Commun. math. Phys. 2, 55—58 (1966)

Embedding of a Relativistic Charged Particle

Solution with a Real Singularity into a Pseudo-Euclidean Space

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Received October 15, 1965

Abstract. A charged particle following the Reissner-Weyl vacuum field-distribution shows in its interior a real singularity (the matter-tensor becomes infinite). By embedding the interior submanifold $ds^2 = g_{11} \cdot dr^2 + g_{00} \cdot dt^2$ into a pseudo-Euclidean space $E_3 : ds^2 = dZ_1^2 + dZ_2^2 - dZ_3^2$ one finds that the embedded (r, t)-metric looks like a cone with the top liying in the Z_1, Z_2 plane. The general formulas of embedding the complete manifold into a pseudo-Euclidean space E_6 are discussed.

I. Introduction

The problem of embeddings of non-Euclidean metrics into pseudo-Euclidean spaces is of current interest. The embedding of various relativistic Einstein-Riemannian spaces has been given in the literature [1]. Leaving out the difficult mathematical aspect which concerns the classification of relativistic manifolds, embedding methods seem useful to obtain a geometrical imagination of non-Euclidean metrics as was first pointed out by FRONSDAL [2].

In a previous publication [3] the author considered a complete relativistic solution of a Reissner-Weyl-like particle [4] which was also applied to a curved de Sitter background. The Coulomb-repulsion is balanced by gravitational action, but no Newtonian approximation exists because space-time is strongly non-Euclidean even for an infinitesimal mass of the particle. This is due to a singularity of space-time at the origin of the particle, where the trace of the matter-tensor becomes infinite. The radius of the particle is directly proportional to its mass. To get a better feeling for the character of the singularity we shall embed our solution into a pseudo-Euclidean space. This adds to our knowledge of embedded four-dimensional solutions of Einstein's theory.

II. General formulas of embedding the particle into E_6

From Einstein's equations

$$R_{ik} - g_{ik} \cdot R/2 = -\varkappa (T_{ik} + U_{ik}) - \lambda \cdot g_{ik} ,$$

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(T_{ik}, U_{ik} are the energy tensors of matter and electromagnetic field), we obtained the complete solution [3] of a Reissner-Weyl like-particle (in terms of polar-coordinates)

Exterior (vacuum):

$$-g^{11} = g_{00} = [(1 - r_0/3r)^2 - (4r_0^3/r - r_0^4/r^2 + 5r^2) \cdot (\lambda/15)], \qquad (1)$$
$$g_{22} = -r^2, \; g_{33} = -r^2 \cdot \sin^2 \cdot \vartheta \, .$$

Interior (charged matter)

$$\begin{aligned} &g_{11} = -[4/9 - (8/15)\lambda \cdot r^2]^{-1}, \quad g_{00} = C \cdot r, \\ &g_{22} = -r^2, \quad g_{33} = -r^2 \cdot \sin^2 \cdot \vartheta, \\ &(C = 4/9r_0 - (8/15)\lambda \cdot r_0), \end{aligned} \qquad (2)$$

where r_0 is the particle radius.

There appears to be a "coordinate-singularity" at $r = r_0/3$ (for simplicity we take $\lambda = 0$) which however is of no physical interest since (1) is valid only for $r \geqq r_0$. The interior manifold (2) shows a real singularity as one can see from the time-coefficient g_{00}.

Now we perform the general embedding of the vacuum manifold (1) into a pseudo-Euclidean space E_6 which is minimal. Using the notation $(+ + - - - -)$ the line element is then given by

$$ds^2 = dZ_1^2 + dZ_2^2 - dZ_3^2 - dZ_4^2 - dZ_5^2 - dZ_6^2$$

with two temporal coordinates Z_1, Z_2. ("+" is used for temporal coordinates and "—" for spatial).

The embedding transformations for the vacuum manifold (1) are of the form

$$\begin{aligned} &Z_1 = \sqrt{g_0} \cdot \sin \cdot t \\ &Z_2 = \sqrt{g_0} \cdot \cos \cdot t \\ &Z_3 = \int dr \cdot \left[\frac{(g_{0,r}/2)^2 + 1}{g_0} - 1\right]^{1/2} \\ &\left.\begin{aligned} &Z_4 = r \cdot \cos \cdot \vartheta \\ &Z_5 = r \cdot \sin \cdot \vartheta \cdot \cos \cdot \varphi \\ &Z_6 = r \cdot \sin \cdot \vartheta \cdot \sin \cdot \varphi \end{aligned}\right\} \; Z_4^2 + Z_5^2 + Z_6^2 = r^2, \end{aligned} \qquad (3a)$$

where $g_0 = g_{00}$, $g_{00} \cdot g^{11} = -1$, $g_{0,r}$ means differentiation with respect to the coordinate "r" and "t" is the time coordinate.

The embedding of the interior manifold (2) into E_6 is given by

$$\begin{aligned} &Z_1 = \sqrt{C \cdot r} \cdot \sin \cdot t \\ &Z_2 = \sqrt{C \cdot r} \cdot \cos \cdot t \\ &Z_3 = \int dr \cdot [C/4r + (4/9 - (8/15)\,\lambda \cdot r^2)^{-1} - 1]^{1/2} \\ &\quad Z_4^2 + Z_5^2 + Z_6^2 = r^2 \, . \end{aligned} \qquad (3b)$$

For further investigation we try to draw a picture of the interior manifold. We have to specialize to the subspace $d\vartheta = d\vartheta = 0$ and we get a 2-dimensional surface in a pseudo-Euclidean space E_3. The surface represents the complete interior space-time.

III. The interior surface

The manifold $\vartheta = \text{const}$, $\varphi = \text{const}$. means that the line element of the solution (2) becomes (we take $\lambda = 0$)

$$ds^2 = C \cdot r \cdot dt^2 - (9/4) \cdot dr^2 , \tag{4}$$

where $C = 4/9 r_0$. This can be embedded in the space [1]

$$ds^2 = dZ_1^2 + dZ_2^2 - dZ_3^2 , \tag{5}$$

(which is not a subspace of the Euclidean space E_6 defined before).

From the identity (4) and (5) the following transformation is valid (for simplicity we put the particle-radius $r_0 = 1$):

$$\begin{aligned} Z_1 &= (2/3) \cdot \sqrt{r} \cdot \sin \cdot t \\ Z_2 &= (2/3) \cdot \sqrt{r} \cdot \cos \cdot t \\ Z_3 &= (3/2) \cdot \int dr \cdot [1 + 4/81 r]^{1/2} . \end{aligned} \tag{6}$$

The surface is then defined by the parameter-representation

$$\begin{aligned} &Z_1^2 + Z_2^2 = (4/9) \cdot r \\ &Z_3 = (3/2) \cdot \left[\sqrt{(4/81 + r) \cdot r} + (2/81) \ln \cdot \frac{\sqrt{(4/81 + r)} + \sqrt{r}}{\sqrt{(4/81 + r)} - \sqrt{r}}\right] \\ &0 \leqq r \leqq 1 . \end{aligned} \tag{7}$$

We note that (7) admits to a rotation group in the Z_1, Z_2 plane. It is also invariant under "time" reflection $Z_1 \to -Z_1$, $Z_2 \to -Z_2$. Drawing the

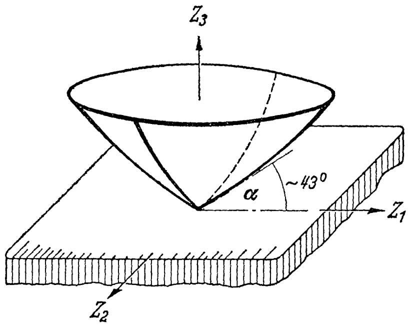

Fig. 1. The surface defined by Eq. (7) of the embedded interior manifold of a charged particle

picture of the embedded manifold schematically one gets Fig. 1 with the numerical values

$$\alpha = \text{arc} \cdot \text{tg} \cdot (15/18) \sim 43^\circ, \quad 0 \leqq Z_3 \leqq 1{,}7, \quad 0 \leqq Z_1, Z_2 \leqq 2/3 .$$

Near the top ($Z_3 \to 0$, the singular point) the surface looks like a cone.

[1] It is an interesting property of (2) that a choice of notation (+ — —) would not permit a complete embedding.

The embedding shows thus, that the topological character of the manifold is completely different from the one of the Schwarzschild line element. In our model space-time is simply connected while the Schwarzschild singularity is well known to have a "throat", i. e. a doubly connected space-time. The singularity encountered in our model is thus far less serious than the one of Schwarzschild's solution.

The author would like to thank Dr. R. U. SEXL for valuable discussions.

References

[1] ROSEN, N. J.: Rev. Mod. Phys. **37**, 204 (1965).
[2] FRONSDAL, C.: Phys. Rev. **116**, 778 (1959).
[3] EFINGER, H. J.: Z. Physik 188, 31 (1965).
[4] REISSNER, H.: Ann. Phys. (Leipzig) **50** (1916).