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# Analytic Structure in $\lambda$ of the $S$-Matrix for Strongly Singular Potentials

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**Abstract.** The asymptotic behaviour in the $\lambda$-plane of solutions of the Schrödinger equation for scattering on singular potentials is investigated. The asymptotic behaviour of the Jost functions and the $S$-matrix is obtained. Furthermore, the general analytic form in the $\lambda$-plane of the Jost functions and the $S$-matrix is established. Some properties of the distribution of poles of the $S$-matrix are proved.

## 1. Introduction

It is well known that for strongly singular potentials repulsive at the origin, the Jost functions are entire functions and the $S$-matrix is a meromorphic function in the $\lambda$-plane [1], [2]. But the asymptotic properties of these functions and the asymptotic distribution of their zeros and respective poles, has not yet been investigated in a general way. For real $k$ several results have been obtained already in Reference [3].

In this paper we intend to establish the general analytic representation in the $\lambda$-plane for the Jost functions and the $S$-matrix in the sense of the well-known Hadamard's factorization theorem for entire functions of finite order. Also the asymptotic distribution of poles of the $S$-matrix in the $\lambda$-plane will be considered.

In Section 2 we consider the behaviour of the Jost solutions and the regular solution of the Schrödinger equation for large $\lambda$ and establish that their order is one. In Section 3 we prove that the Jost function, defined by the Wronskian of the Jost solution and the regular solution, is of order one and infinite type and that it can be expressed as a canonical product of genus one in the $\lambda$-plane for any $k$, $\mathrm{Re}\, k > 0$, with an infinite number of zeros accumulating asymptotically in certain angles. For

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instance, if $k$ is in the first quadrant of the $k$-plane, the zeros $\lambda_n$ of the Jost function $f_-(\lambda, k)$, which correspond to the poles of the $S$-matrix, are in the first and the third quadrant with their arguments approaching asymptotically $\pi/2$ and $3\pi/2$ respectively. It follows also from our analysis that the Regge trajectories for singular potentials never cross the origin in the $\lambda$-plane. We also prove the lower and upper bounds on the number of zeros in $|w| \leqq r$. Furthermore, we establish the asymptotic behaviour of the Jost functions and the $S$-matrix. The $S$-matrix tends to unity for large $\lambda$ in the same region of arguments of $\lambda$ and $k$ as in the case of regular potentials (the shaded region in Fig. 3). We did not investigate the exact asymptotic behaviour of $S - 1$ in this paper. For real $k$ this has been done in Reference [3].

The asymptotic behaviour of the Jost function for real $\lambda$, fixed imaginary $k$, and a special kind of singular potentials has been investigated previously in the Ref. [4]. The problems solved in our present paper, however, require an investigation of asymptotic properties for any angle in the $\lambda$-plane and for any fixed $k$.

## 2. The asymptotic behaviour of the Jost solutions and the regular solution

Before starting the investigation of the behaviour of the solutions of the Schrödinger differential equation, we have to specify more precisely the class of potentials considered. Here the potential $V(z)$ is assumed to be a regular analytic function in the half-plane $\mathrm{Re}\, z > 0$, real on the positive real axis. At the origin on the positive real axis it has to satisfy the following three conditions (compare Reference [3]):

$$V(x) > C x^{-\alpha}, \quad C > 0, \quad \alpha > 2, \quad 0 \leqq x \leqq x_0, \tag{2.1}$$

$$\int_0^{x_0} \frac{1}{\sqrt{V(x)}} \left[\frac{1}{x^2} + \left(\frac{V'(x)}{V(x)}\right)^2 + \frac{|V''(x)|}{V(x)}\right] dx < \infty, \tag{2.2}$$

$$x^2 V(x) \quad \text{is monotonic in} \quad [0, x_0], \tag{2.3}$$

where $x_0$ is a positive constant. The first condition requires that at the origin the potential is repulsive and singular more than $1/x^{2+\varepsilon}$, $\varepsilon$ arbitrarily small. The other two conditions exclude potentials with undesirable oscillatory features when approaching the origin along the real axis. Finally, at infinity in the half-plane $\mathrm{Re}\, z > 0$ we shall assume the following asymptotic condition

$$\lim_{z \to \infty} z^2 V(z) \quad \text{exists for} \quad z = r \exp i\tau, \quad |\tau| < \frac{\pi}{2}. \tag{2.4}$$

For such potentials the Jost solutions $f_\pm(\lambda, k, z)$ exist and they are defined by their asymptotic behaviour at infinity: $f_\pm(\lambda, k, z) \sim \exp(\mp ikz)$

as $\mathrm{Re}\, z \to \infty$. The regular solution $\varphi(\lambda, k, z)$ also exists and it is defined by its behaviour at the origin (compare Reference [2]):

$$\lim_{x \to 0} \frac{\varphi(\lambda, k, x)}{\varphi_0(x)} = 1 \,, \quad x \text{ real} \,, \tag{2.5}$$

where

$$\varphi_0(x) = \frac{1}{\sqrt[4]{V(x)}} \exp\left( - \int\limits_{x}^{x_0} \sqrt{V(y)}\, dy \right) . \tag{2.6}$$

We shall investigate now the rate of growth in $\lambda$ of these solutions. The order $\varrho$ of a function $f(\lambda)$ analytic and regular in an angle $\omega_1 \leqq \arg\lambda \leqq \omega_2$, a measure of its rate of growth, is defined in the following way [5]:

$$\varrho = \limsup_{|\lambda| \to \infty} \frac{\log\log M(|\lambda|)}{\log|\lambda|} \,, \quad \omega_1 \leqq \arg\lambda \leqq \omega_2 \,. \tag{2.7}$$

Here $M(r)$ is the maximum of the function $|f(\lambda)|$ for $|\lambda| \leqq r$ in the angle $\omega_1 \leqq \arg\lambda \leqq \omega_2$. Our main task in this section is to find the maximum $M(|\lambda|)$ of the Jost solutions and of the regular solution. An upper bound on the function $M(|\lambda|)$ can be obtained relatively easily as shown at the end of the Appendix. In the whole $\lambda$-plane $M(|\lambda|) < K \exp C|\lambda|^2$, where the quantities $K$ and $C$ depend on the variables $z$ and $k$ only. We shall need this rough estimate later. We shall need also the exact form of the function $M(|\lambda|)$, which we shall be able to derive only for certain angles. Fortunately, these angles will be larger than $\pi/2$. This fact, the evenness of the solutions in $\lambda$ and the mentioned upper bound of the function $M(|\lambda|)$ will be sufficient for the extension of the properties obtained for these angles to the whole $\lambda$-plane.

Let us find the function $M(|\lambda|)$ for the Jost solutions first; it is closely related to their asymptotic behaviour. The first step towards the proof of the asymptotic behaviour of the Jost solutions is to choose an auxiliary differential equation as the asymptotic substitute for the Schrödinger equation. It must contain those terms of the Schrödinger equation which are dominant for $\lambda \to \infty$, and for all $z$. Both solutions of this auxiliary equation must be known to us in an explicit form and be simple enough to allow simple estimates. The solutions of such equation will be used for constructing the Green's function and the corresponding integral equation for the Jost solutions. We expect that the Jost solutions will approach asymptotically a properly chosen solution of the auxiliary equation. This method, close to the application of the W. K. B. method to the scattering problem [6], has been described in detail in Reference [3]. Here we shall make use of the following expressions from this reference. We can write $f_\pm(\lambda, k, z) = \chi_\pm(\lambda, k, z)\, g_\pm(\lambda, k, z)$, where $\chi_\pm(\lambda, k, z)$

are the auxiliary functions:

$$\chi_\pm(\lambda,k,z) = \frac{1}{\binom{+}{i}\sqrt[4]{1-\frac{\lambda^2}{k^2z^2}}} \times \\ \times \exp\left\{\mp i k z + \int\limits_{C_\pm(z,\infty)} \left(\sqrt{\frac{\lambda^2}{\xi^2}-k^2} \mp ik\right) d\xi\right\}. \tag{2.8}$$

The functions $g_\pm(\lambda, k, z)$ are the solutions of the integral equations:

$$g_\pm(\lambda, k, z) = 1 + \int\limits_{C_\pm(z,\infty)} H_\pm(\lambda, k, z, z')\, g_\pm(\lambda, k, z')\, dz', \tag{2.9}$$

where

$$H(\lambda, k, z, z') = \frac{1}{2\sqrt{\frac{\lambda^2}{z'^2}-k^2}}\left(1-\exp\left\{-2\int\limits_{C_\pm(z,z')}\sqrt{\frac{\lambda^2}{\xi^2}-k^2}\, d\xi\right\}\right) \times \\ \times \left(V(z') - \frac{\lambda^2k^2 + \frac{k^4z'^2}{4}}{(\lambda^2-k^2z'^2)^2}\right). \tag{2.10}$$

The paths $C_\pm(z, \infty)$ are defined in the Appendix and they are drawn in Fig. 4.

It is shown in the Appendix that the functions $g_\pm(\lambda, k, z)$ tend to unity uniformly in $z$ on the path $(z \neq 0)$, if the arguments $\omega = \arg\lambda$ and $\sigma = \arg k$ are in the shaded regions of Fig. 1 and Fig. 2, respectively. This means that the functions $g_\pm(\lambda, k, z)$ are of order zero in the corresponding shaded regions. Hence, the order of the Jost solutions $f_\pm(\lambda, k, z)$ is given by the order of the functions $\chi_\pm(\lambda, k, z)$. To obtain their order we have to consider the maximum of

$$|\chi_\pm(\lambda, k, z)| = \\ = \frac{1}{\left|\sqrt[4]{\frac{\lambda^2}{z^2}-k^2}\right|}\exp\left\{\mp \operatorname{Re} ikz + \operatorname{Re}\int\limits_{C_\pm(z,\infty)}\left(\sqrt{\frac{\lambda^2}{\xi^2}-k^2} \mp ik\right) d\xi\right\}.$$

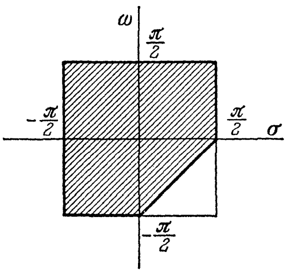
The integral in the exponent can be evaluated explicitly so that

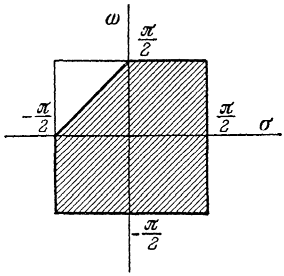
$$|\chi_\pm(\lambda, k, z)| = \frac{1}{\left|\sqrt[4]{\frac{\lambda^2}{z^2}-k^2}\right|} \times \\ \times \exp\left\{-\operatorname{Re}\sqrt{\lambda^2-k^2z^2} \mp \operatorname{Re}\frac{i\pi\lambda}{2} + \operatorname{Re}\lambda\log\left(\frac{\lambda}{kz}+\sqrt{\frac{\lambda^2}{k^2z^2}-1}\right)\right\}.$$

The leading term in this expression for large $\lambda$ is of the form

$$|\chi_\pm(\lambda, k, z)| \sim \exp\{\cos\omega\,|\lambda|\log|\lambda|\}\,.$$

It follows that the functions $\chi_\pm(\lambda, k, z)$ are of order one and infinite type in the angle $\omega_{2\pm} \leqq \omega \leqq \omega_{1\pm}$. Here $\omega_{1\pm} = \omega_{\max\,\pm} - \varepsilon$ and $\omega_{2\pm} = \omega_{\min\,\pm} + \varepsilon$, $\omega_{\max\,\pm}$ and $\omega_{\min\,\pm}$ being the maximum and the minimum value of $\omega$ in the shaded region in Fig. 1 and Fig. 2 respectively, for a fixed $\sigma$. Hence, the Jost solutions $f_\pm(\lambda, k, z)$ are functions of order one and infinite type in the same angles.




Figs. 1 and 2. The Jost functions are of order one and infinite type in the shaded regions

Since the Jost solutions are entire even functions of $\lambda$ (see References [1], [2]) we can consider them as functions of the variable $w = \lambda^2$ and define $h_\pm(w, k, z) \equiv f_\pm(\lambda, k, z)$. The functions $h_\pm(w, k, z)$ are then of order 1/2 in the angles $\Delta_\pm$ in the $w$-plane larger than $\pi$, for every $k$, $\operatorname{Re} k > 0$. But we have also a rough estimate of these functions, quoted earlier for the $\lambda$-plane, which is valid in the whole $w$-plane: $|h_\pm(w,k,z)| < \exp(C|w|)$. So we can apply the Phragmén-Lindelöf theorem to the angles $\Delta^C_\pm$ complementary to $\Delta_\pm$. We divide $h_\pm(w, k, z)$ by $E(w) = \exp\{-i\sqrt{w}\,(\log w)^2\}$, a regular analytic function of order 1/2 in $\Delta^C_\pm$ with the cut along the positive real axis. This quotient is a regular analytic function in $\Delta^C_\pm$ (except of the origin) vanishing asymptotically along the boundary of $\Delta^C_\pm$, and bounded by an exponential function $\exp(K|w|)$ in the angle $\Delta^C_\pm$. Then, by the Phragmén-Lindelöf theorem, this quotient vanishes asymptotically in $\Delta^C_\pm$. Hence, the order of $h_\pm(w, k, z)$ in $\Delta^C_\pm$ is 1/2. In this way we have proved that $h_\pm(w, k, z)$ is of order 1/2 and infinite type in the whole $w$-plane and, consequently, $f_\pm(\lambda, k, z)$ is of order one and infinite type in the whole $\lambda$-plane.

Now, let us turn to the regular solution. Again, we choose first an auxiliary differential equation, which now must contain the potential because the boundary condition (2.5) depends upon the potential. From its two independent solutions $\psi_\pm(\lambda, k, x)$ we construct the Green's function and the integral equation for the regular solution. Apart from a normalisation factor the auxiliary solutions $\psi_\pm(\lambda, k, x)$ are equal to the corresponding ones in Ref. [3]. The regular solution, satisfying the boundary condition (2.5) can be written as $\varphi(\lambda, k, x) = \psi_+(\lambda, k, x) \times \times u(\lambda, k, x)$, where

$$\psi_\pm(\lambda, k, x) = \frac{1}{\sqrt[4]{V(x) + \frac{\lambda^2}{x^2} - k^2}} \times$$

$$\times \exp\left\{\mp \int_x^{x_0} \sqrt{V(y) + \frac{\lambda^2}{y^2} - k^2}\, dy \pm \right. \tag{2.11}$$

$$\left. \pm \int_0^{x_0} \left(\sqrt{V(y) + \frac{\lambda^2}{y^2} - k^2} - \sqrt{V(y)}\right) dy\right\}$$

and $u(\lambda, k, x)$ is the solution of the integral equation

$$u(\lambda, k, x) = 1 + \int_0^x H(\lambda, k, x, y)\, u(\lambda, k, y)\, dy\,. \tag{2.12}$$

Here

$$H(\lambda, k, x, y) = \frac{1}{2\sqrt{V(y) + \frac{\lambda^2}{y^2} - k^2}} \times$$

$$\times \left[1 - \exp\left\{-2 \int_y^x \sqrt{V(\xi) + \frac{\lambda^2}{\xi^2} - k^2}\, d\xi\right\}\right] \times \tag{2.13}$$

$$\times \left[-\frac{1}{4y^2} - \frac{5}{16}\left(\frac{V'(y) - \frac{2\lambda^2}{y^3}}{V(y) + \frac{\lambda^2}{y^2} - k^2}\right)^2 + \frac{1}{4}\, \frac{V''(y) + \frac{6\lambda^2}{y^4}}{V(y) + \frac{\lambda^2}{y^2} - k^2}\right].$$

It is shown in the Appendix that the function $u(\lambda, k, x)$ tends to unity as $\lambda$ tends to infinity along a ray, $|\arg\lambda| < \pi/2$, for any fixed $k$ in the $k$-plane. On the other hand, we can find easily the asymptotic bound for the function $\psi_+(\lambda, k, x)$. The first term in the exponent of $\psi_+(\lambda, k, x)$ behaves like $+\lambda \log x$ for large $\lambda$. The second term can be rewritten as follows:

$$\int_0^{x_0} \left(\sqrt{V(y) + \frac{\lambda^2}{y^2} - k^2} - \sqrt{V(y)}\right) dy$$

$$= \int_0^{x_0} \frac{\frac{\lambda^2}{y^2} - k^2}{\sqrt{V(y) + \frac{\lambda^2}{y^2} - k^2} + \sqrt{V(y)}}\, dy\,.$$

We denote the absolute value of this integral by $I(\lambda)$. Using the condition (2.1) we can make the following estimate for $|\arg\lambda| < \pi/2$:

$$I(\lambda) < A\,|\lambda|^2 \int\limits_0^{x_0} \frac{1}{\sqrt{y^4\,V(y) + \lambda^2 y^2}}\,dy\,.$$

If $y^4\,V(y)$ has a positive limit when $y$ tends to zero, we shall replace $y^4\,V(y)$ by the minimal value of the function $y^4\,V(y)$ in the interval $(0, x_0)$. In this case after calculating the integral we find $I(\lambda) < B\,|\lambda|$, where $B$ is some constant. If $y^4\,V(y)$ tends to zero, we can always find a number $\beta$, $\alpha > \beta > 2$, such that $y^\beta\,V(y)$ tends to infinity. Then again we can estimate $y^\beta\,V(y)$ by its minimal value in the interval $(0, x_0)$ and calculate the obtained integral. We have again the estimate $I(\lambda) < B\,|\lambda|$, with another constant $B$. It follows from this that the order of the regular solution in this angle is one at the most. Since the regular solution is an entire function bounded by $\exp(C\,|\lambda|^2)$, we can use as before the Phragmén-Lindelöf theorem and prove that the order of the regular solution is one at most in the whole $\lambda$-plane. But it can be easily checked from (2.11) that the regular solution increases exponentially along the positive real axis, so that its order has to be one.

## 3. The asymptotic behaviour and the structure of the Jost functions and the $S$-matrix

Knowing the asymptotic behaviour of the solutions of the integral equations (2.9) and (2.12) and the bounds on their derivatives, proved partly in Section 2 and partly in the Appendix, we can establish the asymptotic behaviour of the Jost functions. They are defined by the Wronskians

$$\begin{aligned} f_\pm(\lambda, k) &= W(f_\pm(\lambda, k, x),\, \varphi(\lambda, k, x)) \\ &= \chi_\pm(\lambda, k, x)\,\psi_+(\lambda, k, x)\,W(g_\pm(\lambda, k, x), u(\lambda, k, x)) + \qquad (3.1) \\ &\quad + g_\pm(\lambda, k, x)\,u(\lambda, k, x)\,W(\chi_\pm(\lambda, k, x), \psi_+(\lambda, k, x))\,. \end{aligned}$$

The factor in front of the first Wronskian can be easily estimated and proved that it is of order $1/|\lambda|$ compared to the second Wronskian. The first Wronskian is bounded by a constant because both the functions and their derivatives entering the Wronskian are bounded as shown in the Appendix. The factor in front of the second Wronskian tends to unity as shown in Section 2 and the Appendix. The domain of validity of these assertions in the arguments of $\lambda$ and $k$ is the interior of the shaded domains of Fig. 1 and Fig. 2 respectively. Hence, the asymptotic behaviour of the Jost functions is given by the asymptotic behaviour of the second

Wronskian, which can be calculated from (2.8) and (2.11).

$$f_{\pm}(\lambda, k) = \binom{+}{-i} 2\sqrt{k} \exp\left\{\int_0^{x_0} \left(\sqrt{V(x) + \frac{\lambda^2}{x^2} - k^2} - \sqrt{V(x)}\right) dx + \right.$$
$$\left. + \lambda \log\lambda - \lambda \mp i\lambda\frac{\pi}{2} + i\frac{\pi}{4}\right\} (1 + 0(|\lambda|)) . \tag{3.2}$$

This asymptotic formula is valid in the interior of the shaded domain respectively in Fig. 1 (in case of $f_+(\lambda, k)$) and Fig. 2 (in case of $f_-(\lambda, k)$).

From (3.2) we can prove the asymptotic limit of the $S$-matrix

$$S(\lambda, k) = \exp\left(i\pi\left(\lambda - \frac{1}{2}\right)\right) \frac{f_+(\lambda, k)}{f_-(\lambda, k)} \sim 1 \tag{3.3}$$

in the interior of the shaded domain of Fig. 3. This is the same result as in the case of regular potentials.

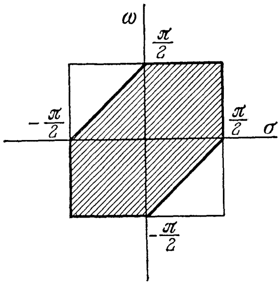



Fig. 3

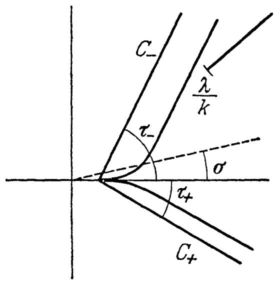



Fig. 4

Fig 3. The $S$-matrix tends to unity in the shaded region

Fig. 4. The integration-pats of the integral equations for the Jost solutions

Now we are able to prove the order of the Jost function in the whole $\lambda$-plane. From the asymptotic behaviour (3.2) we can establish the order in the angle $\Delta_{\pm}$. We know from Section 2 that the first term in the exponent of (3.2) is bounded by $C|\lambda|$. The rest of the exponent behaves asymptotically like $\lambda \log\lambda$. Thus the Jost function is bounded by $N \exp(M|\lambda| \log |\lambda|)$ in the angle $\Delta_{\pm}$. It follows that its order in the angle $\Delta_{\pm}$ is one at most. But we can easily check from (3.2) that for real $\lambda$ the Jost function grows faster than the exponential function. Hence, in fact it is of order one and infinite type in the angle $\Delta_{\pm}$. On the other hand the Jost function is bounded by $K \exp(C|\lambda|^2)$ in the whole $\lambda$-plane. This follows from the definition of the Jost function and the corresponding rough bounds on the solutions of the Schrödinger equation and their derivatives, as shown in the end of the Appendix. Applying now the Phragmén-Lindelöf theorem as in Section 2, we conclude that the Jost function is of order one and infinite type in the whole $\lambda$-plane.

Knowing the order of the Jost function we can establish its analytic representation. As the Jost function is even in $\lambda$ we prefer again to use the new variable $w = \lambda^2$ and to define the entire function $h_\pm(w, k) = f_\pm(\lambda, k)$ which is of order 1/2 in the whole $w$-plane. According to the Hadamard's factorization theorem [5] we can write

$$h_\pm(w, k) = w^{m_\pm} e^{Q_\pm(w)} P_\pm(w) ,$$

where $Q_\pm(w)$ is a polynomial in $w$ of degree $q \leqq \varrho$, $\varrho$ being the order of $h_\pm(w, k)$. $P_\pm(w)$ is the canonical product of genus $p$.

Since in our case $\varrho = \frac{1}{2}$, we have $q = 0$. Furthermore, $m_\pm = 0$. The function $h_\pm(w, k)$ is holomorphic in the polycircle

$$D \equiv \{w, k \mid |w| < r, |k - k_0| < \operatorname{Im} k_0\} ,$$

where $r$ is arbitrarily large, being continuous in both variables together in $D$ and being entire in $w$ and analytic regular in the $k$-plane cut along the imaginary axis [7] (the positive and negative imaginary axis for $h_\pm(w, k)$, respectively). Then there exists the expansion $h_\pm(w, k) = w^{m_\pm} \sum_{n_1, n_2} h_{n_1, n_2 \pm} w^{n_1} (k - k_0)^{n_2}$ uniformly convergent in $D$. Hence the function $h_\pm(w, k)/w^{m_\pm}$ is analytic in $k$ and integers $m_\pm$ cannot depend on $k$. Then it follows from the unitarity that $m_+ = m_- = m$. If $m \neq 0$, both Jost functions vanish for $\lambda = 0$ and consequently the regular solution $\varphi(0, k, x)$ vanishes identically. This contradicts the integral equation (2.12). Hence, $m = 0$, $h_\pm(0, k) \neq 0$ and we conclude that the Regge trajectories for singular potentials cannot cross the origin in the $\lambda$-plane. Finally, the genus $p$ of the canonical product $P_\pm(w)$ must be zero because of the inequality $p \leqq \varrho = \frac{1}{2}$ (compare (2.5.19) of Reference [5]). Hence $P_\pm(w) = \Pi_n (1 - w/w_{n\pm})$, where $w_{n\pm}$ are the zeros of the Jost function. Since $h_\pm(w, k)$ is an entire function of non-integral order it has necessarily an infinite number of zeros (see Theorem 2.9.2 of Reference [5]). After this discussion we can write the general representation of the Jost functions and the $S$-matrix:

$$f_\pm(\lambda, k) = f_\pm(0, k) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_{n\pm}^2(k)}\right) , \tag{3.4}$$

$$S(\lambda, k) = e^{i\pi\lambda} S(0, k) \prod_{n=1}^{\infty} \frac{1 - \dfrac{\lambda^2}{\lambda_{n+}^2(k)}}{1 - \dfrac{\lambda^2}{\lambda_{n-}^2(k)}} , \tag{3.5}$$

$$\lambda_{n+}^*(k) = \lambda_{n-}(k^*) .$$

At the end of this section we want to investigate the asymptotic distribution of zeros of the Jost functions $f_{\pm}(\lambda, k)$ as a consequence of their being even entire functions in $\lambda$ of order one, $k$ fixed, $\mathrm{Re}\, k > 0$. For this purpose we introduce again the variable $w = \lambda^2$ in the expression (3.4) for the Jost functions $h_{\pm}(w, k) = f_{\pm}(\lambda, k)$. This is an entire function of order 1/2 in the $w$-plane. According to the theorem 2.5.12 of Reference [5], the number $n(r)$ of zeros in $|w| \leqq r$ is bounded from above by $C r^{1/2+\varepsilon}$ for every positive $\varepsilon$. The lower bound on $n(r)$ follows from the theorem 2.9.4. and our previous result on the asymptotic maximum $M(r)$ of the Jost functions: $M(r) > C \exp(r^{1/2} \log r)$. We obtain that asymptotically $n(r) > N r^{1/2} \log r$. Hence the number $n(|\lambda|)$ of zeros in the $\lambda$-plane cannot grow asymptotically faster than $|\lambda|^{1+\varepsilon}$, $\varepsilon > 0$, and slower than $|\lambda| \log|\lambda|$.

For $k$ in the first quadrant $(0 < \sigma < \pi/2)$ of the $k$-plane the zeros of $f_{-}(\lambda, k)$ are distributed asymptotically in a region along the imaginary axis which is narrower than any angle, as it can be seen from Fig. 2. Furthermore, because of the eveness of the Jost function and the continuity equation, these zeros can occur only in the first and third quadrant of the $\lambda$-plane. These zeros correspond to the poles of the $S$-matrix. Similar discussion can be done for $k$ from the fourth quadrant $(-\pi/2 < \sigma < 0)$. But, as it can be immediately seen from Fig. 2, in this case we cannot claim that the region of zeros is asymptotically narrower than any angle.

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## Appendix

The Appendix is devoted to the proofs that the functions $g_{\pm}(\lambda, k, z)$ and $u(\lambda, k, x)$ tend to unity, to the discovery of the uniform bounds with respect to $\lambda$ of the derivatives of these functions and finally to the determinations of an upper bound on the Jost and regular solutions.

We rewrite the integral equations (2.9) for the functions $g_{\pm}(\lambda, k, z)$ in the more convenient form for our purposes:

$$g_{\pm}(\lambda, k, z) - 1 = \int\limits_{C_{\pm}(z,\infty)} H_{\pm}(\lambda, k, z, z') + \int\limits_{C_{\pm}(z,\infty)} H(\lambda, k, z, z') \times \times (g_{\pm}(\lambda, k, z') - 1)\, dz' . \tag{A.I}$$

To prove that $g_{\pm}(\lambda, k, z)$ tend to unity we have to find the bound $K(\lambda, k, z')$ of the functions $H_{\pm}(\lambda, k, z, z')$ which is valid for large $\lambda$ and whose integral tends to zero as $\lambda$ increases. Then the proof follows from the estimates of the integral equations (A.I)

$$\begin{aligned}|g_\pm(\lambda, k, z) - 1| &< \int\limits_{C_\pm(z_1,\infty)} K(\lambda, k, z')\,|dz'| \times \\ &\times \exp\Big(\int\limits_{C_\pm(z_1,\infty)} K(\lambda, k, z')\,|dz'|\Big)\,, \quad z \in C(z_1, \infty)\,.\end{aligned} \tag{A.II}$$

Certainly, the desired bound cannot be found for every choice of path. The limiting values $\lim\limits_{\lambda\to\infty} g_\pm(\lambda, k, z) = 1$ in the shaded region of Fig. 1 and Fig. 2 respectively can be proved for any fixed $z$, $\mathrm{Re}\, z > 0$, but we restrict ourselves to real $z$, $z > 0$, because we do not need more in the investigations of the Jost functions. Let us choose some fixed $x$ on the positive real axis and draw two rays from the point $x$: $z = x + \varrho \exp i\tau_\pm$ where $-\pi/2 < \tau_+ < \min(-\sigma, \omega - \sigma)$, $\max(-\sigma, \omega - \sigma) < \tau_- < \pi/2$ (see Fig. 4). Generally the cuts $z = \pm\frac{\lambda}{k}u$, $u > 1$ of the square root $(\lambda^2/z^2 - k^2)^{1/2}$ cross the mentioned rays. But for $\lambda$ large enough this does not happen. As we are interested in large $\lambda$ we can put our integration paths $C_\pm(x, \infty)$ of the equations (A.I) on the defined rays from the point $x$. We prove now that this choice of the integration paths enables us to obtain the bound $K(\lambda, k, z')$. First we consider the exponential function in the definition (2.10) of the functions $H_\pm(\lambda, k, z, z')$. The exponent is

$$-\int\limits_{C(z_1,z_2)} \sqrt{\frac{\lambda^2}{z^2} - k^2}\; dz = -\int\limits_{s_1}^{s_2} \sqrt{\frac{\lambda^2}{s^2\, e^{2i\tau}} - k^2}\; ds\, \frac{e^{i\tau_\pm}}{\cos(\tau_\pm - \tau)}\,, \tag{A.III}$$

where $\tau$ is the argument of the point $z = s e^{i\tau}$. It is easy to see that the real part of (A.III) is negative for $z_1, z_2 \in C_+$ if $\sigma < 0$, $\omega > 0$ and for $z_1, z_2 \in C_-$ if $\sigma > 0$, $\omega < 0$. For other choices of the variables $\sigma$ and $\omega$ the real part of (A.III) is not negative generally except for $z_1, z_2$ large enough, so that we have to change the paths $C_\pm$ near the real axis. They start from the point $x$ along the curves $r = x\exp(\pm A\tau)$, $\tau \gtrless 0$, to the points $z_\pm$ determined by $\tau = \tau_\pm$ respectively. The rest $C_\pm(z_\pm, \infty)$ are laid on the straight lines inclined by $\tau_\pm$ respectively (see Fig. 4). Now the real part of (A.III) is negative on $C_\pm$ for $-\frac{\pi}{2} + \sigma < \pm\,\omega <$ $< \mathrm{arc\,tg}\, A$ respectively and $A$ is as large as we want. We cannot obtain the whole range $|\omega| < \frac{\pi}{2}$ because the branch point $\lambda/k$ pinches the paths $C_\pm$ to the negative and positive imaginary axis respectively. In this way the exponential function in the expression (2.10) for $H_\pm(\lambda, k, z, z')$ can be estimated by unity for $\sigma$ and $\omega$ in the shaded region of Fig. 1 and Fig. 2 respectively. We divide now the paths $z = r e^{i\tau}$, $z \in C_\pm$ into two parts $0 \leqq r < |\lambda/k|$ and $|\lambda/k| \leqq r < \infty$. We can find the estimates of the functions $H_{\pm}(\lambda, k, z, z')$ on these parts of the integration paths

$$\begin{aligned} K_1(\lambda, k, z) &= \frac{x+r}{|\lambda|}\left(\frac{A_1}{(x + r\cos\tau_{\pm})^{\alpha}} + \frac{B_1}{|\lambda|^2}\right) \quad \text{for} \quad r < \left|\frac{\lambda}{k}\right|, \\ K_2(\lambda, k, z) &= \frac{A_2}{r^{\alpha}} + \frac{B_2}{r^2} \quad \text{for} \quad \left|\frac{\lambda}{k}\right| \leqq r < \infty, \end{aligned} \tag{A.IV}$$

where $\alpha$ comes from the estimate of the potential for large $z$ and hence $\alpha \geqq 2$. The constants $A_i$ and $B_i$ depend on the variable $k$ only. We insert the estimate (A.IV) into the inequality (A.II) and obtain that $g_{\pm}(\lambda, k, z)$ tend to unity uniformly with respect to $z$ from the integration paths $C_{\pm}(x, \infty)$ and the error is of order $1/|\lambda|$.

It is easy now to find the bounds of the derivatives $\frac{d}{dx} g_{\pm}(\lambda, k, x)$ uniformly with respect to large $\lambda$ in the shaded regions of Fig. 1 and Fig. 2 respectively.

$$\frac{d}{dz} g_{\pm}(\lambda, k, z) = -\int\limits_{C_{\pm}(z,\infty)} \left(\frac{\frac{\lambda^2}{z^2} - k^2}{\frac{\lambda^2}{z'^2} - k^2}\right)^{1/2} \exp\left\{-2 \int\limits_{C_{\pm}(z,z')} \sqrt{\frac{\lambda^2}{\xi^2} - k^2}\, d\xi\right\} \times$$
$$\times \left(V(z') - \frac{\lambda^2 k^2 + k^4 z'^2/4}{(\lambda^2 - k^2 z'^2)^2}\right) g_{\pm}(\lambda, k, z')\, dz'.$$

We know the estimates of the functions under the integral sign from before except of the square root $(\lambda^2/z^2 - k^2)^{1/2}$. This one can be estimated by $C|\lambda|$ so that we have

$$\left|\frac{d}{dz} g_{\pm}(\lambda, k, z)\right| < K, \quad z \in C_{\pm}(x, \infty)$$

and $K$ depends on the variables $k$ and $x$ only.

We turn now to the proof that the solution $u(\lambda, k, x)$ of the integral equation (2.12) tends to unity when $\lambda$ tends to infinity along the rays $\lambda = |\lambda| \exp i\omega$, $|\omega| < \pi/2$ for every fixed $k$ and $x$. It can be proved that the same property holds for every $z$, $\operatorname{Re} z > 0$, instead of only for real $z$. The proof is much more complicated because of the choice of the integration path and for our purposes the proof for real $z$ is sufficient. (The reader can see an analogous complication in the case of Jost solutions in the Reference [3]). We rewrite the integral equation (2.12) in the form

$$u(\lambda, k, x) - 1 = \int_0^x H(\lambda, k, x, y)\, dy + \int_0^x H(\lambda, k, x, y)\,(u(\lambda, k, y) - 1)\, dy, \tag{A.V}$$

where the function $H(\lambda, k, x, y)$ is defined by (2.13). Again we look for the bound $K(\lambda, k, y)$ of the function $H(\lambda, k, x, y)$ such that the integral $\int_0^w K(\lambda, k, y)\, dy$ tends to zero when $\lambda$ tends to infinity in the half plane $\operatorname{Re}\lambda > 0$, $w$ being any positive fixed number. The exponential function in (2.13) can be estimated by unity for large $\lambda$ because the exponent has a negative real part. Namely, the square root in the exponent for positive $x$ has a positive real part if $\lambda$ is large enough in the half plane $\mathrm{Re}\,\lambda > 0$ for any fixed $k$. The other factors in the expression (2.13) can be estimated as in Reference [3]. Here we prefer to quote the results only rather than to derive them, since the proofs can be found in Reference [3]. We define the point $x_1$ as the solution of the equation $x^2 V(x) = |\lambda|$. There exists only one solution $x_1$ for large $\lambda$ because of (2.3). Any fixed interval $(0, w)$ can be divided into three parts $(0, x_1)$, $(x_1, x_0)$ and $(x_0, w)$. The third part disappears if $w \leqq x_0$. Then the function $H(\lambda, k, x, y)$ can be estimated by the functions $K_i(\lambda, k, y)$ in the corresponding intervals respectively, where

$$\begin{aligned} K_1(\lambda, k, x) &= \frac{C_1}{\sqrt{V(x)}} R(x) && \text{in } (0, x_1)\,, \\ K_2(\lambda, k, x) &= \frac{C_2}{\sqrt{|\lambda|\, V(x)}} R(x) && \text{in } (x_1, x_0)\,, \\ K_3(\lambda, k, x) &= \frac{C_3 x}{|\lambda|} && \text{in } (x_0, w)\,. \end{aligned} \tag{A.VI}$$

The function $R(x)$ is the expression in the brackets of (2.2). We estimate the integral equation (A.V) by

$$|u(\lambda, k, x) - 1| < \int_0^w K(\lambda, k, y)\, dy \exp \int_0^w K(\lambda, k, y)\, dy$$

and using the expressions (A.VI) we conclude in the same way as in Reference [3] that the function $u(\lambda, k, x)$ tends to unity for large $\lambda$ $\mathrm{Re}\,\lambda > 0$ and fixed $k$.

The bound on the derivative: $\left|\frac{d}{dx} u(\lambda, k, x)\right| < C(x)$, where $C(x)$ is finite for $x > 0$ uniformly with respect to large $\lambda$ can be found from the equation (A.V) and the boundedness of $u(\lambda, k, x)$ for large $\lambda$.

At the end of the Appendix we find the rough bounds in the whole $\lambda$-plane of the regular solution and its derivative. The regular solution defined by the boundary condition can be naturally factorized in the way $\varphi(\lambda, k, x) = \varphi_0(x)\, v(\lambda, k, x)$, where the function $v(\lambda, k, x)$ is the solution of the integral equation from Ref. [2]

$$\begin{aligned} v(\lambda, k, x) = 1 + \int_0^x \frac{1}{2\sqrt{V(y)}} \left(1 - \exp\left(-2 \int_y^x \sqrt{V(z)}\, dz\right)\right) \times \\ \times \left(\frac{\lambda^2 - 1/4}{y^2} - \frac{5}{16}\left(\frac{V'(y)}{V(y)}\right)^2 + \frac{1}{4}\frac{V''(y)}{V(y)}\right) v(\lambda, k, y)\, dy\,. \end{aligned} \tag{A.VII}$$

The kernel of this integral equation can be estimated as in Reference [2] so that we have the integral inequality

$$|v(\lambda, k, x)| < 1 + \int_0^x \left(|\lambda|^2 \frac{1}{y^2 \sqrt{V(y)}} + CR(y)\right) |v(\lambda, k, y)| \, dy \, ,$$

where $R(x)$ is the expression in the brackets (2.2). Finally we obtain

$$|\varphi(\lambda, k, x)| = \varphi_0(x) \, |v(\lambda, k, x)| < C_1 \exp(C_2 |\lambda|^2) \, . \qquad \text{(A.VIII)}$$

The derivative $\frac{d}{dx} \varphi(\lambda, k, x)$ has all the dependence on $\lambda$ in the function $v(\lambda, k, x)$ and its derivative. Therefore we have to estimate only the derivative

$$\frac{d}{dx} v(\lambda, k, x) = - \sqrt{V(x)} \int_0^x \frac{1}{\sqrt{V(y)}} \exp\left(-2 \int_y^x \sqrt{V(z)} \, dz\right) \times$$
$$\times \left(\frac{\lambda^2 - 1/4}{y^2} - \frac{5}{16} \left(\frac{V'(y)}{V(y)}\right)^2 + \frac{1}{4} \frac{V''(y)}{V(y)}\right) v(\lambda, k, y) \, dy \, .$$

Knowing the estimate of the function $v(\lambda, k, x)$ expressed by (A.VIII) we conclude that $\left|\frac{d}{dx} v(\lambda, k, x)\right| < K \, |\lambda|^2 \exp(C_2 |\lambda|^2)$. This means that we can write

$$\left|\frac{d}{dx} \varphi(\lambda, k, x)\right| < C_3 \exp(C_4 |\lambda|^2) \, ,$$

where $C_4$ is somewhat larger than $C_2$.

We do not want to repeat the whole procedure for the Jost solutions. But one can do it easily in the same way, using the integral equations of the Reference [6] for the Jost solutions since there the variable $\lambda^2$ appears in the kernels only in the same way as in the equation (A.VII).

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