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Energy and Momentum as Observables in Quantum Field Theory

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Abstract. The spectrum condition implies that energy and momentum are limits of local observables.

I. Introduction and results

In the algebraic theory of local observables the inhomogeneous Lorentz group appears only as a group of automorphisms. Such a setup does in general not guaranty the implementability of these automorphisms by unitary operators in the representations space of the local observables. Even if we assume the existence of such unitary operators we have no reason to believe that they give rise to a continuous representation of this group. In other words, for such a representation energy, momentum and angular momentum cannot be defined. The importance of these quantities for the physical interpretation forces us to postulate the existence of a strongly continuous representation of the inhomogeneous Lorentz group which fulfils the spectrum condition. From this assumption arises immediately the question whether energy, momentum and angular momentum are observables.

Before we go on in our discussion let us try to state the problem more precisely. The above postulate implies the existence of a least one continuous representation of the inhomogeneous Lorentz group which implements the automorphisms of the local observables. However, such a representation need not be the only representation which fulfils these requirements. In particular it might happen that one of the representations consists of elements of the v. Neumann algebra generated by all local observables. When this is the case the energy, momentum and angular momentum are observable quantities.

The purpose of this note is to show that at least energy and momentum are observable quantities. This is true for general reasons which have nothing to do with the special properties of local observables but only with the spectrum condition. We will prove the following

Theorem. Let $\mathfrak{A}$ be a v. Neumann algebra and G be an n-parametric Abelian group. Assume we have a strongly continuous unitary representation $U(g)$ of G such that

1) $U(g)\,\mathfrak{A}\,U^{-1}(g) = \mathfrak{A}$ for all $g \in G$.

2) The spectrum of the representation is contained in a proper cone (apex not necessarily at the origin).

Then

a) The center of $\mathfrak{A}$ is pointwise invariant.

b) There exists a strongly continuous unitary representation $V(g) \in \mathfrak{A}$ and $V(g)\,A\,V^{-1}(g) = U(g)\,A\,U^{-1}(g)$ for every element $A \in \mathfrak{A}$.

This theorem solves the above mentioned problem for the translations ans shows that at least energy and momentum are observable quantities. Statement a) of this theorem is known and proved first by ARAKI ([1] Prop. 1). But our method of proof will be different from ARAKI's. Our proof is based on a new result in the theory of v. Neumann algebras which says that every norm-continuous connected group of automorphisms consists of inner automorphisms. This result is due to KADISON, RINGROSE [2], [3] and SAKAI [4]. By reduction to our method G. F. DELL'ANTONIO (private communication) showed that a weakly continuous one-parameter group of automorphisms of a v. Neumann algebra is induced by a strongly continuous representation of the group inside the algebra if it satisfies certain conditions implying the semi-boundedness of the spectrum.

For the homogeneous part of the Lorentz group we do not know whether it can be chosen to be an inner group of automorphisms. However, there are indications that this does not follow from general reasoning. This means that in order to decide this question one has to take into account the special algebraic structure of the local observables.

II. Proofs

Lemma 1. Let $\mathfrak{A}$ be a v. Neumann algebra and ϕ be an automorphism of $\mathfrak{A}$. Then the following statements 1 and 2 resp. 1′ and 2′ are equivalent:

1) ϕ is unitarily implementable (ϕ is spatial).

1′) ϕ is unitarily implementable by a unitary operator in the v. Neumann algebra (ϕ is spatial and inner).

2) resp. 2′) There exists an increasing sequence of projections in the v. Neumann algebra with

a) $\phi(E_n) = E_n$,

b) the central supports F_n of E_n tend to $\mathbf{1}$ i. e. $\lim_{n\to\infty} F_n = \mathbf{1}$,

c) resp. c′) For each algebra $E_n \mathfrak{A} E_n$ the automorphism ϕ restricted to this algebra is unitarily implementable resp. is unitarily implementable by an element of $E_n \mathfrak{A} E_n$.

Proof. From 1 resp. 1′ follows 2 resp. 2′ by setting $E_n = \mathbf{1}$.

Let us assume now 2 resp. 2′ and denote by U_n the unitary operator defined in $E_n\mathfrak{H}$ which implements ϕ restricted to $E_n\mathfrak{A}E_n$. Define in $F_n\mathfrak{H}$ the operator V_n by

$$V_n \sum_\nu A_\nu E_n \psi_\nu = \sum_\nu \phi(A_\nu) U_n E_n \psi_\nu; \quad A_\nu \in \mathfrak{A}.$$

We have:

$$\begin{aligned}\left\| V_n \sum_\nu A_\nu E_n \psi_\nu \right\|^2 &= \left(\sum_\nu \phi(A_\nu) U_n E_n \psi_\nu, \sum_\mu \phi(A_\mu) U_n E_n \psi_\mu \right) \\ &= \sum_{\nu\mu} (E_n \psi_\nu, U_n^{-1} \phi(E_n A_\nu^* A_\mu E_n) U_n E_n \psi_\mu) \\ &= \sum_{\nu\mu} (E_n \psi_\nu, A_\nu^* A_\mu E_n \psi_\mu) = \left\| \sum_\nu A_\nu E_n \psi_\nu \right\|^2.\end{aligned}$$

Hence V_n is isometric and linear. Now V_n maps$F_n\mathfrak{H} = \mathfrak{A}E_n\mathfrak{H}$ onto $F_n\mathfrak{H}$ or V_n is unitary in $F_n\mathfrak{H}$. If, in particular, 2c′ holds, i.e. $U_n \in E_n\mathfrak{A}E_n$ then V_n is in $F_n\mathfrak{A}F_n$.

Let us now define $V = \sum_n V_n(F_n - F_{n-1})$. This expression defines a unitary operator and we have for A in the v. Neumann algebra

$$\begin{aligned} V A V^{-1} &= \sum V_n F_n (\mathbf{1} - F_{n-1}) A (\mathbf{1} - F_{n-1}) F_n V_n^{-1} \\ &= \sum \phi(F_n(\mathbf{1} - F_{n-1}) A (\mathbf{1} - F_{n-1}) F_n) \\ &= \sum F_n(\mathbf{1} - F_{n-1}) \phi(A) = \phi(A).\end{aligned}$$

If all V_n are from $F_n\mathfrak{A}$ then V is an element in $\mathfrak{A}$. This proves the lemma.

Lemma 2. Let $\mathfrak{A}$ be a v. Neumann algebra and ϕ an automorphism of $\mathfrak{A}$. Assume we have an increasing family of projections E_n with

a) $\phi(E_n) = E_n$,

b) The central supports F_n of E_n tend to $\mathbf{1}$.

Assume ϕ restricted to $E_n\mathfrak{A}E_n$ leaves each central element of this algebra fixed, then ϕ leaves each central element of $\mathfrak{A}$ fixed.

Proof. Let G be a central projection of $E_n\mathfrak{A}E_n$. Then the weakly closed two-sided ideal in $\mathfrak{A}$ generated by G is clearly invariant under ϕ. Now each weakly closed two-sided ideal in $\mathfrak{A}$ is of the form $F\mathfrak{A}$, where F is a central projection of $\mathfrak{A}$ ([5], I, § 4, 6). Since $F\mathfrak{A}$ is invariant under ϕ we see that $\phi(F) = F$. Let now F be any central projection of $\mathfrak{A}$ then FE_n is central projection of $E_n\mathfrak{A}E_n$. Hence FE_n is invariant. Is F_n the central support of E_n then the two-sided ideal generated by FE_n is $FE_n\mathfrak{A}$. Hence FF_n is invariant. Since this holds for all n we have F is invariant.

Lemma 3. Let $\mathfrak{A}$ be a v. Neumann algebra and G be a Lie group. Assume we have for every $g \in G$ an automorphism ϕ_g of $\mathfrak{A}$. Assume, moreover, we have a strongly continuous unitary representation of G such that $U(g) A U^{-1}(g) = \phi_g(A)$ for all $g \in G$ and all $A \in \mathfrak{A}$. Let us assume we have an increasing sequence E_n of projections in $\mathfrak{A}$ such that

1) $\phi_g(E_n) = E_n$ for all g,

2) The central supports F_n of E_n converge to $\mathbf{1}$,

3) $U(g)\,E_n = U_1^n(g)\,U_2^n(g)\,E_n$ with $U_1^n(g) \in E_n \mathfrak{A} E_n$, $U_2^n(g) \in \mathfrak{A}' E_n$ and $U_1^n(g)$ is a strongly continuous representation of G in $E_n \mathfrak{H}$.

Then $U(g) = U_1(g)\,U_2(g)$ with $U_1(g) \in \mathfrak{A}$, $U_2(g) \in \mathfrak{A}'$ and $U_1(g)$ is a strongly continuous representation of G.

Proof. Since $U(g)E_n$ and $U_1^n(g)E_n$ are strongly continuous representations we have that also $U_2^n(g)E_n$ is a strongly continuous representation of G. Let F_n be the central support of E_n then there exist uniquely defined unitary operators $\hat{U}_2^n(g)F_n$ in $F_n\mathfrak{A}'$ with $\hat{U}_2^n(g)E_n = U_2^n(g)$. Since $U_2^n(g)$ is strongly continuous we have a dense set $\mathscr{D}_0$ in $E_n\mathfrak{H}$ of vectors which are simultaneously analytic vectors for every one-parametric subgroup of G. Now $\mathfrak{A}\mathscr{D}_0$ is such a dense set in $F_n\mathfrak{H}$. Hence the representation $\hat{U}_2^n(g)$ in $F_n\mathfrak{H}$ is strongly continuous (see e.g. NELSON [6]). But this implies that also $\hat{U}_1^n(g) = F_n U(g)\,\hat{U}_2^n(g^{-1}) \in \mathfrak{A}F_n$ is strongly continuous. Defining $U_1(g) = \sum (F_n - F_{n-1})\,\hat{U}_1^n(g)$ we see that $U_1(g) \in \mathfrak{A}$ and is strongly continuous.

Lemma 4. Let $\mathfrak{A}$ be a v. Neumann algebra and G be an n-parametric Abelian group, such that for every $g \in G$ we have an automorphism ϕ_g of $\mathfrak{A}$. Assume we have a strongly continuous unitary representation of G which implements ϕ_g. If the spectrum of the representation is contained in a proper cone then there exists an increasing family of projections $E_n \in \mathfrak{A}$ with

1) E_n tends to $\mathbf{1}$,

2) $\phi_g(E_n) = E_n$ for all g,

3) ϕ_g restricted to $E_n\mathfrak{A}E_n$ is norm-continuous.

Proof. Since the cone C is a proper cone we can introduce coordinates $p_1 \ldots p_n$, such that C is contained in $p_1 \geqq 0$, $a\,p_1^2 - \sum_2^n p_i^2 > 0$ with $0 < a < \infty$. Since we have a continuous group representation we might write $U(x) = \int_C e^{ipx} dE_p$ $x \in G$. Let us define the projections G_λ by $G_\lambda = \int_{C,\,p_1<\lambda} dE_p$ and the projections E_λ by $E_\lambda\mathfrak{H} = \overline{\mathfrak{A}' G_\lambda \mathfrak{H}}$. E_λ is clearly a projection from $\mathfrak{A}$. We want to show that E_λ has the properties of this lemma.

1) Since $G_\lambda \to \mathbf{1}$ for $\lambda \to \infty$ follows $E_\lambda \to \mathbf{1}$ because $G_\lambda \leqq E_\lambda$.

2) The space $G_\lambda\mathfrak{H}$ is from its construction invariant under all $U(x)$. Since $U(x)$ implements also an automorphism for $\mathfrak{A}'$ we have that $E_\lambda\mathfrak{H}$ is also invariant under the action of $U(x)$. Hence the projections E_λ have to commute with $U(x)$ for all $x \in G$.

3) Now we want to demonstrate that $U(x)$ gives norm-continuous automorphisms on the algebra $E_\lambda \mathfrak{A} E_\lambda$. Consider the matrix element for $A' \in \mathfrak{A}'$, $A \in \mathfrak{A}$, $x \in G$ and $\psi, \phi \in \mathfrak{H}$:

$$\left(\sum_\nu A'_\nu G_\lambda \psi_\nu,\ U(x)\, A\, U^{-1}(x) \sum_\mu A'_\mu G_\lambda \varphi_\mu\right) = f(x)\,.$$

This function is bounded for all x by

$$|f(x)| \leqq \|A\| \left\|\sum_\nu A'_\nu G_\lambda \psi_\nu\right\| \left\|\sum_\mu A'_\mu G_\lambda \phi_\mu\right\|.$$

On the other hand we see that

$$f(x) = \left(\sum_\nu A'_\nu G_\lambda \psi_\lambda, \sum_\mu A'_\mu U(x)\, A\, U^{-1}(x)\, G_\lambda \phi_\mu\right)$$

is the boundary value of an analytic function holomorphic in the tube $\operatorname{Im} x_1 > 0, \frac{1}{a}(\operatorname{Im} x_1)^2 - \sum_2^n (\operatorname{Im} x_i)^2 > 0$ and we have:

$$\begin{aligned} |f(x+iy)| &\leqq \sum_{\mu\nu} \|A'^*_\mu A'_\nu G_\lambda \psi_\nu\| \, \|U(-x-iy)\, G_\lambda\| \, \|\phi_\mu\| \leqq \\ &\leqq M e^{2\lambda y_1}\,. \end{aligned}$$

In the same way we get:

$$f(x) = \left(U(x)\, A^*\, U^{-1}(x)\, G_\lambda \psi_\nu, \sum_{\nu\mu} A'^*_\nu A'_\mu G_\lambda \phi_\mu\right)$$

is the boundary value of an analytic function holomorphic in the tube $\operatorname{Im} x_1 < 0, \frac{1}{a}(\operatorname{Im} x_1)^2 - \sum_2^n (\operatorname{Im} x_i)^2 > 0$ and is bounded there by

$$|f(x+iy)| \leqq M' e^{2\lambda |y_1|}\,.$$

Since both functions have the same boundary values we find by the "edge of the wedge"-theorem (e.g. [7], 2—5) that $f(x)$ is an entire function bounded by $|f(x)| \leqq \max(M, M')\, e^{2\lambda \|\operatorname{Im} x\|}$.

Using now the Phragmen-Lindelöf theorem (e.g. [8] 5.6.) we find

$$|f(x)| \leqq \|A\| \left\|\sum_\nu A'_\nu G_\lambda \psi_\nu\right\| \left\|\sum_\mu A'_\mu G_\lambda \phi_\mu\right\| e^{2\lambda \|\operatorname{Im} x\|}.$$

Using now Schwartz's lemma (e.g. [9] III § 6) we get for a dense set of vectors in $E_\lambda \mathfrak{H}$

$$|(\psi, U(x) E_\lambda A E_\lambda U^{-1}(x) \phi) - (\psi, E_\lambda A E_\lambda \phi)| \leqq 2\|x\|\, \|A\|\, \|\psi\|\, \|\phi\|\, e^{2\lambda}$$

with $\|x\| = \sqrt{\sum_1^n |x_i|^2}$ or

$$\|U(x) E_\lambda A E_\lambda U^{-1}(x) - E_\lambda A E_\lambda\| \leqq 2\|A\|\, \|x\|\, e^{2\lambda} \qquad \text{q.e.d.}$$

Proof of the theorem. G is an n-parametric Abelian group and $U(g)$ a strongly continuous representation of it. The spectrum of this representation is by assumption contained in a proper cone. Since we can shift the apex by multiplication with a one-dimensional representation of G we can assume that the apex is at the origin. Now, by Lemma 4, we can find an increasing family of projections $E_n \in \mathfrak{A}$ with $E_n \to \mathbf{1}$, $\phi_g(E_n) = E_n$ and ϕ_g acts norm-continuously on the algebra $E_n \mathfrak{A} E_n$. From the result of KADISON, RINGROSE and SAKAI [2], [3], [4] follows the existence of a norm-continuous inner representation $U_1^n(g) \subset E_n \mathfrak{A} E_n$ of G which implements ϕ_g. The theorem follows now from Lemma 3.

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