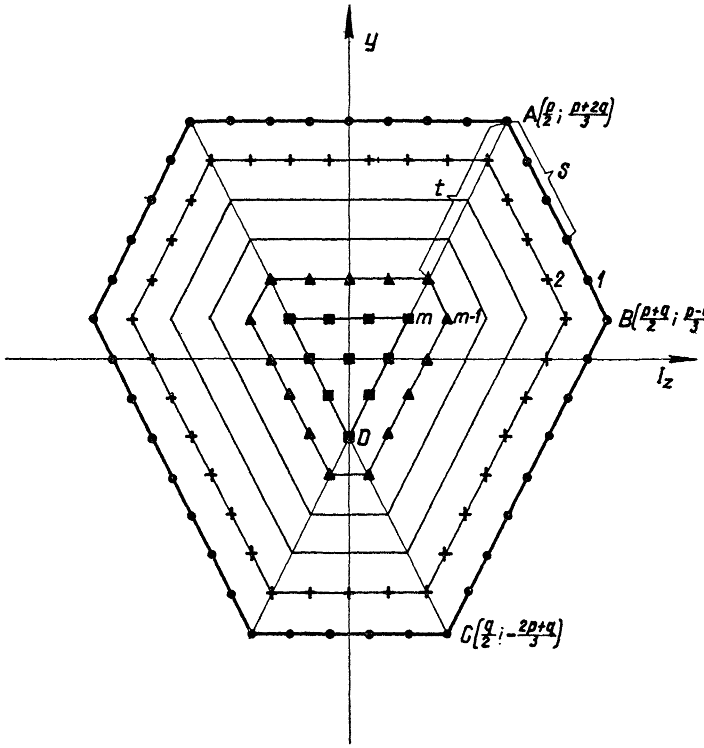

Fig. 1. Eigenvalue diagram for an irreducible representation (p, q). The numbers denote the eigenvalue multiplicity. The maximum multiplicity is $m = 1 + \min(p, q)$

The matrix elements (2) being expressed in the variables x, y instead of I, Y, we shall adopt the former to label the matrix elements of the I.T. Using the Wigner-Eckart theorem for SU_2 we find [17]:

$$\begin{aligned} &(\mu_3; I_3, I_{z_3}, Y_3 \,|T^{\mu_2}_{I_2, I_{z_2}, Y_2}|\, \mu_1, I_1, I_{z_1}, Y_1) \\ &= \delta_{Y_3, Y_1 + Y_2}\, C^{I_1, I_2, I_3}_{I_{z_1}, I_{z_2}, I_{z_3}} (\mu_1, x_1, y_1 \,|\mu_2, x_2, y_2|\, \mu_3, r) , \end{aligned} \tag{8}$$

where we have used the well known triangle rule:

$$I_3 = I_1 + I_2 - r \quad (r = 0, 1, \ldots, 2 \min(I_1, I_2)) \tag{9}$$

and the additivity of the hypercharge. The rather asymmetric notation of the new matrix element is more fashionable; we shall return to a symmetric notation in the final result.

We must also have in view the selection rules for μ which are given by SPEISER's [17] graphical multiplication rules or by the rather intricate expressions in [18, 19, 20, 21, 22] and [9].

We shall mention only the following relations:

$$a_1 + a_2 - a_3 = \tau; \quad b_1 + b_2 - b_3 = \sigma \tag{10}$$

where τ and σ are nonnegative integers while the multiplicity of the equivalent representations is given by the expression:

$$M = N + 1 - n \tag{11}$$

where

$$N = b_2 - c_2 + \sum_{i=1}^{3} u_i \theta(-u_i) \tag{12}$$

with

$$u_1 = a_3 - a_1 + c_2; \quad u_2 = b_1 - b_3 + c_2; \quad u_3 = c_1 - c_3 + c_2 \tag{13}$$

and n is a nonnegative integer which vanishes for $p_1 \geqq p_2$, $q_1 \geqq q_2$ and whose concrete expression can be deduced from the above mentioned papers.

A careful examination of the commutation relations (1) suggests the following way of solving the system of equations which determine the matrix elements of the I.T.:

a) Find the matrix elements with $r = 0$ of the tensor $T^{\mu_2}_{\frac{1}{2}(p_2+q_2), \frac{1}{2}(p_2+q_2), c_2}$ (B in Fig. 1).

b) Determine by recurrence the matrix elements with $r = 0$ of the tensor $T^{\mu_2}_{\frac{p_2}{2}, \frac{p_2}{2}, b_2}$ (running from B to A).

c) Solve the finite difference equations which determine all the matrix elements of the tensor $T^{\mu_2}_{\frac{p_2}{2}, \frac{p_2}{2}, b_2}$ (A in Fig. 1).

d) Obtain by recurrence the tensors $T^{\mu_2}_{\frac{1}{2}(p_2+s), \frac{1}{2}(p_2+s), b_2-s}$ (running from A to B).

e) Determine by recurrence the matrix element of the tensors $T^{\mu_2}_{\frac{1}{2}(p_2+s-t), \frac{1}{2}(p_2+s-t), b_2-s-t}$ (running parallel to BC).

III. The matrix elements of the tensor $T^{\mu_2}_{\frac{p_2}{2}, \frac{p_2}{2}, b_2}$

Let us consider the matrix elements $\left(\mu_3, I_1 + \frac{1}{2}(p_2 + q_2 + 1), I_{z_1} + \frac{1}{2}(p_2 + q_2 + 1), Y_1 + c_2 \pm 1 | \ldots | \mu_1, I_1, I_{z_1}, Y_1\right)$ of the C.R.:

$$\left[K_+, T^{\mu_2}_{\frac{1}{2}(p_2+q_2), \frac{1}{2}(p_2+q_2), c_2}\right] = 0 \tag{14a}$$

$$\left[L_-, T^{\mu_2}_{\frac{1}{2}(p_2+q_2), \frac{1}{2}(p_2+q_2), c_2}\right] = 0 \tag{14b}$$

and the change of function

$$(\mu_1, x_1, y_1 \,|\mu_2, a_2, b_2|\, \mu_3, 0) = \tag{15}$$
$$\left[\frac{(x_1+y_1+1)!\, f(a_3, b_3, c_3; x_1+a_2)\, f(b_3, a_3, -c_3; y_1+b_2)}{(x_1+y_1+a_2+b_2+1)!\, f(a_1, b_1, c_1; x_1)\, f(b_1, a_1, -c_1; y_1)}\right]^{1/2} F(\mu, x_1, y_1)$$

where

$$f(a, b, c; x) = \frac{(b+x+1)!\,(c+x)!}{(a-x)!} \tag{16}$$

we obtain the system of finite difference equations

$$F(\mu, x_1+1, y_1) = F(\mu, x_1, y_1) \tag{17a}$$

$$F(\mu, x_1, y_1+1) = F(\mu, x_1, y_1) \tag{17b}$$

with the obvious solution

$$F(\mu, x_1, y_1) = F(\mu)\,. \tag{18}$$

In the last expressions μ stands for μ_1, μ_2, μ_3. Consider now the matrix elements $\left(\mu_3, I_1 + \frac{1}{2}(p_2+s+1), I_{z_1} + \frac{1}{2}(p_2+s+1), Y_1 + b_2 - s \pm \pm 1 \,|\, \ldots \,|\, \mu_1, I_1, I_{z_1}, Y_1\right)$ of the C.R.:

$$\left[K_+, T^{\mu_2}_{\frac{1}{2}(p_2+s), \frac{1}{2}(p_2+s), b_2-s}\right] = 0 \quad (0 \leqq s \leqq q_2) \tag{19a}$$

$$\left[L_-, T^{\mu_2}_{\frac{1}{2}(p_2+s), \frac{1}{2}(p_2+s), b_2-s}\right] = - \tag{19b}$$
$$-\,[(s+1)(q_2-s)]^{1/2}\, T^{\mu_2}_{\frac{1}{2}(p_2+s+1), \frac{1}{2}(p_2+s+1), b_2-s-1}$$

we make the change of function

$$(\mu_1, x_1, y_1 \,|\mu_2, a_2, c_2+s|\, \mu_3, 0)$$
$$= \left[\frac{(b_2-c_2-s)!\,(x_1+y_1+1)!\, f(a_3, b_3, c_3; x_1+a_2)\, f(b_3, a_3, -c_3; y_1+c_2+1)}{s!\,(x_1+y_1+a_2+c_2+s+1)!\, f(a_1, b_1, c_1; x_1)\, f(b_1, a_1, -c_1; y_1)}\right]^{1/2} \times \tag{20}$$
$$\times\, G_s(\mu, x_1, y_1)$$

and find the system of finite difference equations

$$G_s(\mu, x_1+1, y_1) = G_s(\mu, x_1, y_1) \tag{21a}$$

$$G_s(\mu, x_1, y_1+1) - G_s(\mu, x_1, y_1) = -\,G_{s+1}(\mu, x_1, y_1)\,. \tag{21b}$$

Since from (15), (18) and (20) it results that the function $G_{b_2-c_2}(\mu, x_1, y_1)$ is independent of x_1 and y_1, for $s = 0$ we get:

$$G_0(\mu, x_1, y_1) = \sum_{\gamma=0}^{b_2-c_2} T'_\gamma(\mu)\, y_1^\gamma \tag{22}$$

where $T'_\gamma(\mu)$ are undetermined constants.

Let us take now the matrix elements $\left(\mu_3, I_1 + \frac{1}{2}(p_2+1) - r, I_{z_1} + \frac{1}{2}(p_2+1), Y_1 + b_2 + 1 \,|\ldots|\, \mu_1, I_1, I_{z_1}, Y_1\right)$ of the C.R.:

$$\left[K_+, T^{\mu_2}_{\frac{p_2}{2}, \frac{p_2}{2}, b_2}\right] = 0 \tag{23}$$

and make the change of function:

$$\begin{aligned} &(\mu_1, x_1, y_1\,|\mu_2, a_2, c_2|\,\mu_3, r) \\ &= \left[\frac{(b_2-c_2)!\,(x_1+y_1+1)\,(x_1+y_1+a_2+c_2-r+1)!}{((x_1+y_1+a_2+c_2-2r+1)!)^2\,(x_1+y_1-r)!}\right]^{1/2} \times \\ &\times \left[\frac{f(a_3, b_3, c_3; x_1+a_2-r)\, f(b_1, a_1, -c_1; y_1)}{f(a_1, b_1, c_1; x_1)\, f(b_3, a_3, -c_3; y_1+c_2-r)}\right]^{1/2} f_r(\mu, x_1, y_1) \end{aligned} \tag{24}$$

we obtain then the equations

$$\begin{aligned} &r^{-1/2}(a_2+c_2+1-r)^{1/2}\,[f_r(\mu, x_1+1, y_1) - f_r(\mu, x_1, y_1)] \\ &= \frac{f_{r-1}(\mu, x_1, y_1-1)}{(x_1+y_1+1)\,(x_1+y_1)} - \\ &- \frac{f_{r-1}(\mu, x_1, y_1)}{(x_1+y_1+a_2+c_2-2r+3)\,(x_1+y_1+a_2+c_2-2r+2)}\,. \end{aligned} \tag{25}$$

Equations (20) and (22) give

$$f_0(\mu, x_1, y_1) = f_0(\mu, y_1) = \frac{f(b_3, a_3, -c_3; y_1+c_2)}{f(b_1, a_1, -c_1; y_1)} \sum_{\gamma=0}^{b_2-c_2} T'_\gamma(\mu)\, y_1^\gamma\,. \tag{26}$$

One observes that for $r = a_2 + c_2 + 1$ the left side of (25) vanishes so that we obtain an identity which is easily verified. Consider the function

$$f_{a_2+c_2}(\mu, x_1, y_1) = \frac{(x_1+y_1-a_2-c_2+1)!\,(x_1+y_1-a_2-c_2)!}{(x_1+y_1+1)!\,(x_1+y_1)!} H(x_1, y_1)\,. \tag{27}$$

From (25) we then obtain for $r = a_2 + c_2 + 1$:

$$H(x_1, y_1+1) = H(x_1, y_1) = H(x_1)\,. \tag{28}$$

The general solution of the system of equations (25) is

$$\begin{aligned} f_r(x_1, y_1) = &\sum_{l=0}^{r} \sum_{k=0}^{r-l} (-1)^k\,[(a_2+c_2-r+l)!\,(r-l)!]^{1/2} \times \\ &\times \frac{(x_1+y_1-k)!\,(x_1+y_1+a_2+c_2-2r+2l+1)!}{k!\,(r-k-l)!\,(x_1+y_1+a_2+c_2+l+1-r-k)!} \frac{1}{(x_1+y_1)!} \varphi_l(y_1-k) \end{aligned}$$

where $\varphi_0 = f_0(\mu, y_1)$ and $\varphi_l(y_1)$ $(l = 1, 2, \ldots, r)$ are arbitrary functions. One can see by direct calculation that for the supplementary condition (28) we have $\varphi_l(y_1) \equiv 0$ if $l \neq 0$ and $f_0(\mu, y_1)$ must be a polinomial of degree p_2 in y_1. From the Speiser-Goldberg selection rules we then obtain some relations between the constants $T'_\gamma(\mu)$ so that instead of (26) we have

$$f_0(\mu, y_1) = \frac{f(b'_1, a'_1, -c'_1; y_1)}{f(b_1, a_1, -c_1, y_1)} \sum_{\gamma=0}^{N} T_\gamma(\mu)\, y_1^\gamma \tag{29}$$

where

$$a'_1 = a_1 + u_1 \theta(u_1); \quad b'_1 = b_1 - u_2 \theta(u_2); \quad c'_1 = c_1 - u_3 \theta(u_3); \tag{30}$$

$T_\gamma(\mu)$ are arbitrary constants and N is given by (12).

Summarising the results, we have

$$(\mu_1, x_1, y_1\,|\mu_2, a_2, c_2|\,\mu_3, r) = \sum_{\gamma=0}^{N} T_\gamma(\mu)\,(\mu_1, x_1, y_1\,|\mu_2, a_2, c_2|\,\mu_3, r)_\gamma \tag{31}$$

where:

$$\begin{aligned}
&(\mu_1, x_1, y_1\,|\mu_2, a_2, c_2|\,\mu_3, r)_\gamma \\
&\quad = \Big[(a_2 + c_2 - r)!\, r!\,(x_1 + y_1 + 1)\, \frac{(x_1 + y_1 + a_2 + c_2 + 1 - r)!}{(x_1 + y_1 - r)!} \times \\
&\quad \times \frac{f(a_3, b_3, c_3; x_1 + a_2 - r)\, f(b_1, a_1, -c_1, y_1)}{f(a_1, b_1, c_1; x_1)\, f(b_3, a_3, -c_3; y_1 + c_2 - r)}\Big]^{1/2} \times \\
&\quad \times \sum_{k=0}^{r} (-1)^k \frac{(x_1 + y_1 - k)!\,(y_1 - k)^\gamma}{k!\,(r-k)!\,(x_1 + y_1 + a_2 + c_2 + 1 - r - k)!} \times \\
&\quad \times \frac{f(b'_1, a'_1, -c'_1; y_1 - k)}{f(b_1, a_1, -c_1; y_1 - k)}\,.
\end{aligned} \tag{32}$$

IV. The matrix elements for an arbitrary I. T.

Let us consider the matrix elements $\left(\mu_3, I_1 + \frac{1}{2}(p_2 + s + 1) - r,\right.$ $I_1 + \frac{1}{2}(p_2 + s + 1),\, Y_1 + b_2 - s - 1| \ldots |\, \mu_1, I_1, I_{z_1}, \left.Y_1\right)$ of the C.R (19b) and make the change of functions:

$$\begin{aligned}
&(\mu_1, x_1, y_1\,|\mu_2, a_2, c_2 + s|\,\mu_3, r)_\gamma \\
&\quad = \Big[r!\,(a_2 + c_2 + s - r)!\,(a_2 + c_2 + s + 1)\,(x_1 + y_1 + 1) \times \\
&\quad \times \frac{(x_1 + y_1 + a_2 + c_2 + s + 1 - r)!}{(x_1 + y_1 - r)!} \frac{f(a_1, b_1, c_1; x_1)}{f(a_3, b_3, c_3; x_1 + a_2 - r)} \times \\
&\quad \times \frac{f(b_3, a_3, -c_3; y_1 + c_2 + s - r)}{f(b_2, a_2, -c_2; c_2 + s)\, f(b_1, a_1, -c_1; y_1)}\Big]^{1/2} g^\gamma_{s,r}(\mu, x_1, y_1)\,.
\end{aligned} \tag{33}$$

We obtain the recurrence relation

$$g^{\gamma}_{s+1,r}(\mu, x_1, y_1) \tag{34}$$
$$= \frac{g^{\gamma}_{s,r}(\mu, x_1, y_1) + g^{\gamma}_{s,r-1}(\mu, x_1, y_1)}{x_1 + y_1 + a_2 + c_2 + s + 2 - 2r} - \frac{g^{\gamma}_{s,r}(\mu, x_1, y_1 + 1) + g^{\gamma}_{s,r-1}(\mu, x_1 - 1, y_1)}{x_1 + y_1 + 1}$$

in which $g^{\gamma}_{0,r}(\mu, x_1, y_1)$ is known from (32) and (33). The solution of the recurrence relation is:

$$g^{\gamma}_{s,r}(\mu, x_1, y_1)$$
$$= \sum_{m_{i,\alpha} \geqq 0} (-1)^{m_{0,3} + m_{0,4}} \delta_{\varepsilon_0, s} \theta(m_{0,2} + m_{0,4} + a_2 + c_2 - r) \prod_{i=1}^{s} \delta_{\varepsilon_i, s-i} \times$$
$$\times (\delta_{m_{i-1,\alpha} - m_{i,\alpha}, 0} + \delta_{m_{i-1,\alpha}, 1}) \times$$
$$\times \frac{(x_1 + y_1 + s + 1 - 2r + a_2 + c_2 + m_{i,2} - m_{i,1})^{m_{i,1} - m_{i-1,1} + m_{i,2} - m_{i-1,2}}}{(x_1 + y_1 + 1 + m_{i,3} - m_{i,4})^{m_{i-1,3} - m_{i,3} + m_{i-1,4} - m_{i,4}}} \times$$
$$\times g^{\gamma}_{0, r - m_{0,2} - m_{0,4}}(\mu, x_1 - m_{0,4}, y_1 + m_{0,3})$$
$$(\alpha = 1, 2, 3, 4; \quad i = 0, 1, \ldots, s); \quad \varepsilon_i = \sum_{\alpha=1}^{4} m_{i,\alpha} \,. \tag{35}$$

Choosing $I_{z_1} = I_1 - r$ and taking the matrix elements

$$\Big(\mu_3, I_1 + \frac{1}{2}(p_2 + s - t - 1) - r, I_1 +$$
$$+ \frac{1}{2}(p_2 + s - t - 1) - r, Y_1 + b_2 - s - t - 1 \,|\cdots|\, \mu_1, I_1\, I_1 - r, Y_1\Big)$$

of the C.R.

$$\left[K_-, T^{\mu_2}_{\frac{1}{2}(p_2+s-t), \frac{1}{2}(p_2+s-t), b_2-s-t}\right]$$
$$= \left[\frac{(s+1)(q_2 - s)(p_2 + s + 2)}{(p_2 + s - t + 1)(p_2 + s - t + 2)}\right]^{1/2} \times$$
$$\times T^{\mu_2}_{\frac{1}{2}(p_2+s-t+1), \frac{1}{2}(p_2+s-t-1), b_2-s-t-1} + \tag{36}$$
$$+ \left[\frac{(p_2 - t)(p_2 + q_2 - t + 1)(t + 1)}{p_2 + s - t + 1}\right]^{1/2} \times$$
$$\times T^{\mu_2}_{\frac{1}{2}(p_2+s-t-1), \frac{1}{2}(p_2+s-t-1), b_2-s-t-1},$$

we make the change of function

$$(\mu_1, x_1, y_1 \,|\mu_2, a_2 - t, c_2 + s|\, \mu_3, r)_\gamma$$
$$= [(r!)^{-1}(a_2 + c_2 + s - t + 1)(x_1 + y_1 + 1)(x_1 + y_1 - r)!]^{1/2} \times$$
$$\times \Big[(x_1 + y_1 + a_2 + c_2 + s - t - r + 1)!\,(a_2 + c_2 + s - t - r)! \times$$
$$\times \frac{f(a_2, b_2, c_2; a_2 - t)\, f(a_1, b_1, c_1; x_1)}{f(b_2, a_2, -c_2; c_2 + s)\, f(a_3, b_3, c_3; x_1 + a_2 - r - t)} \times \tag{37}$$
$$\times \frac{f(b_3, a_3, -c_3; y_1 + c_2 + s - r)}{f(b_1, a_1, -c_1; y_1)}\Big]^{1/2} h^{\gamma}_{t,s,r}(\mu, x_1, y_1)$$

and find the recurrence relation

$$\begin{aligned} h^{\gamma}_{t,s,r}(\mu, x_1, y_1) = & - h^{\gamma}_{t-1,s+1,r+1}(\mu, x_1, y_1) + (a_2 + c_2 + s - t + 2) \times \\ & \times [h^{\gamma}_{t-1,s,r}(\mu, x_1, y_1) + (x_1 + y_1 + 1)^{-1} \times \\ & \times (h^{\gamma}_{t-1,s,r+1}(\mu, x_1, y_1 + 1) - h^{\gamma}_{t-1,s,r}(\mu, x_1 - 1, y_1)] , \end{aligned} \tag{38}$$

where $h^{\gamma}_{0,s,r}(\mu, x_1, y_1)$ is known from (33)–(35) and (38).

The solution of the recurrence equation is:

$$\begin{aligned} h^{\gamma}_{t,s,r}(\mu, x_1, y_1) = & \sum_{n_{j,\beta} \geqq 0} (-1)^{n_{0,1} + n_{0,4}} \delta_{\omega_0, t} \theta(x_1 + y_1 - r - n_{0,1} - n_{0,4}) \times \\ & \times \theta(b_2 - c_2 - s - n_{0,1}) \prod_{j=1}^{t} \delta_{\omega_j, t-j} (\delta_{n_{j-1,\beta} - n_{j,\beta}, 0} + \delta_{n_{j-1,\beta} - n_{j,\beta}, 1}) \times \\ & \times \frac{(a_2 + c_2 + s + n_{j,1} - j + 2)^{1 + n_{j,1} - n_{j-1,1}}}{(x_1 + y_1 + 1 - n_{j,4} + n_{j,3})^{n_{j-1,3} + n_{j-1,4} - n_{j3} - n_{j,4}}} \times \\ & \times h^{\gamma}_{0, s + n_{0,1}, r + n_{0,1} + n_{0,3}} (\mu, x_1 - n_{0,4}, y_1 + n_{0,3}) \end{aligned} \tag{39}$$

$$(\beta = 1, 2, 3, 4;\ j = 0, 1, \ldots, t),\quad \omega_j = \sum_{\beta=1}^{4} n_{j,\beta}$$

and so all the matrix elements are known.

V. Final result

Noting that $x_1 + x_2 - x_3 = y_1 + y_2 - y_3 = r$, we shall write the final result in a more symmetric manner:

$$\begin{aligned} & (\mu_3, I_3, I_{z,3}, Y_3 \,| T^{\mu_2}_{I_2, I_{z_2}, Y_2} |\, \mu_1; I_1, I_{z_1}, Y_1) \\ & \qquad = C^{I_1, I_2, I_3}_{I_{z_1}, I_{z_2}, I_{z_3}} (\mu_1, x_1, y_1 \,| \mu_2, x_2, y_2 |\, \mu_3, x_3, y_3) . \end{aligned} \tag{40}$$

The expression derived for the I_z-independent matrix element is

$$\begin{aligned} & (\mu_1, x_1, y_1 \,| \mu_2, x_2, y_2 |\, \mu_3, x_3, y_3) \\ & \qquad = \sum_{\gamma=0}^{N} T_\gamma(\mu_1, \mu_2, \mu_3) (\mu_1, x_1, y_1 \,| \mu_2, x_2, y_2 \,| \mu_3, x_3, y_3)_\gamma \end{aligned} \tag{41}$$

The coefficients T_γ are related to the reduced matrix element. The formula for the (nonorthonormalized) isoscalar coefficients is:

$$\begin{aligned} & (\mu_1, x_1, y_1 \,| \mu_2, x_2, y_2 |\, \mu_3, x_3, y_3)_\gamma \\ & \qquad = \Big[(x_1 + y_1 + 1)(x_2 + y_2 + 1)(y_1 - x_2 + x_3)!\,(-x_1 + y_2 + x_3)! \times \\ & \times \frac{(x_1 + x_2 + y_3 + 1)!\, f(a_1, b_1, c_1; x_1)\, f(a_2, b_2, c_2; x_2)\, f(b_3, a_3, -c_3; y_3)}{(x_1 + x_2 - x_3)!\, f(a_3, b_3, c_3; x_3)\, f(b_1, a_1, -c_1; y_1)\, f(b_2, a_2, -c_2; y_3)} \Big]^{1/2} F_\gamma(\mu, x, y) \end{aligned} \tag{42a}$$

where

$$F_\gamma(\mu, x, y) = \sum_{n_{0,\alpha}} \sum_{m_{0,\beta}} (-1)^{n_{0.1} + n_{0,4} + m_{0,3} + m_{0,4}} \delta_{\omega_0, a_2 - x_2} \delta_{\varepsilon_0, y_2 - c_2 + n_{0,1}} \times$$

$$\times\theta(b_2-y_2-n_{0,1})\,\theta(a_2+c_2-x_1-x_2+x_3-n_{0,1}-n_{0,3}+m_{0,2}+m_{0,4})\times$$
$$\times\varphi(n_{0,\alpha};\mu,x,y)\,\psi(n_{0,\alpha},m_{0,\beta};\mu,x,y)\,\chi_\gamma(\mu;x_1-n_{0,4}-m_{0,4},\qquad (42\text{b})$$
$$y_1+n_{0,3}+m_{0,3},x_3+n_{0,2}+m_{0,2},y_3-m_{0,1})\frac{(x_1+x_2-x_3+n_{0,1}+n_{0,3})!}{(y_1-x_2+x_3-n_{0\,1}-n_{0,4})!}$$

and

$$\varphi(n_{0,\alpha};\mu,x,y)=\sum_{n_{i,\alpha}}\prod_{i=1}^{a_2-x_2}\delta_{\omega_i,a_2-x_2-i}(\delta_{n_{i-1,\alpha}-n_{i\alpha},0}+\delta_{n_{i-1,\alpha}-n_{i,\alpha},1})\times$$
$$\times\frac{(x_2+y_2+2n_{i,1}+n_{i,2}+n_{i,3}+n_{i,4}+2)^{1+n_{i,1}-n_{i-1,1}}}{(x_1+y_1+n_{i,3}-n_{i,4}+1)^{n_{i-1,3}+n_{i-1,4}-n_{i,3}-n_{i,4}}};\qquad (42\text{c})$$
$$(\alpha=1,2,3,4)$$

$$\psi(n_{0,\alpha},m_{0,\beta};\mu,x,y)=\sum_{m_{j,\beta}}\prod_{j=1}^{y_2-c_2+n_{0,1}}\delta_{\varepsilon_j,y_2-c_2+n_{0,1}-j}\,(\delta_{m_{j-1,\beta}-m_{j,\beta},0}+\qquad (42\text{d})$$
$$+\delta_{m_{j-1,\beta}-m_{j,\beta},1})\frac{(x_3+y_3+n_{0,2}-m_{j,2}-m_{j,1}+1)^{m_{j,1}+m_{j,2}-m_{j-1,1}-m_{j-1,2}}}{(x_1+y_1+n_{0,3}-n_{0,4}+m_{j,3}-m_{j,4}+1)^{m_{j-1,3}+m_{j-1,4}-m_{j,3}-m_{j,4}}};$$
$$(\beta=1,2,3,4)$$

$$\chi_\gamma(\mu,x_1,y_1,x_3,y_3)=\frac{f(a_3,b_3,c_3;x_3)\,f(b_1,a_1,-c_1;y_1)}{f(a_1,b_1,c_1;x_1)\,f(b_3,a_3,-c_3;y_3)}\times\qquad (42\text{e})$$
$$\times\sum_k(-1)^k\frac{(x_1+y_1-k)!\,(y_1-k)^\gamma f(b_1',a_1',-c_1';y_1-k)}{k!(x_1-x_3+a_2-k)!\,(y_1+x_3+c_2-k+1)!\,f(b_1,a_1,-c_1;y_1-k)}.$$

We note that if the selection rules correspond to a multiplicity smaller than $N+1$, the functions appearing in (40) are no longer linearly independent.

The expression given above may be used to compute the C.G. coefficients if we use the orthonormality relations [17]. The computation of the general expression for the C.G. coefficients, a very difficult task, must be done for each concrete case individually.

VI. The special I. T. corresponding to the representation $(0, q)$

The matrix elements of these tensors which are multiplicity-free can be obtained directly without making use of the general formula given in Sec. V. This is preferable because it is difficult to observe the simplifications which occur in this case in the intricate expression (42). We shall return to Sec. III observing that in the present case the sum (31) contains a single term:

$$\left(\mu_1,x_1,y_1\middle|\mu_2,\frac{q_2}{3},\frac{-q_2}{3}\middle|\mu_3,r\right)=T_0(\mu)\left(\mu_1,x_1,y_1\middle|\mu_2,\frac{q_2}{3},\frac{-q_2}{3}\middle|\mu_3,r\right)_0\qquad (43)$$

while eq. (32) gives

$$\left(\mu_1,x_1,y_1\middle|\mu_2,\frac{q_2}{3},\frac{-q_2}{3}\middle|\mu_3,r\right)_0$$
$$=\delta_{r,0}\left[\frac{f(a_3,b_3,c_3;a_2+x_1)\,f(b_1,a_1,-c_1;y_1)}{f(a_1,b_1,c_1;x_1)\,f(b_3,a_3,-c_3;y_1+c_2)}\right]^{1/2}.\qquad (44)$$

$T_0(\mu)$ is an undetermined constant. To determine the matrix elements of the other tensors we shall use a method other than that of Sec. IV. From the commutation relation (19b) and the change of functions (33) we shall determine the matrix elements $(\mu_1, x_1, y_1 \mid \mu_2, a_2, c_2 + s \mid \mu_3, 0)_0$.

The recurrence relation (34) with $r = 0$ is:

$$g_{s+1,0}(x_1, y_1) = \frac{g_{s,0}(x_1, y_1)}{x_1 + y_1 + s + 2} - \frac{g_{s,0}(x_1, y_1 + 1)}{x_1 + y_1 + 1} . \tag{45}$$

We make the change of function:

$$g_{s,0}(x_1, y) = \frac{(x_1 + y_1)!}{(x_1 + y_1 + s + 1)!} u_s(x_1, y_1) \tag{46}$$

and obtain the recurrence relation

$$u_{s+1}(x_1, y_1) = u_s(x_1, y_1) - u_s(x_1, y_1 + 1) \tag{47}$$

which has the solution:

$$u_s(x_1, y_1) = \sum_{k=0}^{s} (-1)^k \frac{s!}{k!(s-k)!} u_0(x_1, y_1 + k) . \tag{48}$$

From (46) and (48) we find finally:

$$g_{s,0} = \frac{(x_1 + y_1)!}{(x_1 + y_1 + s + 1)!} \sum_{k=0}^{s} (-1)^k \frac{s!}{k!(s-k)!} \times \tag{49}$$
$$\times (x_1 + y_1 + k + 1)\, g_{0,0}(x_1, y_1 + k) .$$

The function $g_{0,0}(x_1, y_1)$ is derived from (33) and (44):

$$g_{0,0}(x_1, y_1) = \frac{1}{(q_2!)^{1/2}(x_1 + y_1 + 1)} \frac{f(a_3, b_3, c_3; a_2 + x_1)\, f(b_1, a_1, -c_1; y_1)}{f(a_1, b_1, c_1; x_1)\, f(b_3, a_3, -c_3; c_2 + y_1)} . \tag{50}$$

Using the relations (33), (49) and (50), the matrix elements $(\mu_1, x_1, y_1 | \mu_2, a_2, c_2 + s | \mu_3, 0)_0$ are completely determined. To find the other ones we consider the C.R.:

$$\left[K_+, T^{(0,q_2)}_{\frac{s}{2}, \frac{s}{2}, \frac{2q_2}{3} - s}\right] = 0 \tag{51}$$

and the same method as in Sec. III. We than obtain the matrix elements $\left(\mu_1, x_1, y_1 \middle| \mu_2, \frac{q_2}{3}, y_2 \middle| \mu_3, x_3, y_3\right)$ expressed in terms of $(\mu_1, x_1, y_1 | \mu_2, a_2, c_2 + s | \mu_3, 0)$ derived above.

Expressed in the variables x and y the final result is:

$$\left(\mu_1, x_1, y_1 \middle| \mu_2, \frac{q_2}{3}, y_2 \middle| \mu_3, x_3, y_3\right) \tag{52}$$
$$= T(\mu) \Big[(x_1 + y_1 + 1)\,(x_1 + x_2 + y_3 + 1)!\,(-x_1 + y_2 + x_3)! \times$$
$$\times (q_2 - x_2 - y_2)! \frac{(x_1 + x_2 - x_3)!\, f(a_3, b_3, c_3; x_3)\, f(b_1, a_1, -c_1; y_1)}{(y_1 - x_2 + x_3)!\, f(a_1, b_1, c_1; x_1)\, f(b_3, a_3, -c_3; y_3)}\Big]^{1/2} \times$$
$$\times \sum_{k,l} \frac{(-1)^{k+l}}{k!\,l!} [(x_1 + x_2 - x_3 - k)!\,(x_2 + y_2 - l)!]^{-1} \times$$
$$\times \frac{(x_1 + y_1 - k)!}{(x_1 + x_2 + y_3 - k)!} \frac{f(b_1, a_1, -c_1; y_1 - k + l)\, f(b_3, a_3, -c_3; y_1 + y_2 - k)}{f(b_3, a_3, -c_3; y_1 + c_2 - k + l)\, f(b_1, a_1, -c_1; y_1 - k)} .$$

To compare the present result for the I.T. corresponding to a representation $(0, q)$ with Moshinsky's one [6] corresponding to a representation $(q, 0)$ we have to use the symmetry properties of the C.G. coefficients [17]. We then observe that the former is more convenient, containing fewer terms.

Acknowledgement. The authors would like to express their gratitude to Professor V. Novacu for critical discussions.

References

[1] Gelfand, I. M., and M. L. Zetlin: Doklady Akad. Nauk S.S.S.R. **71**, 825 (1950).
[2] Biedenharn, L. C.: Physics Letters, **3**, 69 (1962).
[3] Harvey, M., and J. P. Elliot: Proc. Roy. Soc. London, **272** A, 557 (1963).
[4] Baird, G. E., and L. C. Biedenharn: J. Math. Phys. **4**, 436 (1963).
[5] Pursey, D. L.: Proc. Roy. Soc. London **275** A, 284 (1963).
[6] Moshinsky, M.: Rev. Mod. Phys. **34**, 813 (1962).
[7] Kurian, J. G., D. Lurie, and A. J. Macfarlane: J. Math. Phys. **6**, 722 (1965).
[8] Baird, G. E., and L. C. Biedenharn: J. Math. Phys., in press.
[9] Pandit, L. K., and N. Mukunda: J. Math. Phys., in press.
[10] Sawada, S., and M. Yinezawa: Prog. Theor. Phys. (Kyoto) **23**, 662 (1960).
[11] Rashid, M. A.: Nuovo Cimento **26**, 118 (1962).
[12] Tarjanne, P.: Ann. Acad. Sci. Fennicae, A VI Physica, 105 (1962).
[13] McNamee, P., and F. Chilton: Rev. Mod. Phys. **36**, 1005 (1964).
[14] Bányai, L., N. Marinescu, I. Raszillier, and V. Rittenberg: Phys. Letters, **14**, 156 (1965).
[15] Lurié, D., and A. J. Macfarlane: J. Math. Phys. **5**, 565 (1964).
[16] Ginibre, J.: Nuovo Cimento **30**, 406 (1963).
[17] de Swart, J. J.: Rev. Mod. Phys. **35**, 916 (1963).
[18] Goldberg, H.: Nuovo Cimento **27**, 532 (1963).
[19] Moshinsky, M.: J. Math. Phys. **4**, 1128 (1963).
[20] Simoni, A., and B. Vitale: Nuovo Cimento **33**, 1199 (1964); **34**, 1101 (1964) and in press.
[21] Coleman, S.: J. Math. Phys. **5**, 1342 (1964).
[22] Mandelzweig, V. B.: Zh. Eksperim I. Teor. Fiz. **47**, 1836 (1964).