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Abstract

Let f be an element of the subspace $(L^p,l^q)^\alpha(\mathbb{R}^d)$ $(1 \le p \le \alpha \le q \le 2)$ of the Wiener amalgam space $(L^p,l^q)(\mathbb{R}^d)$. We give sufficient conditions for Lebesgue integrability of the Fourier transform of f. These conditions are in terms of the $(L^p,l^q)^\alpha(\mathbb{R}^d)$ integral modulus of continuity of f. As an application, we obtain that if $1 \le \alpha \le q \le 2$ with $\frac{1}{\alpha} - \frac{1}{q} < \frac{1}{d}$ and $N = [\frac{d}{\alpha}] + 1$, then the Fourier inversion theorem can be applied to the elements of the Sobolev space $W^N((L^1,l^q)^\alpha(\mathbb{R}^d))$.

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1 Introduction

Let d be a positive integer. $\mathbb{R}^d$ is endowed with its usual scalar product $(x,\xi) \mapsto x.\xi = \sum_{j=1}^{d} x_j\xi_j$, euclidean norm $x \mapsto |x| = \sqrt{\sum_{j=1}^{d} x_j^2}$ and Lebesgue measure $E \mapsto |E|$.

For $1 \leq \alpha \leq \infty$ we denote by $L^\alpha = L^\alpha(\mathbb{R}^d)$ the classical Lebesgue space endowed with its usual norm $\|.\|_\alpha$ and by α' the conjugate exponent of α : $\frac{1}{\alpha'} + \frac{1}{\alpha} = 1$ with the convention $\frac{1}{\infty} = 0$.

We define the Fourier transform $\mathcal{F}(f) = \widehat{f}$ of an element f of L^1 by

$$\widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)e^{-ix.\xi}dx, \qquad \xi \in \mathbb{R}^d. \tag{1.1}$$

The Fourier transform has a well known extension, also denoted by $T \mapsto \mathcal{F}(T) = \widehat{T}$, to the space $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ of tempered distributions. This extension has turned to be of great usefulness in a variety of areas in Analysis such as: signal theory, probability, partial differential equations, approximation theory, singular integral operators, number theory (see [15],[23],[2],[1],[7],[10],[12],[21],[16], [5], [4]).

It is known that $\mathcal{F}$ is a bijection of $\mathcal{S}'$ onto itself. However, this space is very large and very often it is useful to have some knowledge on the image or the preimage by $\mathcal{F}$ of a pecular subspace of $\mathcal{S}'$. In this regard the following proposition contains results well known and widedly used.

Proposition 1.1. *1) For $1 \leq \alpha \leq 2$, $\mathcal{F}(L^\alpha)$ is contained in $L^{\alpha'}$ and there is a real constant M_α satisfying*

$$\|\widehat{f}\|_{\alpha'} \leq M_\alpha \|f\|_\alpha, \quad f \in L^\alpha \tag{1.2}$$

(Hausdorff-Young theorem).

2) $\mathcal{F}(L^1) \subset C_0$, where

$$C_0 = \left\{ g : \mathbb{R}^d \to \mathbb{C} / \ g \text{ continuous and } \lim_{|x|\to\infty} |g(x)| = 0 \right\}$$

and if f is an element of L^1 such that $\widehat{f}$ belongs to L^1 then

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \widehat{f}(y)e^{ix.y}, \quad a.e\ x \in \mathbb{R}^d \tag{1.3}$$

(Fourier inversion theorem).

3) $\mathcal{F}(L^2) = L^2$ and for any element f of L^2

$$\|\widehat{f}\|_2 = \|f\|_2 \tag{1.4}$$

(Plancherel identity)

$$\lim_{R\to\infty}\int_{\mathbb{R}^d}\left|(2\pi)^{-\frac{d}{2}}\int_{Q(0,R)}\widehat{f}(y)e^{ix.y}dy-f(x)\right|^2dx=0.$$

In view of the results above the following question is worthy:
what condition on an element f of L^α $(1\le\alpha\le2)$ is sufficient for $\widehat{f}$ to belong to a Lebesgue space L^β with $\beta<\alpha'$?

Titchmarsh has given to this question an answer using the notion of modulus of continuity.

Let $(X, \|.\|_X)$ be a complex normed space wich is included in L^1_{loc} and translation invariant, that is:

$$\tau_u f\in X,\qquad for\ \ f\in X \text{ and } u\in\mathbb{R}^d,$$

where

$$\tau_u f(x)=f(x-u).$$

Then the modulus of continuity $\omega_X(.)$ in X is defined by

$$\omega_X(f,t)=\sup\left\{\|\tau_u f-f\|_X/u\in\mathbb{R}^d,|u|\le t\right\},\quad f\in X,\ \ t\in\mathbb{R}_+.$$

The result of Titchmarsh may be stated as follows.

Proposition 1.2. *([21], Theorem 84) Let* $1<\alpha\le2$ *and* f *be an element of* $L^\alpha(\mathbb{R})$.
The condition

$$\textit{for some real number }\ \theta>0,\ \ \omega_{L^\alpha}(f,t)=O(t^\theta)\tag{1.5}$$

implies

$$\widehat{f}\textit{ belongs to } L^\beta \textit{ for } \frac{1}{\alpha'}\le\frac{1}{\beta}<\frac{1}{\alpha'}+\theta.\tag{1.6}$$

In the sequel we shall take $X=(L^1,l^q)^\alpha$ with $1\le\alpha\le q\le2$ (see section 2 for definition and some properties of this space).

We stress that L^α is a subspace of X wich is proper when $1<\alpha<q$ (see point 2) of Proposition 2.3).

The main results of the present paper are weighed Lebesgue norm inequalities between the Fourier transform $\widehat{f}$ and the modulus of continuity $\omega_X(f,.)$ of an element f of X (see Proposition 3.4 and Proposition 3.10).

From these inequalities, we deduce an extension of Proposition 1.2 to the framework of X (see Corollary 3.5). This result, in turn, enables us to obtain sufficient conditions on an element f of X in order to apply points 2) and 3) of Proposition 1.1. As an application we obtain Sobolev type theorems for X-Hölder spaces (see Corollary 3.6) and Sobolev spaces $W^N(X)$ (see Corollary 3.11).

The remaining of the paper is organized as follows. Section 2 is devoted to some pre-requisites on the space X. Section 3 contains our results and applications.

2 Background on $(L^p, l^q)^\alpha$

In [19] and [20] Szeptycki has given an interesting answer to the following question:

What may be said about the set of those elements of $\mathcal{S}'$ whose Fourier transforms are representable by functions in terms of Wiener amalgam spaces?

This answer has been given in terms of Wiener amalgam spaces which constitute the framework of our work and is defined as follows

By $L^1_{loc} = L^1_{loc}(\mathbb{R}^d)$ we denote the space of equivalence classes (modulo equality almost everywhere) of measurable complex functions on $\mathbb{R}^d$ which are locally in L^1.

For any positive real number r we set:

$I^r_k = \prod_{j=1}^{d}[k_j r, (k_j+1)r), \qquad k = (k_1, ..., k_d) \in \mathbb{Z}^d.$

If $1 \leq p, q \leq \infty$ then

- for any element f of L^1_{loc} and any real number $r > 0$,

$$_r\|f\|_{p,q} = \begin{cases} \left[\sum_{k\in\mathbb{Z}^d}\left(\|f\chi_{I^r_k}\|_p\right)^q\right]^{\frac{1}{q}} & if \ \ q < \infty, \\ \sup_{k\in\mathbb{Z}^d}\|f\chi_{I^r_k}\|_q & if \ \ q = \infty, \end{cases}$$

where, for any subset E of $\mathbb{R}^d$,

$$\chi_E(x) = \begin{cases} 1 & if \ \ x \in E, \\ 0 & otherwise. \end{cases}$$

- $(L^p, l^q) = (L^p, l^q)(\mathbb{R}^d) = \left\{f \in L^1_{loc}(\mathbb{R}^d)/_1\|f\|_{p,q} < \infty\right\}.$

The Wiener amalgam spaces (L^p, l^q) $(1 \leq p, q \leq \infty)$ have been introduced by N. Wiener (see [22]) and widely studied (see [11] and the references therein). Because of their properties stated below, they offer a well suited framework for the study of the Fourier transformation as an operator from $\mathcal{S}' \cap L^1_{loc}$ to L^1_{loc}.

Proposition 2.1. *([13],[19],[20]) Let us assume* $1 \leq p, q \leq \infty$.

1. $((L^p, l^q)\ ,\ _1\|\,.\,\|_{p,q})$ *is a Banach space included in* $\mathcal{S}'$ *and satisfying*
   $\bullet (L^p, l^p) = L^p$,
   $\bullet\ L^p \cup L^q \subset (L^p, l^q)$ *when* $p < q$.

2. *If* V *is a subspace of* $\mathcal{S}' \cap L^1_{loc}$ *wich is solid, that is:*
   $\forall f, g \in L^1_{loc} \quad [f \in V \text{ and } |g| \leq f] \Rightarrow g \in V$,
   then the following conditions are equivalent:
   $(i)\ \mathcal{F}(V) \subset L^1_{loc}$,
   $(ii)\ V \subset (L^1, l^2)$.

3. *If* $1 \leq p, q \leq 2$ *then* $\mathcal{F}((L^p, l^q))$ *is included in* $(L^{q'}, l^{p'})$ *and there is a real number* $M_{p,q}$ *such that*

$$_1\|\widehat{f}\|_{q',p'} \leq M_{p,q}\ _1\|f\|_{p,q}, \quad f \in (L^p, l^q) \tag{2.1}$$

   (Hausdorff-Young theorem).

Our results shall be stated in the framework of subspace $(L^p, l^q)^\alpha$ $(1 \leq p \leq \alpha \leq q \leq +\infty)$ of amalgam Wiener space (L^p, l^q) which has been introduced by I. Fofana in [8] and is defined as follows.

Definition 2.2. For $1 \leq p \leq \alpha \leq q \leq \infty$, we set

$$(L^p, l^q)^\alpha = (L^p, l^q)^\alpha(\mathbb{R}^d) = \left\{ f \in L^1_{loc}\ \ /\ \ \|f\|_{p,q,\alpha} = \sup_{r>0} r^{d(\frac{1}{\alpha} - \frac{1}{p})}\ _r\|f\|_{p,q} < \infty \right\}.$$

Let us recall some of their properties.

Proposition 2.3. *([6],[8],[9]) Let us assume that* $1 \leq p \leq \alpha \leq q \leq \infty$.

1. $((L^p, l^q)^\alpha, \|.\|_{p,q,\alpha})$ *is a complex Banach space included in* (L^p, l^q)*, translation invariant and there is a real number* $C(p,q)$ *such that*

$$\|\tau_u f\|_{p,q,\alpha} \leq C(p,q)\|f\|_{p,q,\alpha}\ , \qquad f \in (L^p, l^q)^\alpha. \tag{2.2}$$

   .

2. $\bullet \|f\|_{p,q,\alpha} \leq \|f\|_\alpha\ , \quad f \in L^\alpha$,
   $\bullet$ *if* $p < \alpha < q$ *then* $(L^p, l^q)^\alpha$ *contains properly* L^α,
   $\bullet$ *if* $\alpha = p$ *or* $\alpha = q$ *then* $(L^p, l^q)^\alpha = L^\alpha$ *with equivalent norms.*

3. *If* $1 \leq p \leq \alpha \leq 2$ *then* $\mathcal{F}((L^p, l^q)^\alpha)$ *is included in* $(L^{q'}, l^{p'})^{\alpha'}$ *and there is a real number* $M_{p,q}$ *such that:*

$$\|\widehat{f}\|_{q',p',\alpha'} \leq M_{p,q}\|f\|_{p,q,\alpha}\ , \quad f \in (L^p, l^q)^\alpha. \tag{2.3}$$

   (Hausdorff-Young theorem)

4. *When $p < \infty$ and f belongs to the Sobolev space $W^1((L^p, l^q)^\alpha)$, that is, f and all its distributional partial derivatives $\frac{\partial f}{\partial x_j}$ $(1 \le j \le d)$ are in $(L^p, l^q)^\alpha$, then*

$$\|\tau_u f - f\|_{p,q,\alpha} \le C \||\nabla f|\|_{p,q,\alpha} |u| , \quad u \in \mathbb{R}^d$$

where C is a real number not depending on f.

3 Integrability of the Fourier transform and modulus of continuity

We begin this part with a notation and two preparatory lemmas.

Notation 3.1. For any integer k, we set

$$T_k = [-2^k, 2^k]^d \backslash [-2^{k-1}, 2^{k-1}]^d.$$

Lemma 3.2. *For any integer k, we have*

$$T_k = \bigcup_{l \in L_k} I_l^{2^{k-1}} \tag{3.1}$$

and
a) L_k has $2^d(2^d - 1)$ elements,
b) for any element l of L_k, there exists a vector $u(l)$ of $\mathbb{R}^d$ satisfying:

$$|u(l)| = 2^{-k}, \tag{3.2}$$

$$\frac{1}{2} \le x.u(l) \le 1, \;\; x \in I_l^{2^{k-1}}. \tag{3.3}$$

Proof. Let k be any integer.
a) It is clear that $T_k = \bigcup\limits_{l \in L_k} I_l^{2^{k-1}}$, with

$$L_k = \{-2, -1, 0, 1\}^d \backslash \{-1, 0, \}^d$$

and L_k has $2^d(2^d - 1)$ elements.
b) Let l be any element of L_k.
There exists an element $j(l)$ of $\{1, 2, ..., d\}$ such that $l_{j(l)}$ belongs $\{-2, 1\}$.
1^{rst}case: $j(l) = -2$.

We notice that, for any element $x = (x_1, x_2, ..., x_d)$ of $I_l^{2^{k-1}}$, we have

$$x_j = 2^{k-1} l_j + t_j \text{ with } 0 \le t_j < 2^{k-1}, \quad j \in \{1, 2, ..., d\}$$

$$x.(-e_{j(l)}) = 2^k - t_{l(j)} \text{ with } 0 \le t_{j(l)} < 2^{k-1}$$

and therefore, setting $u(l) = -2^{-k}e_{j(l)}$, we obtain

$$2^{-1} < x.u(l) \leq 1 \text{ and } |u(l)| = 2^{-k}.$$

2^{nd}case $j(l) = 1$

Proceeding in the same manner as in the first case, we obtain that the vector $u(l) = 2^{-k}e_{j(l)}$ satisfies

$$|u(l)| = 2^{-k} \quad \text{and} \quad 2^{-1} \leq x.u(l) < 1, \quad x \in I_k^{2^{k-1}}.$$

□

Lemma 3.3. *Suppose that* $1 \leq \alpha \leq p \leq 2$. *Then there is a real constant* B *such that for any element* f *of* $X = (L^1, l^q)^\alpha$ *and for any integer* k,

$$\|\widehat{f}\chi_{T_k}\|_{q'} \leq B2^{kd(\frac{1}{\alpha}-\frac{1}{q})}\omega_X(f, 2^{-k+1}). \tag{3.4}$$

Proof. Let k be any integer.

Recall the following well-known results :

•

$$(\tau_{-u}\widehat{f-\tau_u}f)(x) = 2i\sin(x.u)\widehat{f}(x), \quad u \in \mathbb{R}^d \text{ and a.e } x \in \mathbb{R}^d,$$

• there is a real number $A > 0$ such that:

$$A\theta \leq \sin\theta, \quad \theta \in [0, 1).$$

Using the notations and results of the Lemma 3.2, we have, for any element l of L_k

$$2^{-1}A \leq Ax.u(l) \leq \sin(x.u(l)), \qquad x \in I_l^{2^{k-1}}$$

$$A|\widehat{f}(x)| \leq |2i\sin(x.u(l))\widehat{f}(x)|, \qquad \text{a.e} \qquad x \in I_l^{2^{k-1}}$$

$$|\widehat{f}(x)| \leq A^{-1}|(\tau_{-u(l)}\widehat{f-\tau_{u(l)}}f)(x)|, \qquad \text{a.e} \qquad x \in I_l^{2^{k-1}}$$

$$\|\widehat{f}\chi_{I_l^{2^{k-1}}}\|_{q'} \leq A^{-1}\|(\tau_{-u(l)}\widehat{f-\tau_{u(l)}}f)\chi_{I_l^{2^{k-1}}}\|_{q'}.$$

$$\|\widehat{f}\chi_{I_l^{2^{k-1}}}\|_{q'} \leq A^{-1}2^{(k-1)d(\frac{1}{\alpha}-\frac{1}{q})}\|\tau_{-u(l)}\widehat{f-\tau_{u(l)}}f\|_{q',\infty,\alpha'} \quad \text{(by Definition 2.2)}$$

$$\|\widehat{f}\chi_{I_l^{2^{k-1}}}\|_{q'} \le A^{-1}M_{1,q}C(1,q)2^{(k-1)d(\frac{1}{\alpha}-\frac{1}{q})}\|f-\tau_{2u(l)}f\|_{1,q,\alpha} \quad \text{by} \quad (2.3)$$

$$\|\widehat{f}\chi_{I_l^{2^{k-1}}}\|_{q'} \le A^{-1}M_{1,q}C(1,q)2^{(k-1)d(\frac{1}{\alpha}-\frac{1}{q})}\omega_X(f,2^{-k+1}).$$

Let $B = 2^d(2^d-1)A^{-1}M_{1,q}C(1,q)2^{d(\frac{1}{q}-\frac{1}{\alpha})}$.
From the relation (3.1) and the result obtained above, we deduce

$$\|\widehat{f}\chi_{T_k}\|_{q'} \le B2^{kd(\frac{1}{\alpha}-\frac{1}{q})}\omega_X(f,2^{-k+1}).$$

□

A first form of our result reads as follows.

Proposition 3.4. *Suppose that* $1 \le \alpha \le q \le 2$. *Then there exists a real constant* C *such that, for all element* f *of* $X = (L^1, l^q)^\alpha$ *and any element* β *of* $(0,q']$

$$\int_{|x|>\sqrt{d}} |\widehat{f}(x)|^\beta \le C^\beta \int_{|x|\ge 1}\left[|x|^{-\frac{d}{\alpha'}}\omega_X(f,2\sqrt{d}|x|^{-1})\right]^\beta dx. \tag{3.5}$$

Proof. Let us consider an integer $k \ge 0$.
By Hölder's inequality , we have

$$\int_{T_k}|\widehat{f}(x)|^\beta dx \le \|\widehat{f}\chi_{T_k}\|_{q'}^\beta |T_k|^{1-\frac{\beta}{q'}} = \|\widehat{f}\chi_{T_k}\|_{q'}^\beta \left[2^{kd}-2^{(k-1)d}\right]^{-\frac{\beta}{q'}}|T_k|.$$

From relation (3.4), we deduce

$$\begin{aligned}\int_{T_k}|\widehat{f}(x)|^\beta dx &\le \left\{B2^{kd(\frac{1}{q'}-\frac{1}{\alpha'})}\omega_X(f,2^{-k+1})\left[2^{kd}(2^d-1)\right]^{-\frac{1}{q'}}\right\}^\beta |T_k| \\ &\le \left\{B(2^d-1)^{-\frac{1}{q'}}2^{-\frac{kd}{\alpha'}}\omega_X(f,2^{-k+1})\right\}^\beta |T_k|.\end{aligned}$$

We notice that for x in T_k we have

$$2^{k-1} \le |x| \le \sqrt{d}2^k, \text{ that is , } 2^{k-1} \le |x| \text{ and } 2^{-k} \le \sqrt{d}|x|^{-1}$$

and therefore

$$2^{k-1} \le |x| \text{ and } \begin{cases} 2^{\frac{-kd}{\alpha'}} \le (\sqrt{d})^{\frac{d}{\alpha'}}|x|^{\frac{-d}{\alpha'}} \\ 2^{-k+1} \le 2\sqrt{d}|x|^{-1}. \end{cases}$$

So, since $t \mapsto \omega_X(f,t)$ is non decreasing, we have

$$\int_{T_k} |\widehat{f}(x)|^\beta dx \leq C^\beta \int_{T_k} \left[|x|^{-\frac{d}{\alpha'}} \omega_X(f, 2\sqrt{d}|x|^{-1}) \right]^\beta dx$$

with $C = B(2^d - 1)^{-\frac{1}{q'}} (\sqrt{d})^{\frac{d}{\alpha'}}$.

It is easy to see that $\left\{ x \in \mathbb{R}^d / |x| > \sqrt{d} \right\} \subset \bigcup_{k \geq 1} T_k \subset \left\{ x \in \mathbb{R}^d / |x| \geq 1 \right\}$.

Therefore from the inequality above we get

$$\int_{|x| > \sqrt{d}} |\widehat{f}(x)|^\beta \leq C^\beta \int_{|x| \geq 1} \left[|x|^{-\frac{d}{\alpha'}} \omega_X(f, 2\sqrt{d}|x|^{-1}) \right]^\beta dx.$$

□

As an immediate consequence of this proposition, we have the following result.

Corollary 3.5. *Suppose that* $1 < \alpha \leq q \leq 2$, θ *is a positive real number and* f *is an element of* $X = (L^1, l^q)^\alpha$ *such that*

$$\omega_X(f,t) = O(t^\theta), \quad \text{when } t \text{ goes to } 0. \tag{3.6}$$

Then

$$\widehat{f} \in L^\beta \quad \text{for} \quad \frac{1}{q'} \leq \frac{1}{\beta} < \frac{1}{\alpha'} + \frac{\theta}{d}. \tag{3.7}$$

Proof. Consider a real β verifying (3.7). There is a real number $D > 0$ such that:

$$\int_{|x| > \sqrt{d}} \left[|x|^{-\frac{d}{\alpha'}} \omega_X(f, 2\sqrt{d}|x|^{-1}) \right]^\beta dx \leq \sigma(\mathcal{S}^{d-1}) D \int_{\sqrt{d}}^\infty t^{-(\frac{d}{\alpha'} + \theta)\beta + d - 1} dt$$

and $\int_{\sqrt{d}}^\infty t^{-(\frac{d}{\alpha'} + \theta)\beta + d - 1} dt < \infty$ because of

$$-\left(\frac{d}{\alpha'} + \theta \right) \beta + d = \beta d \left[-\left(\frac{1}{\alpha'} + \frac{\theta}{d} \right) + \frac{1}{\beta} \right] < 0.$$

Therefore, according to (3.5),

$$\int_{|x| > \sqrt{d}} |\widehat{f}(x)|^\beta dx < \infty.$$

Moreover, since $\beta \leq q'$, Hölder inequality and of (2.3) imply ,

$$\int_{|x| \leq \sqrt{d}} |\widehat{f}(x)|^\beta dx < \infty.$$

□

It is clear from point 2) of Proposition 2.3 that, taking $d = 1$ and $p = \alpha$ in Corollary 3.5, we recover Proposition 1.2.

Corollary 3.5 has the following consequence.

Corollary 3.6. *Suppose that*

(i) $1 < \alpha \leq q \leq 2$ *and* θ *is a positive real number such that*

$$0 < \frac{1}{\alpha} - \frac{\theta}{d} < \frac{1}{2}.$$

(ii) f *belongs to* $X = (L^1, l^q)^\alpha$ *and satisfies (3.6).*

Then

$$f \in L^p \ \textit{for} \ \frac{1}{\alpha} - \frac{\theta}{d} < \frac{1}{p} \leq \frac{1}{2}.$$

Proof. According to the hypotheses

$$\frac{1}{q'} \leq \frac{1}{2} < \frac{1}{\alpha'} + \frac{\theta}{d} \leq 1.$$

Let β an element of $[1,2]$ satisfying

$$\frac{1}{\beta} < \frac{1}{\alpha'} + \frac{\theta}{d}.$$

According to the Corollary 3.5, $\widehat{f}$ belongs to L^β.

Using the Fourier transformation in $\mathcal{S}'$ and Fourier inversion theorem in $\mathcal{S}$, we obtain

$$f(x) = \widehat{\widehat{f}}(-x) \ \text{ a.e } \ x \in \mathbb{R}^d.$$

Therefore, according to Hausdorff-Young's theorem f belongs to $L^{\beta'}$.

In addition, we have

$$0 \leq \frac{1}{\alpha} - \frac{\theta}{d} < \frac{1}{\beta'} \leq \frac{1}{2}.$$

So, just take $p = \beta'$. □

Remark 3.7. It is well known that if $1 < \alpha \leq \infty$ and f belongs to L^α then the following assertions are equivalent:

(i)

$$\omega_{L^\alpha}(f,t) = O(t) \ \text{ when } \ t \ \text{ goes to } \ 0$$

(ii) $f \in W^{1,\alpha}$ (the partial distributional derivatives $\frac{\partial f}{\partial x_j}$ $(1 \leq j \leq d)$ of f belong to L^α) (see [3]).

Therefore, taking $\theta = 1$ and $q = \alpha$ in the Corollary 3.6, we obtain the following implication:

$$\left(1 < \alpha \leq 2 \text{ and } 0 < \frac{1}{\alpha} - \frac{1}{d} < \frac{1}{2},\ f \in W^{1,\alpha}\right) \Rightarrow \left(f \in L^p \text{ for } p \in [\alpha, \alpha^*[\text{ with } \frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{1}{d}\right).$$

So, Corollary 3.6 may be viewed as a Sobolev-type theorem for $(L^1, l^q)^\alpha$- Hölder spaces.

In the sequel we shall establish a weighted version of Proposition 3.4. The weight that we shall use belongs to a special class introduced by F. Moricz in [14] and is defined as follows.

Definition 3.8. Let us consider a real number $\gamma \geq 1$. A positive and locally integrable function λ on $\mathbb{R}^d \backslash \{0\}$ belongs to $\mathcal{A}_\gamma$ if there is a real number $\mathcal{K}_\gamma$ such that:

$$\|\lambda\chi_{T_k}\|_\gamma \leq \mathcal{K}_\gamma 2^{\frac{kd(1-\gamma)}{\gamma}} \|\lambda\chi_{T_{k-1}}\|_1, \quad k \in \mathbb{N}. \tag{3.8}$$

Example 3.9. *a*) Suppose that λ is a positive and locally integrable function on $\mathbb{R}^d \backslash \{0\}$ such that, for some real number K,

$$\sup\{\lambda(x) \ / \ x \in T_k\} \leq K \inf\{\lambda(x) \ / \ x \in T_{k-1}\}, \quad k \in \mathbb{Z}.$$

Then, we have for any real $\gamma \geq 1$ and any integer k

$$\begin{aligned} \|\lambda\chi_{T_k}\|_\gamma &\leq \sup\{\lambda(x) \ / \ x \in T_k\} |T_k|^{\frac{1}{\gamma}} \\ &\leq K \inf\{\lambda(x) \ / \ x \in T_{k-1}\} |T_k|^{\frac{1}{\gamma}} \\ &\leq K |T_k|^{\frac{1}{\gamma}} |T_{k-1}|^{-1} \|\lambda\chi_{T_{k-1}}\|_1. \end{aligned}$$

Therefore

$$\|\lambda\chi_{T_k}\|_\gamma \leq K 2^{\frac{d(\gamma-1)}{\gamma}} (2^d - 1)^{\frac{1-\gamma}{\gamma}} 2^{\frac{kd(1-\gamma)}{\gamma}} \|\lambda\chi_{T_{k-1}}\|_1.$$

So, λ belongs to $\mathcal{A}_\gamma$ for any real $\gamma \geq 1$.

b) Let us consider a real number c and set

$$\lambda(x) = |x|^c, \quad x \in \mathbb{R}^d \backslash \{0\}.$$

We have, for any integer k

$$\sup\{\lambda(x) \ / \ x \in T_k\} = \begin{cases} (\sqrt{d}2^k)^c, & \text{if } c \geq 0 \\ 2^{(k-1)c}, & \text{if } c \leq 0 \end{cases}$$

and

$$\inf\{\lambda(x) \ / \ x \in T_{k-1}\} = \begin{cases} 2^{(k-2)c}, & \text{if } c \geq 0 \\ (\sqrt{d}2^{k-1})^c, & \text{if } c \leq 0. \end{cases}$$

Therefore, there exists a real number K such that

$$\sup\{\lambda(x) \ / \ x \in T_k\} \leq K \inf\{\lambda(x) \ / \ x \in T_{k-1}\}, \quad k \in \mathbb{Z}$$

These inequalities and the result obtained at the point a), imply that λ belongs to $\mathcal{A}_\gamma$ for any real $\gamma \geq 1$.

The announced result reads as follows.

Proposition 3.10. *Suppose that:* $1 \leq \alpha \leq q \leq 2$, $0 < \beta \leq q'$, $\gamma = \frac{q}{q-\beta q+\beta}$ *and* λ *belongs to* $\mathcal{A}_\gamma$. *Then there exists a real constant* C *such that, for all element* f *of* $X = (L^1, l^q)^\alpha$

$$\int_{|x|>\sqrt{d}} |\widehat{f}(x)|^\beta \lambda(x)dx \leq C^\beta \int_{|x|\geq\frac{1}{2}} \left[|x|^{-\frac{d}{\alpha'}} \omega_X(f, \sqrt{d}|x|^{-1})\right]^\beta \lambda(x)dx \tag{3.9}$$

Proof. We notice that

$$\frac{\beta}{q'} + \frac{1}{\gamma} = 1 \text{ and therefore } \frac{1-\gamma}{\gamma} = -\frac{\beta}{q'}$$

So, using Hölder inequality, (3.4) and (3.8) we get

$$\begin{aligned}\int_{T_k} |\widehat{f}(x)|^\beta \lambda(x)dx &\leq \|\widehat{f}\chi_{T_k}\|_{q'}^\beta \|\lambda\chi_{T_k}\|_\gamma \\ &\leq \left[B2^{kd(\frac{1}{\alpha}-\frac{1}{q})}\omega_X(f, 2^{-k+1})2^{-\frac{kd}{q'}}\right]^\beta \mathcal{K}_\gamma \int_{T_{k-1}} \lambda(x)dx. \\ &\leq \left[B2^{-\frac{kd}{\alpha'}}\omega_X(f, 2^{-k+1})\right]^\beta \mathcal{K}_\gamma \int_{T_{k-1}} \lambda(x)dx\end{aligned}$$

Note also that, for any element x of T_{k-1}, we have

$$2^{-k+1} \leq \sqrt{d}|x|^{-1} \quad \text{and therefore} \quad 2^{-\frac{kd}{\alpha'}} \leq (2^{-1}\sqrt{d})^{\frac{d}{\alpha'}}|x|^{-\frac{d}{\alpha'}}$$

So, since $\omega_X(f, .)$ is non decreasing, we have

$$\int_{T_k} |\widehat{f}(x)|^\beta \lambda(x)dx \leq \left[(2^{-1}\sqrt{d})^{\frac{d}{\alpha'}} B\right]^\beta \mathcal{K}_\gamma \int_{T_{k-1}} \left[\omega_X(f, \sqrt{d}|x|^{-1})|x|^{-\frac{d}{\alpha'}}\right]^\beta \lambda(x)dx.$$

The above inequality being true for any positive integer k, we have

$$\int_{|x|>\sqrt{d}} |\widehat{f}(x)|^\beta \lambda(x)dx \leq C \int_{|x|\geq\frac{1}{2}} \left[\omega_X(f, \sqrt{d}|x|^{-1})|x|^{-\frac{d}{\alpha}}\right]^\beta \lambda(x)dx$$

where $C = \left[(2^{-1}\sqrt{d})^{\frac{d}{\alpha'}} B\right]^\beta \mathcal{K}_\gamma$. □

From point 2) of Proposition 2.3, it is easy to see that Theorem 1 of [14] is a special case of Proposition 3.10 (the one where $d = 1$ and $p = \alpha$).
A consequence of Proposition 3.10 is the following result.

Corollary 3.11. *Suppose that* $1 \le \alpha \le q \le 2$ *with* $\frac{1}{\alpha} - \frac{1}{q} < \frac{1}{d}$ *and* $f \in W^N(X)$ *where* $N = [\frac{d}{\alpha}] + 1$ *and* $X = (L^1, l^q)^\alpha$.

Then $\check{f}$ *belongs to* L^1 *and*

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \check{f}(y) e^{-ix.y} dy, \quad a.\ e.\ \ x \in \mathbb{R}^d.$$

Proof.
• Let be $\theta \in \mathbb{N}^d$ with $|\theta| = N - 1$.
We have

$$D^\theta f \in X \ \text{ and } \ \frac{\partial}{\partial x_j}(D^\theta f) \in X \text{ for any } j \in \{1, 2, ..., d\}$$

So, there is a real number D independant of f and θ such that

$$\forall t > 0, \qquad \omega_X(D^\theta f, t) \le D|||\nabla(D^\theta f)|||_{1,q,\alpha} t \tag{3.10}$$

(see Proposition 3.2 of [6])
By hypothesis

$$\begin{cases} \frac{d}{\alpha} < N \text{ and therefore } 1 + \frac{1}{d} - \frac{N}{d} < 1 + \frac{1}{d} - \frac{1}{\alpha} = \frac{1}{d} + \frac{1}{\alpha'} \\ \frac{1}{\alpha} - \frac{1}{q} < \frac{1}{d} \text{ and therefore } \frac{1}{q'} < \frac{1}{d} + \frac{1}{\alpha'}. \end{cases}$$

Choosing β such that $1 + \frac{1}{d} - \frac{N}{d} < \frac{1}{\beta} < \frac{1}{d} + \frac{1}{\alpha'}$ and $\frac{1}{q'} \le \frac{1}{\beta}$, we get from inequalities (3.5) and (3.10).

$$\int_{|x| > \sqrt{d}} |\overset{\vee}{(D^\theta f)}(x)|^\beta dx \le C^\beta \left[D|||\nabla(D^\theta f)|||_{1,q,\alpha} \sqrt{d}\right]^\beta \int_{|x| \ge 1} |x|^{-\beta(\frac{d}{\alpha'}+1)} dx.$$

In addition, since $\frac{1}{\beta} - \frac{1}{\alpha'} < \frac{1}{d}$, we have

$$\int_{|x|>1} |x|^{-\beta(\frac{d}{\alpha'}+1)} dx = \sigma(S^{d-1}) \int_1^\infty t^{-\beta(\frac{d}{\alpha'}+1)+d-1} dt = \sigma(S^{d-1}) \int_1^\infty t^{-d\beta[\frac{1}{d} - (\frac{1}{\beta} - \frac{1}{\alpha'})]-1} dt < \infty$$

So there is a number E independant of f and θ such that

$$\int_{|x| > \sqrt{d}} |\overset{\vee}{(D^\theta f)}(x)|^\beta dx \le E^\beta |||\nabla(D^\theta f)|||^\beta_{1,q,\alpha}. \tag{3.11}$$

• Since X is a subspace of $\mathcal{S}'$, it is known that

$$(-i.)^{\theta}\check{f} = (\overset{\vee}{D^{\theta} f}) \quad \text{in } \mathcal{S}',\ \theta \in \mathbb{N}^d,\ |\theta| = N-1.$$

We have also

$$\begin{aligned}
|x|^{\beta(N-1)} &= \left[\left(\sum_{j=1}^{d} x_j^2\right)^{\frac{1}{2}}\right]^{\beta(N-1)} \\
&\leq \left[\sum_{j=1}^{d} |x_j|\right]^{\beta(N-1)} \\
&\leq \left[\sum_{|\theta|=N-1} |x^{\theta}|\right]^{\beta} \\
&\leq A(N)^{\beta} \sum_{|\theta|=N-1} |x^{\theta}|^{\beta}, \quad x \in \mathbb{R}^d.
\end{aligned}$$

where $A(N) = \left(\sharp\{\theta \in \mathbb{N}^d / \ |\theta| = N-1\}\right)^{\frac{1}{\beta'}}$

So, we have:

$$\begin{aligned}
\int_{|x|>\sqrt{d}} \left[|x|^{N-1}|\check{f}(x)|\right]^{\beta} dx &\leq A(N)^{\beta} \sum_{|\theta|=N-1} \int_{|x|\geq 1} |(-ix)^{\theta}\check{f}(x)|^{\beta} dx \\
&= A(N)^{\beta} \sum_{|\theta|=N-1} \int_{|x|>\sqrt{d}} |\ (\overset{\vee}{D^{\theta} f})\ (x)|^{\beta} dx.
\end{aligned}$$

From this inequality and (3.11), we get

$$\int_{|x|>\sqrt{d}} \left[|x|^{N-1}|\check{f}(x)|\right]^{\beta} dx \leq A(N)^{\beta} E^{\beta} \sum_{|\theta|=N-1} |||\nabla(D^{\theta} f)|||_{1,q,\alpha}^{\beta}$$

Therefore, using Hölder inequality , we get

$$\begin{aligned}
\int_{|x|>\sqrt{d}} |\check{f}(x)| dx &\leq \left(\int_{|x|>\sqrt{d}} |x|^{(1-N)\beta'} dx\right)^{\frac{1}{\beta'}} \left(\int_{|x|>\sqrt{d}} \left[|x|^{N-1}|\check{f}(x)|\right]^{\beta} dx\right)^{\frac{1}{\beta}} \\
&\leq A(N) E \left(\int_{|x|>\sqrt{d}} |x|^{(1-N)\beta'} dx\right)^{\frac{1}{\beta'}} \left(\sum_{|\theta|=N-1} |||\nabla(D^{\theta} f)|||_{1,q,\alpha}^{\beta}\right)^{\frac{1}{\beta}}
\end{aligned}$$

We notice that $\frac{N-1}{d} - \frac{1}{\beta'} > 0$, and therefore

$$\int_{|x|>\sqrt{d}} |x|^{(1-N)\beta'} dx = \sigma(S^{d-1}) \int_{\sqrt{d}}^{\infty} t^{(1-N)\beta'+d-1} dt = \sigma(S^{d-1}) \int_{\sqrt{d}}^{\infty} t^{-\beta d[-\frac{1}{\beta'}+\frac{N-1}{d}]-1} dt < \infty.$$

So, there exists a real number K_1 not dependant of f, such that

$$\int_{|x|>\sqrt{d}} |\check{f}(x)|dx \leq K_1 \sum_{|\theta|=N-1} |||\nabla(D^\theta f)|||_{1,q,\alpha}$$

Moreover, by Hölder inequality and (1.6), we have

$$\int_{|x|\leq\sqrt{d}} |\check{f}(x)|dx \leq |B(0,\sqrt{d})|^{\frac{1}{q}} \left(\int_{|x|\leq\sqrt{d}} |\check{f}(x)|^{q'} dx\right)^{\frac{1}{q'}} \leq K_2 ||f||_{1,q,\alpha}$$

where K_2 is a real number not dependent of f.

From these inequalities it follows that there exists a real number K not dependent of f, such that

$$\int_{\mathbb{R}^d} |\check{f}(x)|dx \leq K\left[||f||_{1,q,\alpha} + \sum_{|\theta|=N-1} |||\nabla(D^\theta f)|||_{1,q,\alpha}\right].$$

Thus $\check{f}$ belongs $L^1(\mathbb{R}^d)$ and, by Fourier inversion theorem,

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \check{f}(y)e^{-ix.y}dy, \text{ a. e. } x \in \mathbb{R}^d.$$

□

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