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Iterated Crossed Box Diagram in the Complex Angular Momentum Plane and Bethe-Salpeter Equation*

By

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Abstract. The analytic properties of the iterated crossed box diagram are considered in the complex angular momentum plane by using the partial wave Bethe-Salpeter equation.

It is shown that the kernel of the Bethe-Salpeter equation is of the Hilbert-Schmidt type in the domain

$$\Delta(l, s) \equiv \{l, s; \mathrm{Re}\, l > -3/2; \mathrm{Im}\, s \neq 0\}$$

having a simple pole at $l = -1$.

This pole produces in the whole amplitude a singularity found by Gribov and Pomeranchuk since the kernel is not of finite rank.

I. Introduction

In the last few years [1] the theory of the complex angular momentum has been a very important and powerful tool for treating the scattering amplitude at high energies.

One way [2] to introduce, in relativistic scattering theory, the concept of the complex angular momentum is by an analytic continuation of the partial wave amplitude into the complex angular l plane. Then the uniqueness [1] of such an analytic continuation requires that the continued amplitude is bounded and analytic in some region of the complex l plane. Generally, that can be satisfied if for the original amplitude before the analytic continuation the validity of a dispersion relation has been assumed.

Thus an analytically continued partial wave amplitude will be analytic in at least a half-plane $\mathrm{Re}\, l > l_0$, where l_0 is connected with the necessary number of subtractions in a dispersion relation.

We consider the scattering of identical neutral spinless particles with equal mass. The relativistic amplitude written as $T(s, t)$ may have,

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of course, both left- and right-hand cut in the t variable with respect to the channel where t is energy. Then we shall say that our scattering amplitude has a third double-spectral function, following the terminology usual for the Mandelstam representation [3]. In that case the partial wave amplitude $T_l(s)$, where s is the total center-of-mass energy squared, will be defined [1] by two different analytic functions of l which coincide respectively with $T_l(s)$ for even and odd values of l.

For the elastic scattering in the s-channel when s is in the elastic region one may continue the two particle unitarity condition into the complex l plane, thus getting generalized two particle unitarity [4].

The important fact which follows when $T_l(s)$ satisfies the generalized two particle unitarity is that $T_l(s)$ is bounded for $s \geqq 4m^2$, where m is the particle's mass.

If one goes to the left of the branch point $s = 4m^2$ then from generalized unitarity it also follows that $T_l(s)$ cannot be unbounded on both Riemann sheets. This means that $T_l(s)$ cannot have an s-independent pole [1] at the point $l = l_1 < l_0$ which one may reach by analytic continuation in the complex l plane.

When the third double-spectral function is present then the individual terms of the perturbation series of the partial wave amplitude will have poles at the negative integer values of l due to the properties of the Q_l function. Here, we have been ignoring possible subtraction in the dispersion relation.

If the whole amplitude satisfies generalized two particle unitarity the perturbation series will then develop singularities, found by Gribov and Pomeranchuk [5], at the negative integer values of l, which will cancel these fixed poles. This simply means that if the such perturbation series is obtained by an iteration process, each term will have poles of increasing order at the above values of l.

The simplest case of the perturbation series [1] which has the above behaviour may be constructed by iterating the crossed box diagram.

This case we shall consider in the complex l plane in some detail in this paper.

When one iterates the crossed box diagram each term will have a pole at $l = -1$ and since the whole amplitude, obtained by summing up all iterated terms, satisfies generalized two particle unitarity this pole should disappear.

One may try to see how this cancellation could happen by using the method for finding the high energy behaviour of Feynman graphs [6, 7], and then summing up the most singular terms.

The result will be that the structure of the coefficient $c_N(s)$ of the variable t tending to infinity, for say N times iterated crossed box diagram, remains unknown and one is unable to sum up the series explicitly.

One possibility of getting over these difficulties is to consider the perturbation series as a Bethe-Salpeter (BS) [8] integral equation whose kernel is the single crossed box diagram. The whole problem will then be reduced to finding a domain $\Delta(l, s)$ in the complex $l \otimes s$ plane where the kernel of the partial wave BS-equation is square integrable, i. e. of the Schmidt type [9, 10], and includes the point $l = -1$.

In Sec. II the symmetrized partial wave BS-integral equation is given.

Sec. III gives the structure of the kernel of the BS-equation for the iterated crossed box diagram.

In Sec. IV we discuss the integrability of the kernel for the crossed box diagram. The domain $\Delta(l, s)$ where the kernel $K_l(s)$ is square integrable is also given.

In Sec. V we consider the singularity of the resolvent $R_l^s(s)$ in the complex l plane when l approaches -1. $R_l^s(s)$ is the resolvent defined by the singular part of the kernel $K_l(s)$.

II. Partial wave BS-equation

We consider elastic scattering of two identical scalar particles of mass m. In the two particle approximation diagram given symbolically by Fig. 1 we have in fact an integral equation of the form

$$T(12; 34) = F(12; 34) + \\ + \int \frac{d^4 p_5 \, d^4 p_6 \, \delta(p_5 + p_6 - p_1 - p_2)}{(p_5^2 - m^2 + i\varepsilon)(p_6^2 - m^2 + i\varepsilon)} F(12; 56)\, T(56; 34)\,. \tag{2.1}$$

$F(12; 34)$ is an irreducible diagram not having two particle intermediate state in the channel defined by the particles (1,2) and (3,4). The above

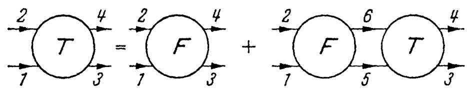

Fig. 1

integral equation is known as an integral equation of BS-type [8]. The BS-equation is an off-mass-shell integral equation and therefore depends upon six independent scalar products of the four-vectors p_i ($i = 1, \ldots, 4$).

Since the integration in the Eq. (2.1) runs over four dimensional space one may consider two scalar products as two independent parameters, usually taken to be complex.

For our further discussion we shall need partial wave projection of Eq. (2.1).

In the center-of-mass system for the particles (1,2) and (3,4) one may introduce the new system of coordinates usual for the BS-equation, that is

$$
\begin{aligned}
&p_1 + p_2 = p_3 + p_4 = W \equiv (\sqrt{s}, 0, 0, 0)\,, \\
&p_1 \equiv (p_{10}, \vec{p}_1) = \left(\sqrt{\frac{s}{2}} + \omega, \vec{q}\right), \\
&p_2 \equiv (p_{20}, \vec{p}_2) = \left(\sqrt{\frac{s}{2}} - \omega, -\vec{q}\right), \\
&p_3 \equiv (p_{30}, \vec{p}_3) = \left(\sqrt{\frac{s}{2}} + \omega', \vec{q}'\right), \\
&p_4 \equiv (p_{40}, \vec{p}_4) = \left(\sqrt{\frac{s}{2}} - \omega', -\vec{q}'\right).
\end{aligned}
\tag{2.2}
$$

If we rewrite Eq. (2.1) in the following form, we can conveniently apply the partial wave projection to obtain the partial wave BS-equation

$$
\begin{aligned}
&T(s;\, \xi, \xi';\, \hat{q}\cdot\hat{q}') = F(s;\, \xi, \xi';\, \hat{q}\cdot\hat{q}') + \int d^2\, \xi''\, q''^2 \int d^2\, \hat{q}'' \,\times \\
&\times \frac{F(s;\, \xi, \xi'';\, \hat{q}\cdot\hat{q}'')\, T(s;\, \xi'', \xi';\, \hat{q}''\cdot\hat{q}')}{\left[q''^2 + m^2 - i\varepsilon - \left(\sqrt{\frac{s}{2}} + \omega''\right)^2\right]\left[q''^2 + m^2 - i\varepsilon - \left(\sqrt{\frac{s}{2}} - \omega''\right)^2\right]}
\end{aligned}
\tag{2.3}
$$

where we have used the abbreviation

$$
\xi \equiv (\omega, q) \tag{2.4}
$$

for the one dimensional space-time vector whose space coordinate varies only through the region $(0, \infty)$. The hats on the q's mean take the corresponding unit vector.

We define the partial wave projection as an integral over the Legendre polynomial P_l of the form

$$
\frac{qq'}{2} \int_{-1}^{1} f(\hat{q}\cdot\hat{q}')\, P_l(\hat{q}\cdot\hat{q}')\, d(\hat{q}\cdot\hat{q}') = f_l\,. \tag{2.5}
$$

Using Eq. (2.5) one easily gets the partial wave BS-equation

$$
\begin{aligned}
&T_l(s;\, \xi, \xi') = F_l(s;\, \xi, \xi') + 2\pi \int d^2\, \xi'' \,\times \\
&\times \frac{F_l(s;\, \xi, \xi'')\, T_l(s;\, \xi'', \xi')}{\left[q''^2 + m^2 - i\varepsilon - \left(\sqrt{\frac{s}{2}} + \omega''\right)^2\right]\left[q''^2 + m^2 - i\varepsilon - \left(\sqrt{\frac{s}{2}} - \omega''\right)^2\right]}.
\end{aligned}
\tag{2.6}
$$

The structure of the Eq. (2.6) shows that it is a singular integral equation [11] because the denominators in Eq. (2.6) can vanish in the region of integration. Under a suitable phase transformation and a very general restriction on the energy parameter s these difficulties can be avoided, but, of course, not for any kernel.

It is convenient first to make a similarity transformation by introducing the function

$$R_l(s;\,\xi,\,\xi') = h(s,\,\xi)\; T_l(s;\,\xi,\,\xi')\, h(s;\,\xi') \tag{2.7}$$

where

$$\begin{aligned} h(s;\,\xi) = \sqrt{2}\,\pi \Bigg\{ & \left[q^2 + m^2 - i\varepsilon - \left(\sqrt{\frac{s}{2}} + \omega \right)^2 \right] \times \\ & \times \left[q^2 + m^2 - i\,\varepsilon - \left(\sqrt{\frac{s}{2}} - \omega \right)^2 \right] \Bigg\}^{-1/2} \end{aligned} \tag{2.8}$$

when Eq. (2.6) gets the simple form

$$R_l(s;\,\xi,\,\xi') = K_l(s,\,\xi,\,\xi') + \int d^2\,\xi''\; K_l(s;\,\xi,\,\xi'')\, R_l(s;\,\xi'',\,\xi')\,. \tag{2.9}$$

Here K_l means

$$K_l(s;\,\xi,\,\xi') = h(s,\,\xi)\, F_l(s;\,\xi,\,\xi')\, h(s;\,\xi')\,. \tag{2.10}$$

One will immediately recognize that Eq. (2.9) represents an integral equation for the resolvent of the kernel $K_l(s;\,\xi,\,\xi')$. Symbolically we can write the Eq. (2.9) as

$$R_l(s) = K_l(s) + K_l(s)\; R_l(s) \tag{2.9}$$

where $R_l(s;\,\xi,\,\xi')$ is then simply expressed in terms of $R_l(s)$ as

$$R_l(s,\,\xi,\,\xi') = \langle \xi\, |R_l(s)|\, \xi' \rangle \tag{2.11}$$

using Dirac's notation.

If kernel (2.10) is symmetric, as it will be in our case, it has optimal bound [12].

III. Structure of the kernel $K_l(s;\,\xi,\,\xi')$ for iterated crossed box diagram

In this Section we shall consider the crossed box diagram given by the Fig. 2. One can express this diagram using the Feynman parametic representation [13] as

$$F(s,\,t;\,p_i^2) = G \int d^4\alpha \frac{1}{[D(\alpha|\,s,\,t;\,p^2) + i\,\varepsilon]^2} \tag{3.1}$$

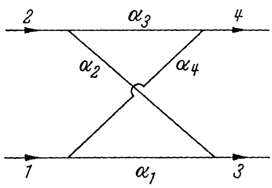

Fig. 2

where G is a constant including the coupling constant.

$$\int d^4\alpha = \int_0^1 \cdots \int_0^1 d\alpha,\,\ldots,\, d\alpha_4\, \delta(\Sigma\alpha_i - 1)\,.$$

The function in the denominator of (3.1) is a known Feynman denominator [6] linear in s, t, p_i^2 and m^2 where s is the total energy in the c. m. system of the particles (1,2) and (3,4), and t is the square of the momentum transfer. The explicit form of D is given in Appendix A. It is only important to notice here that the coefficient $g(\alpha)$ of the variable t in D is not of definite sign in the domain of the α integration.

All the p_i^2's should be considered as functions of the corresponding ω, q, ω', q' and s through the relations (2.2).

Since the sign of the $g(\alpha)$ is not definite we make use of the relation

$$\theta(g) + \theta(-g) = 1 \tag{3.2}$$

which splits the integration in (3.1) into two parts

$$F(s, t; p_i^2) = \frac{G}{2qq'} \cdot \frac{\partial}{\partial m^2} \int \frac{d^4\alpha}{|g|} \left\{ \frac{\theta(-g)}{\sigma(\alpha) - z} + \frac{\theta(g)}{\sigma(\alpha) + z} \right\} \tag{3.3}$$

where

$$\sigma(\alpha) = [D(t=0) + g(\alpha)((\omega - \omega')^2 - q^2 - q'^2)] [2qq' |g|]^{-1} = \frac{\tau(\alpha)}{2qq' |g|} \tag{3.4}$$

and

$$z = (\hat{q} \cdot \hat{q}') \tag{3.5}$$

The partial wave projection of (3.3) is

$$F_l(s; \xi, \xi') = \frac{G}{2} \frac{\partial}{\partial m^2} \int \frac{d^4\alpha}{|g|} \{\theta(-g) + (-)^l \theta(g)\} Q_l(\sigma) . \tag{3.6}$$

For the analytic continuation procedure it is better to consider $F_l^\pm$ instead of F_l, where $F_l^\pm$ is defined as

$$F_l^\pm(s; \xi, \xi') = \frac{G}{2} \frac{\partial}{\partial m^2} \int \frac{d^4\alpha}{|g|} \{\theta(-g) \pm \theta(g)\} Q_l(\sigma) \tag{3.7}$$

and corresponds to the usual decomposition [1, 2] of the partial wave amplitude (3.6) into even and odd partial waves.

It should be noted that on the mass shell, when $p_i^2 = m^2$ and therefore $\omega = \omega' = 0$ and $q^2 = q'^2 = \frac{1}{4}(s - 4m^2)$, F_l^- will be identically zero. Thus the whole amplitude F_l is determined on the mass-shell by the behaviour of F_l^+.

On the mass-shell F_l is given by [22]

$$F_l(s; p_i^2 = m^2)$$
$$= \frac{1}{2\pi^2} \int_{4m^2}^{\infty} dt\, Q_l\left(1 + \frac{t}{2q^2}\right) \int_{4m^2}^{\infty} du \frac{\theta[(u - 4m^2)(t - 4m^2) - 4m^4]}{s + t + u - 4m^2} \varrho(u, t) \tag{3.8}$$

where $\varrho(u, t)$ is the known Mandelstam double-spectral function for the box diagram. This function, as is seen from the integral (3.8), will have only the left-hand cut in the complex s plane which runs from the

$s = -\infty$ up to the $s = -8m^2$ and the discontinuity across this cut is

$$\frac{1}{2i} \Delta \left[q^{-2l} F_l(s; p_i^2 = m^2)\right]$$
$$= \frac{1}{2\pi} \int_{t+(s)}^{t-(s)} dt \frac{1}{(-q^2)^l} Q_l \left(-1 - \frac{1}{2q^2}\right) \varrho(4m^2 - s - t, t) \tag{3.9}$$

where

$$t_\pm(s) = \frac{1}{2}(4m^2 - s) \pm \frac{1}{2}\left[s(s + 8m^2)\right]^{1/2}. \tag{3.10}$$

It is now easy to see that the Eq. (3.9) will have poles of negative integer values of l due to the Q_l function.

Let us now return back to the Eq. (3.7) and consider only the "+" superscript which will be omitted in the further discussion.

Since we can write (14)

$$\frac{\partial}{\partial m^2} Q_l(\sigma) = -\frac{1}{2 q\, q' |g|} \frac{\partial}{\partial \sigma} Q_l(\sigma) \tag{3.11}$$

and

$$\frac{\partial}{\partial \sigma} Q_l(\sigma) = -2^{-l-1} \sqrt{\pi} \frac{\Gamma(l+2)}{\Gamma(l+3/2)} \sigma^{-l-2} H(l, \sigma^{-2}) \tag{3.12}$$

where

$$H(l, \sigma^{-2}) = F\left(\frac{l+3}{2}, \frac{l+2}{2}; l + 3/2; \sigma^{-2}\right) \tag{3.13}$$

the Eq. (37) may be rewritten as

$$F_l(s; \xi, \xi') = C(l)\,(q\, q')^{l+1} \int d^4\alpha\, |g|^l \left[\tau(\alpha)\right]^{-l-2} H(l, \sigma^{-2}) \tag{3.14}$$

where

$$C(l) = \frac{G}{2} \sqrt{\pi} \frac{\Gamma(l+2)}{\Gamma(l+3/2)}. \tag{3.14'}$$

The representation (3.12), for the derivative of the Q_l function is for $|\sigma| > 1$ a one-valued function regular in the complex σ plane with cut along the real axis from 1 to $-\infty$ when l is not an integer.

Therefore H will be a regular function for $|\sigma| > 1$ with properties

$$H \to 1 \quad \text{as} \quad |\sigma| \to \infty$$

and when $l = -2k, -2k - 1$, H turns out to be a polynomial of the order $n = k - 1$. It is shown in the Appendix A that when $\operatorname{Im} s \neq 0$, $\tau(\alpha)$ can never be zero in the domain of the α's integration. If we make the following phase transformation* i. e.,

$$\begin{aligned} \omega &\to \omega\, e^{i\varphi} \\ \omega' &\to \omega'\, e^{i\varphi} \end{aligned} \tag{3.15}$$

* Here we would like to mention that generally speaking the phase transformation does not leave the Hilbert-Schmidt norm invariant if it exists at all. The reason is that the operator which does it is unbounded and the resulting function generally need not be square-integrable.

where φ is a fixed angle satisfying

$$\varphi = \arg \sqrt{s} \tag{3.16}$$

then the singularities of $h(s; \xi)$ [see Eq. (2.8)] also cannot be reached and furthermore $|\sigma|$ can be chosen greater than unity independently of the ξ and ξ'. In that case we can weaken the condition $\operatorname{Im} s \neq 0$ by requiring only $\arg s \neq 0$. We should notice that Eq. (3.14) in the given form will not be regular for $\operatorname{Re} l \leqq -1$ because $g(\alpha)$ vanishes in the domain of the α integration (Appendix B). After an analytical continuation it turns out that F_l has in fact a simple pole at $l = -1$.

Thus we have prepared the ground for finding the domain $\Delta(l, s)$ in the complex $l \otimes s$ plane where the kernel $K_l(s; \xi, \xi')$ will be square-integrable.

IV. Integrability of the kernel $K_l(s)$

The Fredholm theory can be applied if one finds a domain $\Delta(l, s)$ in the complex $l \otimes s$ plane where

$$\|K_l(s)\|^2 = \int d^2 \xi \, d^2 \xi' \, |K_l(s; \xi, \xi')|^2 < \infty \tag{4.1}$$

holds. Then we can assign an analytic [15, 16, 17] operator $K_l(s)$, in the domain $\Delta(l, s)$ to the functions $K_l(s; \xi, \xi')$, if the functions $K_l(s; \xi, \xi')$ are analytic in $\Delta(l, s)$ for each ξ, ξ' and the operator $K_l(s)$ is uniformly bounded in every interior subset of $\Delta(l, s)$.

One can also define a meromorphic operator in $\Delta(l, s)$. We shall consider the Laurent expansion of $K_l(s)$ only in l, keeping s in $\Delta(l, s)$ away from possible singular points of $K_l(s)$. The above definition of operator analyticity will remain the same for an operator meromorphic in Δ except at the finite number of points where $K_l(s)$ could have singularities of the pole type.

From the considerations in the previous Sections, which have led to the conclusion that $\operatorname{Im} s$ should be different from zero in order to avoid singularities inside the domain of the ξ and α integrations, one sees that the norm (4.1) will be finite provided the contributions to the double integral from the regions $\xi, \xi' \to 0$ and ∞ are finite.

The best way to see the behaviour of the kernel $K_l(s; \xi, \xi')$ when ξ and ξ' are both large is to make the following scale transformation [12] on ξ, ξ'

$$\xi, \xi' \to \frac{\xi}{t}, \frac{\xi'}{t}$$

where t is some real parameter.

Under such a scale transformation one has

$$K_l(s, m^2; \xi, \xi') \to K_l\left(s, m^2; \frac{\xi}{t}, \frac{\xi'}{t}\right) = t^6 \, K_l(s\, t^2, m^2\, t^2; \xi, \xi') \,. \tag{4.2}$$

It is easy to see that the asymptotic behaviour of the K_l, when ξ and ξ' are both large, is now equivalent to putting $t \to 0$.

We should also note that under such a transformation when $t \to 0$ the σ goes to a constant. The contribution to the double sum from the region $\xi, \xi' \to \infty$ will not be finite if

$$\lim_{t\to 0} |\sigma(t^2 s, t^2 m^2; \xi, \xi'; \alpha)| \to 1 \quad \begin{array}{l}\text{for } \alpha\text{'s in the region}\\ \text{of the integration}\end{array} \tag{4.3}$$

since in that case $H(l, \sigma^{-2})$, which enters in K_l and in fact is a hypergeometric function, will diverge. In Appendix A it is shown that this is not the case if $\operatorname{Im} s \neq 0$.

The contribution to the double sum (4.1) from the region where $\xi, \xi' \to 0$ will be finite if $\operatorname{Re} l > -3/2$ and if either $l \neq -1$ or $g(\alpha) \neq 0$ — which conclusion follows simply from the Eq. (3.14).

Thus we have established that the kernel of the integral operator $K_l(s)$ satisfies the condition (4.1) when l and s are in the domain $\Delta(l, s)$ given by

$$\Delta(l, s) \equiv \{l, s; \operatorname{Re} l > -3/2, l \neq -1; \operatorname{Im} s \neq 0\}\,. \tag{4.4}$$

It follows also that $K_l(s; \xi, \xi')$ is analytic in $\Delta(l, s)$ for all ξ and ξ'.

The operators satisfying (4.1) form the so called Hilbert-Schmidt class [8, 10]. It can also be verified that $K_l(s)$ belongs to the trace class [8] of Hilbert-Schmidt operators in the same domain (4.4), i. e., $\operatorname{Tr} K_l(s) < \infty$ for $l, s \in \Delta(l, s)$.

The trace of $K_l(s)$ is defined as the integral

$$\operatorname{Tr} K_l(s) = \int d^2 \xi\, K_l(s; \xi, \xi)\,. \tag{4.5}$$

V. Singularities of the resolvent $\boldsymbol{R}_l^s(\mathbf{s}) = \left[1 - \frac{1}{l+1} \boldsymbol{K}^{(1)}(\mathbf{s})\right]^{-1}$ in the l-plane and conclusion

The Hilbert-Schmidt integral operator $K_l(s)$ which is an analytic operator [15—17] in the domain $\Delta(l, s)$ [Sec. IV. (4.4)] of the complex $l \otimes s$ plane is compact [9, 10] i. e. it transforms every bounded subset into a set whose closure is compact [18]. A compact, sometimes called completely continuous, integral operator has essentially discrete spectrum [18], that is to say it has either a finite or a denumerable number of eigenvalues. If $K_l(s)$ possesses symmetric kernel its Hilbert-Schmidt norm (4.1) can also be written in the form

$$\|K_l(s)\|^2 = \Sigma\, \lambda_{ln}^2(s) < \infty \tag{5.1}$$

where the $\lambda_{ln}(s)$ are the eigenvalues of $K_l(s)$. The eigenvalues of a compact operator form a strictly finite sequence away from zero. They may have only zero as an accumulation point if there are infinite number of them.

The number of eigenvalues defines the rank of the compact operator. We have seen that our integral operator $K_l(s)$, square-integrable in the domain $\Delta(l, s)$, is in fact meromorphic operator in $\Delta(l, s)$ having a simple pole at $l = -1$ (Appendix B). This pole appears because $g(\alpha)$ vanishes in the domain of the α integration.

Therefore $K_l(s)$ admits a Laurent expansion around this pole of the form

$$K_l(s) = \frac{1}{l+1} K^{(1)}(s) + K_l^{(2)}(s) \tag{5.2}$$

where the integral operator $K_l^{(2)}(s)$ is regular at $l = -1$. $K^{(1)}(s)$ is an integral operator with kernel

$$K^{(1)}(s; \xi, \xi') = h(s; \xi) F^{(1)}(s; \xi, \xi') h(s; \xi') \tag{5.3}$$

where

$$F^{(1)}(s; \xi, \xi') = G \int d^4\alpha\, \delta(g) \, [D(t=0)]^{-1} . \tag{5.4}$$

We shall denote the resolvent of the integral operator $\lambda K^{(1)}(s)$ by $R_l^s(s)$ where $\lambda = 1/(l+1)$. Then the $R_l(s)$ and the $R_l^s(s)$ will satisfy the so called second resolvent equation [15, 16, 17]

$$1 + R_l(s) = R_l^s(s) - R_l^s(s) K_l^{(2)}(s) [R_l(s) + 1] . \tag{5.5}$$

The formal solution of the Eq. (5.6) one may write in the form

$$R_l(s) = [1 + R_l^s(s) K_l^{(2)}(s)]^{-1} R_l^s(s) - 1 . \tag{5.6}$$

The resolvent $R_l^s(s)$ can be represented as a ratio of two series in powers of λ in the following form [19]

$$\langle \xi | R_l^s(s) | \xi' \rangle = -\frac{1}{\det(1 - \lambda K^{(1)}(s))} \left\langle \xi \left| \frac{\delta}{\delta \lambda K^{(1)}(s)} \det(1 - \lambda K^{(1)}(s)) \right| \xi' \right\rangle \tag{5.7}$$

where δ means functional derivation. We shall consider only the denominator of the Eq. (5.7) which is sufficient to show that an essential singularity [5] is present in $R_l^s(s)$ when $l \to -1$, if $K^{(1)}(s)$ is not an operator of finite rank. $\det(1 - \lambda K^{(1)}(s))$ is in fact a power series in λ [19] i. e.

$$\det(1 - \lambda K^{(1)}(s)) = \sum_{n=0}^{\infty} (-)^{n} \frac{\lambda^n}{n!} \int d^2\xi_1, \ldots, d^2\xi_n \det_{(n)} \| K^{(1)}(s; \xi_i, \xi_j) \| . \tag{5.8}$$

If the integral operator $K^{(1)}(s)$ is of finite rank the series (5.8) will turn out to be a polynomial in λ [20] of order equal to the rank of $K^{(1)}(s)$. In that case, the Eq. (5.8) will have pole type singularity where $l \to -1$. In fact we find that $K^{(1)}(s)$ is an integral operator of infinite rank having thus an infinite number of eigenvalues. The situation now becomes quite different since, when $l \to -1$ and therefore $1/\lambda \to 0$ one will reach the eigenvalues of $K^{(1)}(s)$ which are placed in the neighbourhood of zero and they have an accumulation point there.

To see this we notice that the functions

$$d_n(s) = \int d^2\xi_1, \ldots, d^2\xi_n \det_{(n)} \| K^{(1)}(s; \xi_i, \xi_j) \| \tag{5.9}$$

become equal to zero for all $n > r$, where r is the rank of the operator $K^{(1)}(s)$, since the kernel $K^{(1)}(s; \xi_i, \xi_j)$ is symmetric i. e., satisfies the relation

$$K^{(1)}(s; \xi_i, \xi_j) = K^{(1)}(s; \xi_j, \xi_i) \,, \tag{5.10}$$

it has a finite number of eigenvalues when and only when it is degenerate. The kernel of an integral operator is degenerate if it can be written as a finite sum of products of functions of ξ and functions of ξ' in the form

$$K^{(1)}(\xi, \xi') = \sum_{i=1}^{r} X_i(\xi)\, Y_i(\xi') \,. \tag{5.11}$$

Here $X_1(\xi), \ldots, X_r(\xi)$; $Y_1(\xi'), \ldots, Y_r(\xi')$ are two sets of linearly independent L_2-functions. Therefore, if the kernel has the property that the d_n are identically zero for all $n > r$ it has to be of the form (5.11) and vice-versa.

Since we are not considering the problem of analyticity in the s-plane in this paper we can simplify the situation by putting $\varphi = \pi/2$ [Sec. III., (3.16)] and let $s \to 0$, getting

$$K^{(1)}(0; \xi, \xi') \equiv K^{(1)}(0, \xi^2, \xi'^2) \,. \tag{5.12}$$

The explicit form of the (5.12) is

$$K^{(1)}(0; t, t') = \frac{1}{m^2 + t} F^{(1)}(0; t, t') \frac{1}{m^2 + t'} \tag{5.13}$$

and

$$F^{(1)}(0; t, t') = -2G \int \frac{d\alpha_2\, d\alpha_4}{\sqrt{\lambda(1\,\alpha_1\,\alpha_4)}} \frac{1}{\alpha_2(1 - \alpha_2 + \alpha_4)\, t + \alpha_4(1 - \alpha_4 + \alpha_2)\, t' + m^2} \tag{5.14}$$

where we have put $-\xi^2 = t$ and

$$\lambda(1\,\alpha_2\,\alpha_4) = 1 + \alpha_2^2 + \alpha_4^2 - 2(\alpha_2 + \alpha_4 + \alpha_2\,\alpha_4) \,.$$

Explicit inspection of the Eq. (5.14) shows that (5.14) cannot be written in the form (5.11) at least when r is finite.

Thus we have established that the iterated crossed box diagram has an accumulation of poles in the complex l plane when we approach the point $l = -1$. This conclusion is also in agreement with the result of GRIBOV and POMERANCHUK [5] for the general scattering amplitude possessing a third double-spectral function and satisfying the two-particle generalized unitarity in the s-channel.

Appendix A

Properties of $\sigma(\alpha)$

$\sigma(\alpha)$ introduced in (3.4) is defined as

$$\sigma(\alpha) = \frac{1}{2q\,q'\,|g(\alpha)|} \tau(\alpha) \tag{A.1}$$

where

$$\begin{aligned}
\tau(\alpha) &= D(t=0) + g(\alpha)\left((\omega-\omega')^2 - q^2 - q'^2\right) \\
D(t=0) &= f(\alpha)\, s + \Sigma\, h_i(\alpha)\, p_i^2 - m^2 \\
f(\alpha) &= -\alpha_1\,\alpha_3 \\
h_1(\alpha) &= \alpha_1\,\alpha_2 + \alpha_1\,\alpha_3 \\
h_2(\alpha) &= \alpha_2\,\alpha_3 + \alpha_1\,\alpha_3 \\
h_3(\alpha) &= \alpha_1\,\alpha_4 + \alpha_1\,\alpha_3 \\
h_4(\alpha) &= \alpha_3\,\alpha_4 + \alpha_1\,\alpha_3 \\
g(\alpha) &= \alpha_2\,\alpha_4 - \alpha_1\,\alpha_3\,.
\end{aligned} \tag{A.2}$$

We want to show that $\tau(\alpha)$ can never be zero in the domain of the α integration if and only if $\operatorname{Im} s \neq 0$. Suppose that $\sqrt{s}$ moves from the origin along a ray defined by the angle φ then we turn the contours ω and ω' through the angle φ, which is in fact a phase transformation that, generally, does not leave the norm of an operator invariant [12, 15]. We need such a transformation in order to avoid the singularities of $h(s;\xi)$, [Eq. (2.8)], as well.

The condition on φ is

$$\varepsilon < \varphi = \arg\sqrt{s} < \pi - \varepsilon \tag{A.3}$$

where $\varepsilon > 0$.

Now it is easy to see the $\operatorname{Im}\tau(\alpha) = \tau_2(\alpha)$ is given by the formula

$$\operatorname{Im}\tau(\alpha) = \frac{\operatorname{Im} s}{|s|}\, a(\alpha) = \tau_2(\alpha) \tag{A.4}$$

where $a(\alpha)$ is always positive in the domain of the α integration and has the form

$$\begin{aligned}
a(\alpha) = \alpha_1\,\alpha_2\left(\frac{|\sqrt{s}|}{2}+\omega\right)^2 + \alpha_2\,\alpha_3\left(\frac{|\sqrt{s}|}{2}-\omega\right)^2 + \alpha_1\,\alpha_4\left(\frac{|\sqrt{s}|}{2}+\omega'\right)^2 + \\
+ \alpha_3\,\alpha_4\left(\frac{|\sqrt{s}|}{2}-\omega'\right)^2 + \alpha_3\,\alpha_1(\omega+\omega')^2 + \alpha_2\,\alpha_4(\omega-\omega')^2\,.
\end{aligned} \tag{A.5}$$

The relation for $\operatorname{Re}\tau(\alpha) = \tau_1(\alpha)$ is

$$\operatorname{Re}\tau(\alpha) = \tau_1(\alpha) = \frac{\operatorname{Re} s}{|s|}\, a(\alpha) - b(\alpha) \tag{A.6}$$

where $a(\alpha)$ is the same as $a(\alpha)$ in (A.4) and $b(\alpha)$ is always positive and of the form

$$\begin{aligned}
b(\alpha) = [\alpha_1(\alpha_2+\alpha_3) + \alpha_2(\alpha_3+\alpha_4)]\, q^2 + \\
+ [\alpha_4(\alpha_2+\alpha_3) + \alpha_1(\alpha_3+\alpha_4)]\, q^2 + m^2\,.
\end{aligned} \tag{A.7}$$

We allow that φ can be equal to $\pi/2$ in the condition (A.3), since in that case, although $\operatorname{Im} s = 0$, the $\frac{\operatorname{Re} s}{|s|}$ is equal to -1, and again $\tau(\alpha)$ is always different from zero in the domain of the α integration. If one makes the phase transformation on the ω's with an arbitrary angle ψ then one has to put an extra condition on s

$$|\mathrm{Re}\,\sqrt{s} - \mathrm{ctg}\,\psi\;\mathrm{Im}\,\sqrt{s}| < 2m \tag{A.8}$$

in order to avoid the singularities of $h(s, \xi)$.

If we consider $\det(1 - K_l(s))$ only, the phase transformation turns out to be equivalent to a rotation of the ω, ω' contours of integration by an angle φ.

We should also mention that the absolute value of $\sigma(\alpha)$ is bounded. This follows from the relation

$$|\sigma(\alpha)| = \frac{1}{2q\,q'\,|g(\alpha)|}\sqrt{(\tau_1^2 + \tau_2^2)} = \frac{1}{2q\,q'\,|g(\alpha)|}\sqrt{(a^2 - 2ab\cos 2\varphi + b^2)} \tag{A.9}$$

which says that $|\sigma|$ satisfies an inequality of the form

$$\frac{|a(\alpha) - b(\alpha)|}{2q\,q'\,|g(\alpha)|} < |\sigma(\alpha)| \leqq \frac{a(\alpha) + b(\alpha)}{2q\,q'\,|g(\alpha)|}\,. \tag{A.10}$$

One can show also, starting from (A.10) with a little longer algebra, that for any chosen φ from (A.3) $|\sigma(\alpha)|$ is greater than unity for all ξ, ξ' and α's allowed by the condition $\Sigma\,\alpha = 1$.

Appendix B

Laurent expansion of $K_l(s; \xi, \xi')$ near $l = -1$

The kernel $K(s; \xi, \xi')$ is given by the Eq. (2.10) where

$$F_l(s; \xi, \xi') = C(l)\,(q\,q')^{l+1} \int d^4\alpha\,|g(\alpha)|^l\,\{\tau(\alpha)\}^{-l-2}\,H(l, \sigma^{-2})\,. \tag{B.1}$$

Since $\tau(\alpha)$ can never be zero when $\mathrm{Im}\,s \neq 0$, the only singularities in the region $\mathrm{Re}\,l > -3/2$ come from the vanishing of $g(\alpha)$ in the domain of the α integration. That will happen when $l = -1$ and $\alpha_1\,\alpha_3 = \alpha_2\,\alpha_4$.

Let us consider the function

$$(|g|^l, \varphi_l) = \int d^4\alpha\,|g(\alpha)|^l\,\varphi_l(\alpha) \tag{B.2}$$

where φ^l is analytic at least in the domain $\Delta(l, s)$ defined by Eq. (4.4) where $l = -1$ may be included.

Since $\nabla\,g \neq 0$, that is easily seen from the structure of g (Appendix A), and g is irreducible i. e., cannot be split as a product of the form $g_1\,g_2$, we can introduce a local coordinate system $\{\beta\}$ and then identify g with say β_1.

Transforming the coordinate system $\{\alpha\}$ to $\{\beta\}$ with the Jacobian $J\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = J\begin{pmatrix}\alpha_1, \ldots, \alpha_4\\ \beta_1, \ldots, \beta_4\end{pmatrix} > 0$.

We shall have

$$(|g|^l, \varphi_l) = \int_{-b}^{b} d\beta_1\,|\beta_1|^l\,\Psi_l(\beta_1) \tag{B.3}$$

where

$$\Psi_l(\beta_1) = \int \cdots \int \tilde{\varphi}_l(\beta)\, J\binom{\alpha}{\beta}\, d\beta_2\, d\beta_3\, d\beta_4$$

$$\tilde{\varphi}_l(\beta) = \varphi_l(\alpha) \tag{B.4}$$

$$b = \max_{\alpha} |g(\alpha)| \,.$$

The Laurent development of the $|\beta_1|^l$ near $l = -1$ can be found in [20] and is

$$|\beta_1|^l = 2\frac{\delta(\beta_1)}{l+1} + |\beta_1|^{-1} + (l+1)\,|\beta_1|^{-1}\,\lg|\beta_1| + \cdots \tag{B.5}$$

where

$$(|\beta_1|^{-1}, \Psi) = \int\limits_0^b \frac{\Psi(\beta_1) + \Psi(-\beta_1) - 2\Psi(0)\,\theta(\varepsilon - \beta_1)}{\beta_1}\, d\beta_1$$

with $\varepsilon > 0$.

We see that the Laurent development of (B.3) near $l = -1$ is of the form

$$(|g|, {}^l\varphi_l) = \frac{2}{l+1}\,\Psi_{-1}(0) + \text{terms regular at } l = -1\,. \tag{B.6}$$

Now, we can go back to the original coordinate system $\{\alpha\}$, by noticing that

$$\begin{aligned}\Psi_{-1}(0) &= \int \cdots \int \tilde{\varphi}_{-1}(0, \beta_2\, \beta_3\, \beta_4)\, J\binom{\alpha}{\beta}\Big|_{\beta_1 = 0} d\beta_2\, d\beta_3\, d\beta_4 \\ &= \int \cdots \int \delta(\beta_1)\, \tilde{\varphi}_{-1}(\beta)\, J\binom{\alpha}{\beta}\, d\beta \\ &= \int \cdots \int \delta(g)\, \varphi_{-1}(\alpha)\, d\alpha\,.\end{aligned} \tag{B.7}$$

The equations (B.6) and (B.7) show that (B.2) has a simple pole at $l = -1$ whose residue can be obtained by replacing $|g|^l$ with $\delta(g)$ and by putting $l = -1$ in $\varphi_l(\alpha)$.

From the above discussion it follows that the kernel $K_l(s; \xi, \xi')$ has a simple pole at $l = -1$ and admits the expansion

$$K_l(s; \xi, \xi') = \frac{1}{l+1}\, K^{(1)}(s; \xi, \xi') + K_l^{(2)}(s; \xi, \xi')\,, \tag{B.8}$$

where

$$K_l^{(2)}(s; \xi, \xi') \text{ is regular at } l = -1\,.$$

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