

On the Self-Adjointness of the Operator $-\Delta + V$

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Received August 30, 1965

Abstract. The essential self-adjointness of the operator $-\Delta + V$ is proved, where V is a potential whose main property is the high singularity and repulsiveness at the origin.

1. Introduction

The essential self-adjointness of the operator $-\Delta + V$ will be proved for a class of potentials V which are defined precisely through the conditions (1.1)—(1.5). Before writing these definitions, the qualitative description of the potential will be given briefly. The potential V is a real function $V(x, y, z)$ which is positive at the origin and its singularity there is higher than $1/r^2$ and independent of the way of approaching the origin. Outside the origin the potential $V(x, y, z)$ may possess singularities, such that the square of the potential is a locally integrable function. KATO [1] considered the same problem for another class of the potentials which essentially differ in the behaviour at the origin. The exact definition of the potential is:

The real function $V(x, y, z)$ can be decomposed in the form:

$$V(x, y, z) = V_1(x, y, z) + V_2(x, y, z) + P(r) Q(x, y, z), \quad (1.1)$$

where the four functions on the right of this decomposition satisfy the conditions:

$$\int V_1^2(x, y, z) r^\varepsilon dx dy dz < \infty, \quad \varepsilon > 0, \quad (1.2)$$

$$\limsup_{x,y,z} |V_2(x, y, z)| < \infty, \quad (1.3)$$

$$P(r) = \begin{cases} \frac{1}{r^\alpha} (\delta - r)^3 & \text{or } e^{\frac{1}{r^\beta}} (\delta - r)^3, \quad r \leq \delta, \\ 0 & r > \delta, \end{cases} \quad (1.4)$$

where $\alpha > 2$ or $\beta > 0$,

$$\limsup_{K(d)} |Q(x, y, z) - Q_0| < q(d), \quad d < \delta, \quad (1.5)$$

where $K(d)$ is the sphere of radius d centred at the origin, Q_0 is a positive constant and $q(d)$ is a monotonic continuous function which vanishes when d tends to zero. In the following we put $Q_0 = 1$ without loss of generality.

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Let $L_2(0, \infty)$ be the Hilbert space of functions $f(r)$ in which the scalar product between elements f and g is defined by $(g, f) = \int_0^\infty g^*(r) f(r) dr$.

Let E^{lm} be the one-dimensional space determined by the spherical harmonic $Y_{lm}(\vartheta, \varphi)$. We shall consider the product $L_2^{lm} = E^{lm} \otimes L_2(0, \infty)$ and the sum $L_2 = \sum \oplus L_2^{lm}$ (The obtained space L_2 is isomorphic to the space of functions $f(x, y, z)$ with integrable square of the modulus by the simple isomorphism generated by the multiplication with the function $1/\sqrt{x^2 + y^2 + z^2}$). Then the domains of the symmetric extensions of $-\Delta + V$ will be somewhere in the space L_2 . Let $\mathcal{D}(-\Delta)$ be the domain in L_2 of the symmetric operator $-\Delta$. Let \mathcal{L} be the linear manifold composed of the elements of the form $\exp\{-\lambda(P(x) + P(y) + P(z))\} \times f(x, y, z)$, $\lambda > 0$ and $f \in \mathcal{D}(-\Delta)$. In this way \mathcal{L} becomes a dense linear manifold in the space L_2 . Moreover, \mathcal{L} is contained in the domains $\mathcal{D}(-\tilde{A})$ and $\mathcal{D}(\tilde{V})$ of the closures $-\tilde{A}$ and \tilde{V} of these operators. Hence the operator $H_s = -\Delta + V$ is symmetric on the domain $\mathcal{D}(H_s) = \mathcal{L}$. We want to prove that the closure $H = \tilde{H}_s$ is a self-adjoint operator.

If the potential is spherically symmetric, $V = P$, the symmetric operator $H_1 = H_0 + P$ on \mathcal{L} is essentially self-adjoint. The operator H_1 is reduced by every subspace L_2^{lm} to the operator $I \otimes H_l \equiv I \otimes \left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + P(r)\right)$. Because the function $P(r)$ is singular more than $1/r^2$ at the origin, the Schrödinger differential equation

$$\frac{d^2}{dr^2} \psi(l, k, r) = \left(P(r) + \frac{l(l+1)}{r^2} - k^2\right) \psi(l, k, r) \tag{1.6}$$

always possesses one regular and one irregular solution at both ends for complex k . Hence by STONE [2] follows the self-adjointness of the operator \tilde{H}_1 and thus of the operator \tilde{H}_l .

2. Auxiliary statements

For our proofs we need two special solutions of the differential equation (1.6): The solution $\varphi(l, k, r)$ which is regular at the origin and the Jost solution $u(l, k, r)$ which behaves like $\exp(-ikr)$ for large r . We need also the Wronskian $u(l, k) = W(u(l, k, r), \varphi(l, k, r))$. The uniform estimate of the solutions with respect to the variable l is of special importance to us.

Let us set $s = l + \frac{1}{2}$.

I. For fixed $k = -it$, $t > 0$ and large s the solutions $\varphi(l, t, r)$ and $u(l, t, r)$ are of the form

$$\begin{aligned} \frac{\varphi(l, k, r)}{u(l, k)} &= \frac{1}{2t} \varphi_1(s, t, r) (1 + 0(s)) \\ u(l, k, r) &= \varphi_2(s, t, r) (1 + 0(s)) \end{aligned} \tag{2.1}$$

uniformly with respect to $r \in (0, \infty)$, where $s = l + \frac{1}{2}$, $t = ik$ and

$$\varphi_{1,2}(s, t, r) = \frac{t^{1/2}}{\left(P(r) + \frac{s^2}{r^2} + t^2\right)^{1/4}} \times \times \exp \left\{ \pm tr \mp \int_r^\infty \left(\sqrt{P(x) + \frac{s^2}{x^2} + t^2} - t \right) dx \right\}. \tag{2.2}$$

The proof of this statement for imaginary t is given in Ref. [3]. In our case t is real but nevertheless we can use the proof from the Ref. [3] completely.

II. If $f \in \mathcal{D}(H_l)$ then

$$|f(r)| \leq a \|H_l f\| + b \|f\| \tag{2.3}$$

holds uniformly with respect to r in the interval $(0, \infty)$, where a, b are positive constants independent of l and a can be chosen arbitrarily small.

Let us consider the mapping $f = (H_l - E)^{-1}g$. We shall choose E negative because then we are sure that E belongs to the resolvent spectrum. In our case the resolvent is the integral operator whose kernel is the Green's function

$$G_l(k, r, r') = \begin{cases} \varphi(l, k, r') u(l, k, r) / u(l, k), & r' \leq r, \\ u(l, k, r') \varphi(l, k, r) / u(l, k), & r' > r, \end{cases} \tag{2.4}$$

where $k = -\sqrt{E}$ and the square root is positive for positive E . As the resolvent maps the whole space L_2 onto the domain $\mathcal{D}(H_l)$, we are sure that every $f \in \mathcal{D}(H_l)$ can be represented in the form $f = (H - E)^{-1}g$, where g is some element of L_2 .

$$f(r) = u(l, k, r) \int_0^r \frac{\varphi(l, k, x)}{u(l, k)} g(x) dx + \frac{\varphi(l, k, r)}{u(l, k)} \int_r^\infty u(l, k, x) g(x) dx. \tag{2.5}$$

The estimations of the two integrals on the right-hand side of (2.5) do not differ essentially and we restrict ourselves to the estimation of the first integral. Let us denote this first part by $h(r)$. Accordingly to (2.1) we have the first estimate

$$|h(r)| \leq C P^{-1/4}(r, s, t) \int_0^r \exp \times \times \left\{ -t(r-x) - \int_x^r (P^{1/2}(u, s, t) - t) du \right\} \frac{|g(x)|}{P^{1/4}(x, s, t)} dx, \tag{2.6}$$

where C is a constant independent of s and $P(r, s, t) = P(r) + s^2/r^2 + t^2$. Being positive, the integral in the exponent can be omitted and after

using Schwarz' inequality we obtain $|h(r)| < C\|g\|/2t^{3/2}$. As $g = (H_t - E)f$, we get

$$|h(r)| \leq \frac{C}{2t^{3/2}} \|H_t f\| + \frac{Ct^{1/2}}{2} \|f\|.$$

The same inequality can be obtained for the second part of the relation (2.5). Together they prove the stated inequality (2.3) because the parameter t can be chosen arbitrarily large.

III. Each element $f \in \mathcal{D}(H_1)$ considered as the function $f(x, y, z)$ possesses the estimate

$$|f(x, y, z)| \leq r^{1+\eta} (a \|H_1 f\| + b \|f\|), \quad \eta > 0, \quad (2.7)$$

uniformly with respect to r in the interval $(0, \infty)$, and the constant a can be chosen arbitrarily small.

The element f can be represented uniquely in the form

$$f(x, y, z) = \sum Y_{lm}(\vartheta, \varphi) f_{lm}(r), \quad f_{lm} \in \mathcal{D}(H_l). \quad (2.8)$$

In order to estimate the function (2.8) we must know estimates of the functions $f_{lm}(r)$. We start in the same way as in the proof of the preceding statement up to the formula (2.6). Because of the monotonic character of the function $P(r, s, t)$ the following estimate can be made

$$\begin{aligned} |h_{lm}(r)| &\leq C \left(\frac{r}{s}\right)^{1+\eta} \frac{1}{t^{(1-2\eta)/2}} \int_0^r P^{1/4}(x, s, t) \exp\left(-\int_x^r \sqrt{P(u, s, t)} du\right) \times \\ &\times |g(x)| dx \leq C \cdot \left(\frac{r}{s}\right)^{1+\eta} \frac{1}{t^{(1-2\eta)/2}} \|g\|. \end{aligned}$$

The last inequality follows from the Schwarz inequality. The same can be done for the second part of the function $f_{lm}(r)$. But the estimate is more complicated for this part, because now one must use the monotony of the function $P^{-1/2}(r, s, t) \exp\left\{\varepsilon tr - \varepsilon \int_r^\infty (P^{1/2}(u, s, t) - t) du\right\}$, $\varepsilon < 1$ for t large enough, instead of the monotony of the function $P^{-1/2}(r, s, t)$, as the integration is over the range (r, ∞) . We obtain, finally,

$$|f_{lm}(r)| \leq C \left(\frac{r}{t}\right)^{1+\eta} (a \|H_l f_{lm}\| + b \|f_{lm}\|),$$

and a can be made as small as we like because t can be chosen arbitrarily large. Let us use the abbreviations $x = \{x_{lm}\} = \{Y_{lm}(\vartheta, \varphi)/l^{1+\eta}\}$ and $y = \{y_{lm}\} = \{a \|H_l f_{lm}\| + b \|f_{lm}\|\}$. The function (2.8) can be estimated using Schwarz' inequality

$$|f(x, y, z)| \leq Cr^{1+\eta} \left| \sum x_{lm} y_{lm} \right| \leq Cr^{1+\eta} \|x\| \cdot \|y\|, \quad (2.9)$$

where $\|x\|^2 = \sum |x_{lm}|^2 < \infty$ and $\|y\|^2 = 2a^2 \|H_1 f\|^2 + 2b^2 \|f\|^2$. The inequality (2.9) is the stated inequality (2.7).

IV. $\mathcal{D}(H_l) \subset \mathcal{D}(P_l)$ and for every element $f \in \mathcal{D}(H_l)$ we have

$$\|Pf\| \leq a\|H_l f\| + b\|f\|, \tag{2.10}$$

where a and b are positive constants.

Again the mapping $f = (H_l - E)^{-1}g$ will be used. Now the domain of this mapping will not be the whole space $L_2(0, \infty)$ but, rather, the dense linear manifold \mathcal{X} , whose elements g are bounded functions: $|g(x)| < K(g)$, where $K(g)$ is a finite constant. As we are interested in the norm of the element $Pf = P(H_l - E)^{-1}g$, we must square the Green's function. The mixed product can be estimated by use of the inequality $2 \operatorname{Re} ab < |a|^2 + |b|^2$. As in the last proofs we shall restrict our consideration to the term $|P(r)h(r)|^2$. The other term can be estimated similarly

$$|P(r)h(r)| \leq \frac{P(r)}{P^{1/4}(s, t, r)} \int_0^r \exp \left\{ - \int_x^r P^{1/2}(s, t, u) du \right\} \frac{|g(x)|}{P^{1/4}(s, t, x)} dx. \tag{2.11}$$

From this inequality it is easy to derive $|P(r)h(r)| < C P^{1/4}(r)\|g\|$ and consequently

$$\int_{\frac{\delta}{2}}^{\infty} |P(r)h(r)|^2 dr \leq C^2 \cdot \delta P^{1/2} \left(\frac{\delta}{2} \right) \|g\|^2. \tag{2.12}$$

Hence we have to consider only the integral over $(0, \delta/2)$ in the following part of the proof. For $r \leq \delta/2$ relation (2.11) can be simplified, replacing $P(s, t, r)$ by $P(r)$ because $P(s, t, r) > P(r) > 0$ for $r \leq \delta/2$. For $q \leq \delta/2$, we have

$$\begin{aligned} \int_0^q |P(r)h(r)|^2 dr &\leq \\ &\leq C^2 \int_0^q dr P^{1/2}(r) \left[\int_0^r P^{1/4}(x) \exp \left(- \int_x^r P^{1/2}(u) du \right) |g(x)| dx \right]^2. \end{aligned} \tag{2.13}$$

This integral surely, exists since the element g was chosen from the linear manifold \mathcal{X} . The existence of the integral enables us to change the order of the integration in the following treatment. First, we estimate the bilinear functional represented by the square of brackets. Let us denote it by $I(r)$ and let us use the abbreviation $F(r) = \int_r^{\delta} P^{1/2}(x) dx$.

$$\begin{aligned} I(r) &\leq \frac{1}{2} \exp(2F(r)) \int_0^r dx \int_0^r dy P^{1/4}(x) P^{1/4}(y) \times \\ &\quad \times \exp(-F(x) - F(y)) (|g(x)|^2 + |g(y)|^2) < \\ &< \frac{1}{P^{1/4}(r)} \exp(F(r)) \int_0^r P^{1/4}(x) \exp(-F(x)) |g(x)|^2 dx. \end{aligned}$$

We used here the inequality $\int_0^r P^{1/4}(x) \exp(-F(x)) dx < P^{-1/4}(r) \exp(-F(r))$.

Coming back to (2.13) and changing the order of the integration we have

$$\int_0^q |P(r) h(r)|^2 dr \leq C^2 \int_0^q P^{1/4}(x) e^{-F(x)} |g(x)|^2 dx \int_x^q P^{1/4}(r) e^{F(r)} dr < C^2 \|g\|^2.$$

This inequality combined with (2.12) and analogous relations for the other term of the Green's functions gives $\|Pf\|^2 \leq C\|g\|^2$ for $g \in \mathcal{K}$, where C does not depend on g . Let \mathcal{M} be the range of the mapping $(H_l - E)^{-1}g, g \in \mathcal{K}$. \mathcal{M} is a dense linear manifold in the Hilbert space \mathcal{H} . Then we have $\|Pf\| \leq a\|H_l f\| + b\|f\|, f \in \mathcal{M}$. Because of the self-adjointness of the operators P_l and H_l on the domains $\mathcal{D}(P_l)$ and $\mathcal{D}(H_l)$ respectively, $\mathcal{D}(P_l) \supset \mathcal{D}(H_l)$ and the obtained inequality can be extended to the whole domain $\mathcal{D}(H_l)$ with some constants a and b . This concludes the proof.

3. Self-Adjointness of \tilde{H}_s

I. *The symmetric operator H_s is essentially self-adjoint.* For the proof we use KATO's [1] theorem:

II. *Let H_1 be a self-adjoint operator in L_2 and let U be a symmetric operator in L_2 such that*

$$\mathcal{D} \equiv \mathcal{D}(H_1) \subset \mathcal{D}(U) \tag{3.1}$$

and

$$\|Uf\| \leq a\|H_1 f\| + b\|f\| \quad \text{for any } f \in \mathcal{D}(H_1), \tag{3.2}$$

where a and b are constants such that $0 \leq a < 1$ and $b \geq 0$. Then $H = H_1 + U$ is self-adjoint on $\mathcal{D}(H_1)$.

In our proof H_1 is the self-adjoint operator H_1 on the domain $\mathcal{D}(H_1)$ and U is the symmetric extension of the operator $V - P$ to the domain \mathcal{L} . First we consider the existence of the element $Uf, f \in \mathcal{D}(H_1)$.

$$\begin{aligned} \|Uf\|^2 &\leq \int \frac{dx dy dz}{r^2} P^2(r) |Q(x, y, z) - 1|^2 |f(x, y, z)|^2 + \\ &+ \int \frac{dx dy dz}{r^2} |V_1(x, y, z)|^2 |f(x, y, z)|^2 + \\ &+ \int \frac{dx dy dz}{r^2} |V_2(x, y, z)|^2 |f(x, y, z)|^2. \end{aligned} \tag{3.3}$$

The first integral I_1 on the right-hand side of (3.3) really is extended over the range $K(\delta)$. We divide this range into two disjoint parts $K(d)$ and $K(\delta) - K(d), d < \delta$. It then follows that

$$I_1 \leq q^2(d) \|Pf\|^2 + M P^2(d) \|f\|^2, \quad f \in \mathcal{D}(H_1).$$

$\|Pf\|$ cannot be immediately majorized, as in the statement IV (2.10), because this statement holds for the elements $f \in \mathcal{D}(H_l)$ and the restriction of the operator P to $\mathcal{D}(H_l)$. We can simply enlarge the validity

of (2.10) to any $f \in \mathcal{D}(H_1)$ using the reducibility of the operators P and H_1 in the subspaces L_2^m . For $f \in \mathcal{D}(H_1)$

$$\begin{aligned} \|Pf\|^2 &= \|P \sum f_{lm} \otimes Y_{lm}\|^2 = \sum \|Pf_{lm}\|^2 \leq \\ &\leq 2 \sum (a^2 \|H_1 f_{lm}\|^2 + b^2 \|f_{lm}\|^2) = 2a^2 \|H_1 f\|^2 + 2b^2 \|f\|^2. \end{aligned}$$

We used here (2.10) in the third step of the estimation.

$$I_1 \leq q^2(d) (2a^2 \|H_1 f\|^2 + 2b^2 \|f\|^2) + M P^2(d) \|f\|^2.$$

The second and third integrals I_2 and I_3 respectively in the expression (3.3) are estimated by use of the definitions (1.2) and (1.3) of the potential and by use of the statement III. (2.3) for the integral I_3 .

$$I_2 + I_3 \leq C_1 (a_1 \|H_1 f\| + b_1 \|f\|)^2 + C_2 \|f\|^2.$$

Thus we have obtained

$$\|Uf\| \leq a \|H_1 f\| + b \|f\|, \quad f \in \mathcal{D}(H_1), \quad (3.4)$$

where the positive constants a, b do not depend on f , and a can be taken smaller than unity in accordance with the statement III. and the property (1.5) of the function $q(d)$. At first we remark that Uf exists for every $f \in \mathcal{D}(H_1)$ because of (3.4). Now it is easy to extend the symmetric operator U on \mathcal{L} to the symmetric operator U on $\mathcal{D}(H_1)$ in order to satisfy condition (3.1). Then the inequality (3.4) is the condition (3.2). In this way we have proved the self-adjointness of the operator $H_1 + U$ on $\mathcal{D}(H_1)$ and hence $H = \tilde{H}_s$, where $H_s \equiv H_0 + V \equiv H_1 + U$ on \mathcal{L} . This was the ultimate aim of our proof.

Acknowledgement. The author is grateful to Professor ABDUS SALAM and the IAEA for the hospitality extended to him at the International Centre for Theoretical Physics, Trieste.

References

- [1] KATO, T.: Trans. Am. Math. Soc. **70**, 195 (1951).
- [2] STONE, M. H.: Linear transformations in Hilbert space. Ann. Math. Soc. Colloq. Publ. Vol. XV. New York (1964).
- [3] JAKŠIĆ, B., and N. LIMIĆ: Asymptotic Behaviour of Phase Shift in λ for Strongly Singular Potentials (to be published in J. Math. Phys. vol 7).