Communications in Mathematical Analysis
Volume 22, Number 2, pp. 1–17 (2019) www.math-res-pub.org/cma
ISSN 1938-9787

Practical Stability of Differential Equations with State Dependent Delay and Non-instantaneous Impulses

Michal Fečkan[a,b*]
[a]Department of Mathematical Analysis and Numerical Mathematics
Faculty of Mathematics, Physics and Informatics
Comenius University in Bratislava
Mlynská dolina, 842 48 Bratislava, Slovakia
[b]Mathematical Institute, Slovak Academy of Sciences
Štefánikova 49, 814 73 Bratislava, Slovakia

Snezhana Hristova[†]
Department of Applied Mathematics and Modeling
University of Plovdiv "P. Hilendarski"
4000 Plovdiv, Bulgaria

Krasimira Ivanova[‡]
Department of Mathematics
Department of Applied Mathematics and Modeling
University of Plovdiv "P. Hilendarski"
4000 Plovdiv, Bulgaria

(Communicated by Toka Diagana)

Abstract

In this paper some practical stability results for nonlinear differential equations with non-instantaneous impulses and state dependent delays are presented. The impulses start abruptly at some points and their action continue on given finite intervals. The delay depends on both the time and the state variable which is a generalization of time variable delay. Some sufficient conditions for practical stability and strong practical stability are obtained by the help with the appropriate modification of Razumikhin method and an appropriate definition of the derivative of the Lyapunov function. Examples are given to illustrate the results.

[*]E-mail address: michal.feckan@fmph.uniba.com
[†]E-mail address: snehri gmail.com
[‡]E-mail address: kbr@gmail.com

AMS Subject Classification: 34A37, 34D20.

Keywords: non-instantaneous impulses, differential equations, practical stability.

1 Introduction

One of the most investigated qualitative problems for differential equations is connected with the investigations of various types of stability. Often Lyapunov functions and different modifications of Lyapunov direct method are applied to study stability properties of solutions without their obtaining in a closed form. One type of stability, very useful in real world problems, is so called practical stability problem, introduced by LaSalle and Lefschetz [13] and it deals with the question of whether the system state evolves within certain subsets of the state-space. For instance, an equilibrium point may not be stable in the sense of Lyapunov and yet the system response maybe acceptable in the vicinity of this equilibrium.

There are a few different real life processes and phenomena that are characterized by rapidly changes in their state. We will emphasize on changes which duration of action is not negligible short, i.e., these changes start impulsively at arbitrary fixed points and remain active on finite initially given time intervals. The model of this situation is the non-instantaneous impulsive differential equation. E. Hernandez and D. O'Regan ([7]) introduced this new class of differential equations where the impulses are not instantaneous and they investigated the existence of mild and classical solutions. We refer the reader for some recent results such as existence to [15, 16], to stability [2, 17], to periodic boundary value problems [4].

The state-dependent delays in differential equations are applied to model adequate many problems, such as milling [9], control theory [18], haematopoiesis [5], economics [14]. For a review of recent progress see [6].

In this paper we study an initial value problem (IVP) for a nonlinear system of non-instantaneous impulsive differential equations with state dependent delay. The state dependent delay is a generalization of both a time dependent delay and a constant delay ([3]). Some sufficient conditions for several types of practical stability are obtained. Modified Razumikhin method with piecewise continuous functions, appropriate definition of the Lyapunov functions and comparison results with scalar non-instantaneous equations without delay are applied. We study both cases of bounded state dependent delays and unbounded state dependent delays.

Some examples illustrating the results are given. Some of the obtained sufficient conditions are generalizations of results for practical stability of impulsive functional-differential equations as well as for differential equations with time variable delays.

2 Preliminaries

In this paper we assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $0 < s_i < t_i < s_{i+1}, i = 1, 2, \dots$, and $\lim_{k\to\infty} t_k = \infty$.

Let $s_0 = 0$ and $t_0 \in [0, s_1) \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1})$ be a given arbitrary point. Without loss of generality we will assume that $t_0 \in [0, s_1)$.

The intervals $(t_i, s_{i+1}), i = 0, 1, 2, \ldots, k$ are the intervals on which the differential equation is given and the intervals $(s_i, t_i), i = 1, 2, \ldots, k$ are called impulsive intervals and on these intervals the impulsive conditions are given.

Consider the initial value problem (IVP) for a nonlinear system of non-instantaneous impulsive differential equations with state dependent delay (NIDDE)

$$\begin{aligned} &x'(t) = f(t, x_{\rho(t,x_t)}) \text{ for } t \in \cup_{k=0}^{\infty}(t_k, s_{k+1}], i = 0, 1, 2, \ldots, \\ &x(t) = \Phi_k(t, x(s_k - 0)) \text{ for } t \in (s_k, t_k], k = 1, 2, \ldots, \\ &x(t_0 + t) = \phi(t), \quad t \in E_0, \end{aligned} \tag{2.1}$$

where the functions $f : [0, s_1] \bigcup \cup_{k=1}^{\infty}[t_k, s_{k+1}] \times PC_0 \to \mathbb{R}^n$, $\Phi_k : [s_k, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ ($k = 1, 2, 3, \ldots$), $\rho : [0, s_1] \bigcup \cup_{i=1}^{\infty}[t_i, s_{i+1}] \times PC_0 \to [0, \infty)$, $\phi \in PC_0$, and $r > 0$ is a given number. Here the notation $x_t(s) = x(t + s), s \in [-r, 0]$ is used, i.e. $x_t \in PC_0$ represents the history of the state $x(t)$ from time $t - r$ up to the present time t. Note that for any $t \geq 0$ we let $x_{\rho(t,x_t)}(s) = x(\rho(t, x(t + s))), s \in [-r, 0]$. The initial interval $E_0 \subset (-\infty, 0]$ depends on the properties of the state dependent delay ρ and it will be defined later. The set PC_0 consists of all piecewise continuous functions $\phi : E_0 \to \mathbb{R}^n$ with finite number of points of discontinuity $\tau \in E_0$ at which $\phi(\tau) = \lim_{t\to\tau-0} \phi(t)$, endowed with the norm $||\phi||_0 = \sup_{t\in E_0}\{||\phi(t)|| < \infty : \phi \in PC_0\}$ where $||.||$ is a norm in $\mathbb{R}^n$.

Remark 2.1. The functions Φ_k are called impulsive functions and the intervals $(s_k, t_k], k = 1, 2, \ldots$ are called intervals of non-instantaneous impulses.

Remark 2.2. In the partial case $s_k = t_k$, $k = 1, 2, \ldots$ each interval of non-instantaneous impulses is reduced to a point, and the problem (2.1) is reduced to an IVP for an impulsive differential equation with points of jump t_k and impulsive condition $x(t_k + 0) = I_k(x(t_k - 0)) \equiv \Phi_k(t_k, x(t_k - 0))$.

Remark 2.3. Note NIDDE (1) is a generalization of non-instantaneous impulsive differential equations with both a constant delay and a variable time delay.

Let $J \subset \mathbb{R}^+$ be a given interval. We will use the following classes of functions

$$\begin{aligned} PC(J, \mathbb{R}^n) = \{&u : J \to \mathbb{R}^n : u \in C(J \setminus \cup_{k=1}^{\infty}\{s_k\}, \mathbb{R}^n) : \\ &u(s_k) = u(s_k - 0) = \lim_{t\uparrow s_k} u(t) < \infty, \ u(s_k + 0) = \lim_{t\downarrow s_k} u(t) < \infty, \ k : s_k \in J\}, \\ NPC^1(J, \mathbb{R}^n) = \{&u : J \to \mathbb{R}^n : u \in PC(J, \mathbb{R}^n), \ u \in C^1(J \bigcap \cup_{k=0}^{\infty}(t_k, s_{k+1}], \mathbb{R}^n) : \\ &u'(s_k) = u'(s_k - 0) = \lim_{t\uparrow s_k} u'(t) < \infty, \ k : s_k \in J\}. \end{aligned}$$

Remark 2.4. According to the above description any solution of (2.1) might have a discontinuity at any point $s_k, k = 1, 2, \ldots$.

Introduce the following conditions:

(H1) The function $f \in C([0, s_1] \bigcup \cup_{k=1}^{\infty}[t_k, s_{k+1}] \times PC_0, \mathbb{R}^n)$ and $f(t, 0) \equiv 0$.

(H2) For any $k = 1, 2, \ldots$ the functions $\Phi_k \in C([s_k, t_k] \times \mathbb{R}^n, \mathbb{R}^n)$ and $\Phi_k(t, 0) \equiv 0$.

(H3) The function $\phi \in PC_0$.

(H4) The function $\rho \in C([0,s_1]\bigcup\cup_{i=1}^{\infty}[t_i,s_{i+1}]\times PC_0,[0,\infty))$ and $t-r\le\rho(t,u)\le t$ for any $u\in PC_0$ with $E_0=[-r,0]$.

(H5) The function $\rho \in C([0,s_1]\bigcup\cup_{i=1}^{\infty}[t_i,s_{i+1}]\times PC_0,[0,\infty))$ and $0\le\rho(t,u)\le t$ for any $u\in PC_0$ with $E_0=(-\infty,0]$.

Remark 2.5. In both conditions (H4) and (H5) the inequality $\rho(t,u)\le t$ guarantees the delay in the argument. For example, if $\rho(t,u)=e^{t+u^2}-1$ then the argument will be advanced. But if $\rho(t,u)=1-e^{t+u^2}$ then the argument will be delayed.

In the condition (H4) the inequality $\rho(t,u)\ge t-r$ with $r>0$ guarantees the boundedness of the delay. For example if $\rho(t,u)=1-e^{t+u^2}$ then the argument will be delayed and unbounded, but if $\rho(t,u)=1-e^{-t+u^2}$ then the argument will be both bounded and delayed. The properties of the state delay have a huge influence on the type of the initial interval as well on the Razumikhin condition in the stability sufficient conditions.

Remark 2.6. In the case of unbounded delay in the general case the initial functions have to be defined on $(-\infty,0]$ (see, for example, the book [12]).

Remark 2.7. Note the conditions (H1), (H2) guarantee the existence of the zero solution of IVP for NIDDE (2.1) with the zero initial function $\varphi\equiv 0$.

Define sets:

$$\begin{aligned}\mathcal{K} &= \{\sigma\in C(\mathbb{R}_+,\mathbb{R}_+):\ \text{strictly increasing and } \sigma(0)=0\},\\ S_\lambda &= \{x\in\mathbb{R}^n:\ \|x\|\le A\},\quad A>0,\end{aligned}$$

where $\lambda>0$ is a given number.

We will use the class Λ of Lyapunov functions, defined and used for impulsive differential equations in [10].

Definition 2.8. Let $\alpha<\beta\le\infty$ be given numbers and $\Delta\subset\mathbb{R}^n$ be a given set. We will say that the function $V(t,x):[\alpha-r,\beta]\times\Delta\to\mathbb{R}_+$ belongs to the class $\Lambda([\alpha-r,\beta],\Delta)$ if

- The function $V(t,x)$ is continuous on $[\alpha,\beta)\setminus\{s_k\}\times\Delta$ and it is locally Lipschitz with respect to its second argument.

- For each $s_k\in(\alpha,\beta)$ and $x\in\Delta$ there exist finite limits

$$V(s_k,x)=V(s_k-0,x)=\lim_{t\uparrow s_k}V(t,x)\text{ and }V(s_k+0,x)=\lim_{t\downarrow s_k}V(t,x).$$

For any $t\in[t_k,s_{k+1}],k=0,1,2,\dots$, we define the Dini derivative of the function $V(t,x)\in\Lambda(J,\Delta)$ among the delay non-instantaneous impulsive differential equation (2.1) by

$$D_+V(t,\phi(0),\phi)=\lim_{h\to0^+}\sup\frac{1}{h}\{V(t,\phi(0))-V(t-h,\phi(0)-hf(t,\phi_{\rho(t,\phi_0)-t}))\},\tag{2.2}$$

where $\phi\in PC_0$ and $\phi_0(s)=\phi(s),\ s\in[-r,0]$.

Note for $\phi\in PC_0$ with $E_0=[-r,0]$ if the condition (H4) is satisfied then $\rho(t,\phi_0)-t=\rho(t,\phi(s))-t\in[-r,0]$ for any $t\ge0$ and $s\in[-r,0]$. Therefore, the function $\phi_{\rho(t,\phi_0)-t}=\phi(\rho(t,\phi(s))-t)$ is well defined.

For $\phi\in PC_0$ with $E_0=(-\infty,0]$ if the condition (H5) is satisfied then $\rho(t,\phi_0)-t=\rho(t,\phi(s))-t\in E_0$ for any $t\ge0$ and $s\in[-r,0]$. Therefore, the function $\phi_{\rho(t,\phi_0)-t}=\phi(\rho(t,\phi(s))-t)$ is well defined.

3 Main Results

We give a definition for various types of practical stability of the zero solution of NIDDE (2.1). In the definition below, we denote by $x(t;t_0,\phi) \in NPC^1([t_0,\infty),\mathbb{R}^n)$ any solution of the IVP for NIDDE (2.1). Note the practical stability for non-instantaneous impulsive differential equation is defined and studied following the classical concept of the idea of practical stability ([11]).

Definition 3.1. Let positive constants $\lambda, A : \lambda < A$ be given. The zero solution of the system of NIDDE (2.1) is said to be

(S1) *practically stable w.r.t. (λ,A)*, if there exists $t_0 \in [0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$ such that for any $\phi \in PC_0$ inequality $\|\phi\|_0 < \lambda$ implies $\|x(t;t_0,\phi)\| < A$ for $t \geq t_0$;

(S2) *uniformly practically stable w.r.t. (λ,A)*, if (S1) holds for all initial points $t_0 \in [0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$;

(S3) *practically quasi stable w.r.t. (λ,B,T)*, if there exists $t_0 \in [0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$ such that for any $\phi \in PC_0$ inequality $\|\phi\|_0 < \lambda$ implies $\|x(t;t_0,\phi)\| < B$ for $t \geq t_0 + T$, where the positive constant T is given;

(S4) *uniformly practically quasi stable w.r.t. (λ,B,T)*, if (S3) holds for all initial points $t_0 \in [0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$.

(S5) *strongly practically stable* w.r.t. (λ,A,B,T), if it is practically stable with respect to (λ,A) and practically quasi stable with respect to (λ,B,T), i.e., there exists an initial time $t_0 \in [0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$ such that for any $\phi \in PC_0$ the inequality $\|\phi\|_0 < \lambda$ implies $\|x(t;t_0,\phi)\| < A$ for $t \geq t_0$ and $\|x(t;t_0,\phi)\| < B$ for $t \geq t_0 + T$, where the positive constants $B,T : B < \lambda$ are given;

(S6) *uniformly strongly practically stable* w.r.t. (λ,A,B,T), if (S5) holds for all $t_0 \in [0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$;

(S7) *eventually practically stable w.r.t. (λ,A)*, if there exists $\tau = \tau(\lambda,A) > 0$ such that for any $t_0 \in [\tau,\infty)\bigcap\left([0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)\right)$ and any $\phi \in PC_0$ inequality $\|\phi\|_0 < \lambda$ implies $\|x(t;t_0,\phi)\| < A$ for $t \geq t_0$.

We study the practical stability using the following scalar comparison differential equation with non-instantaneous impulses(NIDE):

$$\begin{aligned} &u^{'} = g(t,u) \text{ for } t \in \cup_{k=0}^{\infty}(t_k,s_{k+1}],\\ &u(t) = \Psi_k(t,u(s_k-0)) \text{ for } t \in (s_k,t_k], k = 1,2,\ldots,\\ &u(t_0) = u_0, \end{aligned} \tag{3.1}$$

where $u \in \mathbb{R}, g : [0,s_1]\bigcup\cup_{k=1}^{\infty}[t_k,s_{k+1}]\times\mathbb{R} \to \mathbb{R}_+, \Psi_k : [s_k,t_k]\times\mathbb{R} \to \mathbb{R}_+$ $(k = 1,2,3,\ldots)$.

We introduce the following conditions.

(H6) The function $g \in C([0,s_1]\bigcup\cup_{k=1}^{\infty}[t_k,s_{k+1}]\times\mathbb{R},\mathbb{R}_+)$, $g(t,0) = 0$.

(H7) For all natural numbers k, the functions $\Psi_k \in C([s_k,t_k]\times\mathbb{R},\mathbb{R}_+)$ are such that $\Psi_k(t,0) = 0$ and $\Psi_k(t,u) \leq \Psi_k(t,v)$ for $u \leq v,\ t \in [s_k,t_k]$.

(H8) There exists a number $K > 0$ such that for any $k = 1,2,\ldots$ the inequality $\Psi_k(s_k,u) < K$ holds for $|u| < K$.

We will assume in the paper that the functions g and Ψ_k are such that for any initial data $(t_0,u_0)\in\mathbb{R}_+\times\mathbb{R}$ the IVP for scalar NIDE (3.1) with $u(t_0)=u_0$ has a solution $u(t;t_0,u_0)\in NPC^1([t_0,\infty),\mathbb{R})$. Also, if the conditions (H6) and (H7) are satisfied then $u(t;t_0,u_0)\geq 0$ for a positive initial value u_0.

We will consider some scalar differential equations with non-instantaneous impulses which could be used as comparison equations.

Example 1. Let $t_0\geq 0$ be an arbitrary point and without loss of generality we can assume $0\leq t_0<s_0$. Consider the following IVP for the scalar differential equation with non-instantaneous impulses

$$\begin{aligned} &u'=0 \text{ for } t\in\bigcup_{k=0}^{\infty}(t_k,s_{k+1}],\\ &u(t)=C_k u(s_k-0) \text{ for } t\in(s_k,t_k],\ k=1,2,\dots,\\ &u(t_0)=u_0, \end{aligned} \tag{3.2}$$

here $C_k>0$ are constants. The solution of (3.2) is

$$u(t;t_0,u_0)=\begin{cases} u_0 & \text{for } t\in(t_0,s_1],\\ u_0\prod_{i=1}^{k}C_i & \text{for } t\in(s_k,s_{k+1}],\ k=1,2,\dots. \end{cases} \tag{3.3}$$

If $\lim_{k\to\infty}\prod_{i=1}^{k}C_i=\infty$ then any solution of (3.2) is unbounded.

If for any natural n there exists $c_n>0$ such that $C_{n+1}=c_nC_n$ then $C_{n+1}=C_1\prod_{i=1}^{n}c_i$. If, for example, $c_n=0.5$ then $\prod_{i=1}^{n}c_i=2.^{-0.5n(1+n)}\leq 0.5$ and the zero solution of (3.2) is uniformly practically stable w.r.t. $(\lambda,\max\{\lambda,0.5C_1\})$ where $\lambda>0$.

If $C_n\leq 1,\ n=1,2,\dots$, then the zero solution of (3.2) is uniformly practically stable w.r.t. (λ,λ) where $\lambda>0$.

In our further investigations we will use the following result:

Lemma 3.2. *([8]) If the scalar function $m\in NPC^1([t_0-r,\infty),\mathbb{R})$ satisfies the inequalities*

$$\begin{aligned} &m'(t)\leq g(t,|m_t|_0) \quad \textit{for } t\in\bigcup_{k=0}^{\infty}(t_k,s_{k+1}],\\ &m(t)\leq\Psi_k(t,m(s_k-0)) \textit{ for } t\in(s_k,t_k],\ k=1,2,\dots, \end{aligned} \tag{3.4}$$

where $g\in C(\bigcup_{k=0}^{\infty}(t_k,s_{k+1}]\times\mathbb{R}_+,\mathbb{R}_+)$, $\Psi_k\in C([s_k,t_k]\times\mathbb{R}_+,\mathbb{R}_+)$ and $u(t;t_0,u_0)$ is a solution of (3.1) with $u_0\geq|m_{t_0}|_0=\sup_{s\in[-r,0]}|m(t_0+s)|$.

Then $m(t)\leq u(t;t_0,u_0)$ for $t\geq t_0$.

In the partial case $\rho(t,x)\equiv x,\ x\in\mathbb{R}$, i.e. $x_\rho(t,x_t)=x(\rho(t,x(t+s)))=x_t$, we obtain the following result for practical stability:

Theorem 3.3. *Assume the following conditions are satisfied:*

1. *The conditions (H1) - (H3), (H6) are satisfied with $n=1$ and $\rho(t,x)\equiv x,\ x\in\mathbb{R}$.*

2. *The functions $\Psi_k\in C([s_k,t_k]\times\mathbb{R},\mathbb{R}_+)$ and $\Psi_k(t,0)\equiv 0$ for $t\in[s_k,t_k],k=1,2\dots$.*

3. *For any point* $t \in [0, s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1}]$ *and any function* $\psi \in PC_0$ *with* $E_0 = [-r, 0]$ *the inequality*
$$|f(t, \psi_0)| \leq g(t, |\psi|_0)$$
holds with $\psi_0(s) = \psi(s)$ *for* $s \in [-r, 0]$.

4. *For any* $k = 1, 2, \dots$ *and any point* $x \in \mathbb{R}$*, the inequality*
$$|\Phi_k(t, x)| \leq \Psi_k(t, |x|), \quad t \in [s_k, t_k]$$
holds.

Then any practical stability property of the zero solution of (3.1) w.r.t. (λ, A) *imply the same practical stability property of the zero solution of NIDDE (2.1) w.r.t.* (λ, A).

Proof. Let the zero solution of (3.1) be practically stable w.r.t. (λ, A). Then there exists $t_0 \in [0, s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1})$ such that for any $u_0 \in \mathbb{R}: \ |u_0| < \lambda$ the inequality
$$|u(t; t_0, u_0)| < A \text{ for } t \geq t_0 \tag{3.5}$$
holds, where $u(t; t_0, u_0)$ is a solution of (3.1). Without loss of generality we can assume $t_0 \in [0, s_1]$.

Let the initial function $\phi \in PC_0$ with $E_0 = [-r, 0]$ be such that $|\phi|_0 \leq \lambda$. Consider the solution $x(t) = x(t; t_0, \phi) \in NPC^1([t_0, \infty), \mathbb{R})$ of (2.1) with $E_0 = [-r, 0]$. Then $x_t \in PC_0$ for any $t \in \cup_{k=0}^{\infty} (t_k, s_{k+1}]$. Let $u_0^* = |\phi|_0$. From the choice of the initial function ϕ we get $u_0^* \leq \lambda$. Therefore, the function $u^*(t)$ satisfies (3.5) for $t \geq t_0$ with $u_0 = u_0^*$, where $u^*(t) = u(t; t_0, u_0^*) \geq 0$ is a solution of (3.1).

Define the function $m(t) = |x(t; t_0, \phi)|$ for $t \geq t_0 - r$. Then,

- for $t \in \cup_{k=0}^{\infty} (t_k, s_{k+1}]$ applying the inequalities $2ab \leq a^2 + b^2$, $\frac{d}{dt}|x(t)| \leq |f(t, x_t)|$, $|m_t|_0 = \sup_{s \in [-r, 0]} |x_t(s)| = |x_t|_0$ and the condition 3 we get
$$m'(t) \leq |f(t, x_t)| \leq g(t, |x_t|_0) = g(t, |m_t|_0).$$

- for $t \in (s_k, t_k], \ k = 1, 2, \dots$ from the condition 4 we have
$$m(t) = |\Phi_k(t, x(s_k - 0))| \leq \Psi_k(t, |x(s_k - 0)|) = \Psi_k(t, m(s_k - 0)),$$

- for $t \in [t_0 - r, t_0]$ we get
$$|m_{t_0}|_0 = sup_{s \in [-r, 0]} |x(t_0 + s; t_0, \phi)| = sup_{s \in [-r, 0]} |\phi(t_0 + s)| = |\phi|_0 = u_0.$$

Therefore, the conditions of Lemma 3.2 are satisfied and from (3.5) we obtain $m(t) = |x(t; t_0, \phi)| \leq u^*(t) < A$ for $t \geq t_0$.

In the case the zero solution of (3.1) is uniformly practically stable/ practically quasi stable/uniformly practically quasi stable/ strongly practically stable/ eventually practically stable, the proofs is similar and we omit them.

□

We will use an extension of Razumikhin method to prove practical stability properties of the nonlinear system of non-instantaneous impulsive delay differential equations with state dependent delay (2.1). For this purpose we need the following comparison result for non-instantaneous impulsive delay differential equations:

Lemma 3.4. *(Lemma 2 [1]) Suppose:*

1. *The function $x(t) = x(t;t_0,\phi) \in NPC^1([t_0,\Theta],\Delta)$ is a solution of the NIDDE (2.1) with $E_0 = [-r,0]$, where $\Delta \subset \mathbb{R}^n$, $\Theta \in (t_p, s_{p+1}]$ is a given number, p is a natural number.*
2. *The condition (H4) is satisfied.*
3. *For all $k = 1,...,p-1$ the condition (H7) is satisfied.*
4. *The the condition (H6) is satisfied on the interval $[t_p,\Theta]\bigcup\cup_{k=0}^{p-1}[t_k,s_{k+1}]$.*
5. *The function $V \in \Lambda([t_0-r,\Theta],\Delta)$ and*
   (i) for any $t \in (t_p,\Theta]\bigcup\cup_{k=0}^{p-1}(t_k,s_{k+1}]$ such that $V(t,x(t)) \geq \sup_{s\in E_0} V(t+s,x(t+s))$ the inequality
   $$D_+V(t,x(t)) \leq g(t,V(t,x(t)))$$
   holds, where
   $$D_+V(t,x(t)) = \lim_{h\to 0^+}\sup\frac{1}{h}\{V(t,x(t))) - V(t-h,x(t)-hf(t,x_{\rho(t,x_t)}))\}; \tag{3.6}$$
   (ii) for any number $k = 1,2,...,p-1$ the inequality
   $$V(t,\Phi_k(t,x(s_k-0))) \leq \Psi_k(t,V(s_k-0,x(s_k-0))) \text{ for } t\in(s_k,t_k]$$
   holds.

If $\sup_{s\in[t_0-r,t_0]} V(s,\phi(s-t_0)) \leq u_0$, then the inequality $V(t,x(t)) \leq r(t)$ for $t \in [t_0,\Theta]$ holds, where $r(t) = r(t;t_0,u_0)$ is the maximal solution of (3.1) with u_0.

Remark 3.5. The result of Lemma 3.4 is also true on the half line, i.e. $\Theta = \infty$.

Remark 3.6. The conditions 4(i) and 4(ii) of Lemma 3.4 are satisfied only for the particular given solution $x^*(t)$ and the condition 4(i) is satisfied only at some particular points t from the studied interval.

In the case the state dependent delay satisfies the less restrictive condition (H5) we obtain the result

Lemma 3.7. *Suppose:*

1. *The conditions 1,3,4,5 of Lemma 3.4 are satisfied with $E_0 = (-\infty,0]$.*

If $\sup_{s\in(-\infty,0]} V(t_0+s,\phi(s)) \leq u_0$, then the inequality $V(t,x(t)) \leq r(t)$ for $t\in[t_0,\Theta]$ holds, where $r(t) = r(t;t_0,u_0)$ is the maximal solution of (3.1) with u_0.

In this paper we will study the connection between the practical stability properties of the system NIDDE (2.1) and the practical stability properties of the scalar NIDE (3.1) .

Theorem 3.8. *Let the following conditions be satisfied:*

1. *The conditions (H1) - (H4), (H6)-(H8) are fulfilled with* $E_0 = [-r, 0]$.
2. *There exists a function* $V(t,x) \in \Lambda([-r,\infty),\mathbb{R}^n)$ *and*
   - *(i) the inequalities*
   $$a(\|x\|) \le V(t,x) \le b(\|x\|), \quad x \in S_A, t \in [-r,\infty)$$
   hold, where $a, b \in \mathcal{K}$, $A = a^{-1}(K)$, *K is the number defined in the condition (H8);*
   - *(ii) for any function* $\psi \in PC_0 : \ \|\psi\|_0 \in S_A$ *with* $E_0 = [-r, 0]$ *and any point* $t \in [0, s_1] \bigcup \cup_{k=1}^{\infty}(t_k, s_{k+1})$ *such that* $V(t+\tau,\psi(\tau)) \le V(t,\psi(0))$ *for* $\tau \in E_0$ *the inequality*
   $$D^+V(t,\psi(0),\psi) \le g(t, V(t,\psi(0)))$$
   holds, where $D^+V(t,\psi(0),\psi)$ *is defined by (2.2);*
   - *(iii) for any* $k = 1, 2, \ldots$ *the inequality*
   $$V(t,\Phi_k(t,y)) \le \Psi_k(t, V(s_k - 0, y)), \quad t \in (s_k, t_{k+1}], \quad y \in S_A$$
   holds.
3. *The zero solution of (3.1) is practically stable (uniformly practically stable) w.r.t.* $(b(\lambda), K)$ *where the constant* λ *is given such that* $0 < \lambda < A, \ b(\lambda) < K$.

Then the zero solution of (2.1) with $E_0 = [-r, 0]$ *is practically stable (uniformly practically stable) w.r.t.* (λ, A).

Proof. Let the zero solution of (3.1) be practically stable w.r.t. $(b(\lambda), K = a(A))$. Therefore, there exists a point $t_0 \in [0, s_1) \bigcup \cup_{k=1}^{\infty}[t_k, s_{k+1})$ such that the inequality $|u_0| < b(\lambda)$ implies the inequality

$$|u(t;t_0,u_0)| < a(A) \ \text{ for } t \ge t_0 \tag{3.7}$$

holds, where $u(t;t_0,u_0)$ is a solution of (3.1) .

Choose the initial function $\phi \in PC_0 : \ \|\phi\|_0 < \lambda$ with $E_0 = [-r, 0]$ and consider the solution $x(t) = x(t;t_0,\phi)$ of system (2.1) with $E_0 = [-r, 0]$ for the initial time t_0 defined above. Let $u_0^* = \sup_{t\in[t_0-r,t_0]} V(t,\phi(t-t_0)) > 0$. From the choice of the initial function ϕ and the properties of the function $b(u)$ applying the condition $2(i)$, we get $u_0^* = \sup_{t\in[t_0-r,t_0]} V(t,\phi(t-t_0)) \le b(\|\phi\|_0) < b(\lambda)$. Therefore, the function $u^*(t)$ satisfies (3.7) for $t \ge t_0$ with $u_0 = u_0^*$, where $u^*(t) = u(t;t_0,u_0^*) \ge 0$ is a solution of (3.1).

We will prove

$$V(t,x(t)) < a(A), \quad t \ge t_0. \tag{3.8}$$

For $t = t_0$ we get $V(t_0,x(t_0)) \le sup_{t\in[t_0-r,t_0]} V(t,\phi(t-t_0)) \le b(\lambda) < a(A)$.

Assume (3.8) is not true and let $t^* = \inf\{t > t_0 : \ V(t,x(t)) \ge a(A)\}$.

Case 1. Let there exist a non-negative integer p such that $t^* \in (t_p, s_{p+1})$. Then the function $x(t)$ is continuous at t^* and $V(t,x(t)) < a(A)$ for $t \in [t_0,t^*)$ and $V(t^*,x(t^*)) = a(A)$. Therefore, $V(t^*,x(t^*)) > V(t,x(t))$ for $t \in [t_0,t^*)$.

Case 1.1. Let $p = 0$. From the condition 2(i) and the choice of the initial function it follows $a(\|x(t)\|) \le V(t, x(t) \le a(A)$, i.e. $x(t) \in S_A$ for $t \in [t_0 - r, t^*]$.

For any fixed number $t \in [t_0, t^*]$ we consider the function $\psi(s) = x(t+s)$, $t \in [t_0, t^*]$. Then $\psi_{\rho(t,\psi_0)-t} = \psi(\rho(t, \psi(s)) - t) = \psi(\rho(t, x(t+s)) - t) = \psi(\xi) = x(t+\xi) = x(\rho(t, x_t)) = x_{\rho(t,x_t)}$ where $\xi = \rho(t, x(t+s)) - t \in [-r, 0]$. Therefore, the equality (2.2) is reduced to (3.6). From the condition 2(ii) of Theorem 3.8 it follows that the condition 4(i) of Lemma 3.4 is fulfilled for $\Delta = S_A$ and $\Theta = t^*$. Therefore $V(t, x(t) \le u^*(t)$ on $[t_0, t^*]$. Thus we get $a(A) = V(t^*, x(t^*)) \le u^*(t^*) < a(A)$. The obtained contradiction proves this case is impossible.

Case 1.2 Let $p \ge 1$. From the condition 2(i) it follows $a(\|x(t)\|) \le V(t, x(t) \le a(A)$, i.e. $x(t) \in S_A$ for $t \in [t_0, t^*]$. Then similar to Case 1.1. all conditions of Lemma 3.4 are satisfied for $\Delta = S_A$ and $\Theta = t^*$ and we obtain a cntradiction.

Case 2. Let there exist a natural number p such that $t^* \in (s_p, t_p)$. From the condition 2(i) it follows $a(\|x(t^*)\|) \le V(t^*, x(t^*)) = a(A)$, i.e. $x(t^*) \in S_A$. Then from the condition 2(iii) we get $V(t^*, x(t^*)) = V(t^*, \Phi_p(t^*, x(s_p - 0))) \le \Psi_p(t^*, V(s_p - 0, x(s_p - 0)))$. From the condition (H8) using the inequality $V(s_p - 0, x(s_p - 0)) < a(A) = K$, we get the contradiction $a(A) \le \Psi_p(t^*, V(s_p - 0, x(s_p - 0))) < K = a(A)$. The obtained contradiction proves this case is impossible.

Case 3. Let there exist a natural number p such that $t^* = s_p$. Then the following two cases are possible.

Case 3.1. Let $V(t, x(t)) < a(A)$ for $t \in [t_0, s_p)$, $V(s + p - 0, x(s_p - 0)) = a(A)$.

Thus, the inclusion $x(t) \in S_A$ for $t \in [t_0, s_p]$ is valid and as in the case 1 for $\Delta = S_A$ and $\Theta = s_p$, we get a contradiction.

Case 3.2. Let $V(t, x(t)) < a(A)$ for $t \in [t_0, s_p]$ and $V(s_p + 0, x(s_p + 0)) \ge a(A)$.

Thus, from the condition 2(i) we get $V(s_p - 0, x(s_p - 0)) < a(A) = K$. From the condition (H8) we have $\Psi_p(s_p + 0, V(s_p - 0, x(s_p - 0))) < K$ which leads to the contradiction $a(A) \le V(s_p + 0, x(s_p + 0)) = V(s_p + 0, \Phi_p(s_p + 0, x(s_p - 0))) \le \Psi_p(s_p + 0, V(s_p - 0, x(s_p - 0))) < K = a(A)$.

The proof of uniformly practical stability is analogous and we omit it.

□

By Theorem 3.8 and Example 1 we obtain the following direct sufficient conditions for uniform practical stability:

Corollary 3.9. *Let the following conditions be satisfied:*

1. *The conditions (H1)-(H4) are fulfilled.*
2. *There exist a function $V(t, x) \in \Lambda([-r, \infty), \mathbb{R}^n)$ and*

   (i) the inequalities

$$a(\|x\|) \le V(t, x) \le b(\|x\|), \quad x \in S_\lambda, t \in [-r, \infty)$$

   hold, where $a, b \in \mathcal{K}$, $\lambda > 0$ is a given number;

(ii) *for any function* $\psi \in PC_0 :\ ||\psi||_0 \in S_\lambda$ *with* $E_0 = [-r,0]$ *and any point* $t \in [0,s_1]\bigcup_{k=1}^{\infty}(t_k,s_{k+1})$ *such that* $V(t+\tau,\psi(\tau)) \le V(t,\psi(0))$ *for* $\tau \in [-r,0]$ *the inequality* $D^+V(t,\psi(0),\psi) \le 0$ *holds;*

(iii) *for any* $k = 1,2,\dots$ *the inequality*

$$V(t,\Phi_k(t,y)) \le C_k V(s_k-0,y), \ \ t \in (s_k,t_{k+1}], \ \ y \in S_\lambda$$

holds with $C_k < 1$.

Then the zero solution of (2.1) with $E_0 = [-r,0]$ *is uniformly practically stable w.r.t.* (λ,λ).

In the case when the state dependent delay satisfies less restrictive condition (H4) we obtain the following sufficient condition with more restriction condition about the Lyapunov function:

Theorem 3.10. *Let the conditions (H1) - (H3), (H5)-(H8) be fulfilled with* $E_0 = (-\infty,0]$, *there exists a function* $V(t,x) \in \Lambda(\mathbb{R},\mathbb{R}^n)$ *and the conditions 2(i), 2(ii), 3 of Theorem 3.8 be satisfied.*

Then the zero solution of (2.1) with $E_0 = (-\infty,0]$ *is practically stable (uniformly practically stable) w.r.t.* (λ,A).

In the case when the condition 2 for the Lyapunov function $V(t,x)$ is satisfied globally, we obtain the following sufficient conditions:

Theorem 3.11. *Let the following conditions be satisfied:*

1. *The condition (H1)- (H4), (H6)-(H8) are fulfilled with* $E_0 = [-r,0]$.
2. *There exists a function* $V(t,x) \in \Lambda([-r,\infty),\mathbb{R}^n)$ *such that*

(i) *the inequalities*

$$a(||x||) \le V(t,x) \le b(||x||), \ \ x \in \mathbb{R}^n, t \in [-r,\infty)$$

hold, where $a,b \in \mathcal{K}$;

(ii) *for any function* $\psi \in PC_0$*with* $E_0 = [-r,0]$ *and any point* $t \in [0,s_1]\bigcup_{k=1}^{\infty}(t_k,s_{k+1})$ *such that* $V(t+\tau,\psi(\tau)) \le V(t,\psi(0))$ *for* $\tau \in E_0$ *the inequality*

$$D^+V(t,\psi(0),\psi) \le g(t,V(t,\psi(0)))$$

holds;

(iii) *for any* $k = 1,2,\dots$ *the inequality*

$$V(t,\Phi_k(t,y)) \le \Psi_k(t,V(s_k-0,y)), \ \ t \in (s_k,t_{k+1}], \ \ y \in \mathbb{R}^n$$

holds.

3. *The zero solution of (3.1) is strongly practically quasi stable (uniformly strongly practically quasi stable) w.r.t.* $(b(\lambda), K, a(B), T)$*, where the positive constants* $\lambda, T: 0 < \lambda < A,\ b(\lambda) \le K$ *are given and the constant* K *is defined in the condition (H8).*

Then the zero solution of (2.1) with $E_0 = [-r, 0]$ *is strongly practically quasi stable (uniformly strongly practically quasi stable) w.r.t.* (λ, A, B, T).

Proof. From the condition 3 the zero solution of (3.1) is practically stable w.r.t. $(b(\lambda), K = a(A))$ and there exist a point $t_0 \in [0, s_1] \cup_{k=1}^{\infty} [t_k, s_{k+1})$ such that the inequality $|u_0| < a(\lambda)$ implies the inequality

$$|u(t; t_0, u_0)| < a(A) \text{ for } t \ge t_0 \tag{3.9}$$

holds, where $u(t; t_0, u_0)$ is a solution of (3.1).

Also,

$$|u(t; t_0, u_0)| < a(B) \text{ for } t \ge t_0 + T \tag{3.10}$$

Note that without loss of generality, we could assume $t_0 + T \ne s_k,\ k = 1, 2, \dots$.

By Theorem 3.8 the zero solution of (2.1) is practically stable w.r.t. (λ, A). So, we need to prove the practical quasi stability of the zero solution of NIDDE (2.1).

Choose the initial function $\phi \in PC_0 : \|\phi\|_0 < \lambda$ with $E_0 = [-r, 0]$ and consider the solution $x(t) = x(t; t_0, \phi)$ of system (2.1) with $E_0 = [-r, 0]$ for the initial time t_0 defined above. Let $u_0^* = \sup_{t \in [t_0 - r, t_0]} V(t, \phi(t - t_0))$. From the choice of the initial function ϕ and the properties of the function $a(u)$ applying the condition 2(i), we get $u_0^* = \sup_{t \in [t_0 - r, t_0]} V(t, \phi(t - t_0)) \le a(\|\phi\|_0) < a(\lambda)$. Therefore, the function $u^*(t)$ satisfies (3.10) for $t \ge t_0 + T$ with $u_0 = u_0^*$, where $u^*(t) = u(t; t_0, u_0^*)$ is a solution of (3.1).

According to the condition 2(ii), the condition 4(i) of Lemma 3.4 is satisfied for the solution $x(t)$, $\Theta = \infty$. From Lemma 3.4 and Remark 3.5 it follows that the inequality $V(t, x(t)) \le u^*(t)$ for $t \ge t_0$ holds. From the condition 2(i) and inequality (3.10), we get $a(\|x(t)\|) \le V(t, x(t)) \le u^*(t) < a(B)$ for $t \ge t_0 + T$, i.e. $\|x(t)\| < B$ for $t \ge t_0 + T$.

The proof of uniform strong practical stability is similar and we omit it.

□

Theorem 3.12. *Let the following conditions be satisfied:*

1. *The conditions (H1) - (H4), (H6)-(H8) are fulfilled.*

2. *The condition 2 of Theorem 3.11 is fulfilled.*

3. *The zero solution of (3.1) is eventually practically stable with respect to* $(b(\lambda), K)$ *where the positive constant* λ *is given such that* $0 < \lambda < a^{-1}(K),\ b(\lambda) \le K$.

Then the zero solution of (2.1) with $E_0 = [-r, 0]$ *is eventually practically stable w.r.t.* (λ, A).

The proof is similar to the one of Theorem 3.11 and we omit it.

In the case when the state dependent delay satisfies less restrictive condition (H4) we obtain the following sufficient condition with more restriction condition about the Lyapunov function:

Theorem 3.13. *Let the conditions (H1)- (H3), (H5)-(H8) be satisfied with* $E_0 = (-\infty, 0]$ *and there exist a function* $V(t,x) \in \Lambda(\mathbb{R}, \mathbb{R}^n)$ *such that the conditions 2(i), 2(ii) and 3 of Theorem 3.11 be fulfilled.*

Then the zero solution of (2.1) with $E_0 = (-\infty, 0]$ *is strongly practically quasi stable (uniformly strongly practically quasi stable) w.r.t.* (λ, A, B, T).

Theorem 3.14. *Let the conditions (H1) - (H3), (H5)-(H8) are fulfilled with* $E_0 = (-\infty, 0]$ *and there exist a function* $V(t,x) \in \Lambda(\mathbb{R}, \mathbb{R}^n)$ *such that the conditions 2(i), 2(ii) of Theorem 3.11 and condition 3 of Theorem 3.12 be fulfilled. Then the zero solution of (2.1) with* $E_0 = (-\infty, 0]$ *is eventually practically stable w.r.t.* (λ, A).

4 Applications

We will consider several partial cases of the studied type of delay and we will apply some of the obtained results to illustrate the practical stability properties.

EXAMPLE 2 (*constant delay*). Consider the IVP for NIDDE:

$$\begin{aligned}
&x'(t) = y(t)\frac{t}{1+t}\Big(x(t) + y^2(t)\Big) + e^{-t}y(t-1),\\
&y'(t) = -0.5x(t)\frac{t}{1+t}\Big(x^2(t) + y^2(t)\Big) + e^{-t}x(t-1) \text{ for } t \in (0,\infty) \cap \cup_{k=0}^{\infty}(2k, 2k+1],\\
&x(t) = a\sin(t)x(2k+1-0),\\
&y(t) = b\sin(t)y(2k+1-0) \text{ for } t \in (2k+1, 2k+2], k = 0,1,2,\dots,\\
&x(s) = \phi_1(s), \quad y(s) = \phi_2(s) \quad s \in [-1,0],
\end{aligned} \tag{4.1}$$

where $x, y \in \mathbb{R}$, $a, b \in (-1,1)$ are given constants.

In this case $s_k = 2k+1$, $t_k = 2k$ for $k = 0,1,2,\dots$, $r = 1$ and $\rho(t,x,y) \equiv t-1$ for all $x, y \in \mathbb{R}$, i.e., the delay is equal to 1.

Let $V(t,x,y) = 1.5(x^2 + 2y^2)$. Then $a(s) = s^2$, $b(s) = 2s^2$.

Let $t \in \cup_{k=0}^{\infty}(2k, 2k+1]$ and $\psi = (\psi_1, \psi_2) \in PC_0$ be such that $V(t, \psi_1(0), \psi_2(0)) > V(t+s, \psi_1(s), \psi_2(s))$ for $s \in [-1,0)$, i.e.,

$$\psi_1^2(0) + 2\psi_2^2(0) > \psi_1^2(s) + 2\psi_2^2(s), \quad s \in [-1,0).$$

In this case $\psi_{1_{\rho(t,(\psi_1)_0,(\psi_2)_0)-t}}(s) = \psi_1(-1)$ and $\psi_{2_{\rho(t,(\psi_1)_0,(\psi_2)_0)-t}}(s) = \psi_2(-1)$ for $s \in [-1,0]$.

Then

$$\begin{aligned}
&D^+V(t, \psi_1(0), \psi_2(0), \psi_1, \psi_2)\\
&= 3e^{-t}\Big(\psi_1(0)(\psi_1)_{\rho(t,(\psi_1)_0,(\psi_2)_0)-t}(s) + 2\psi_2(0)(\psi_2)_{\rho(t,(\psi_1)_0,(\psi_2)_0)-t}(s)\Big)\\
&\leq 1.5e^{-t}\Big(\psi_1^2(0) + (\psi_1(-1))^2 + 2\psi_2^2(0) + 2(\psi_2(-1))^2\Big)\\
&\leq 3e^{-t}\Big(\psi_1^2(0) + 2\psi_2^2(0)\Big) = 2e^{-t}V(t, \psi_1(0), \psi_2(0)).
\end{aligned} \tag{4.2}$$

For any $t \in (2k+1, 2k+2]$, $k = 0,1,2,\dots$, $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
&V(t, a\sin(t)x, b\sin(t)y) \leq 1.5\sin^2(t)\Big(a^2x^2 + 2b^2y^2\Big) \leq \sin^2(t)V(2k+1-0, x, y)\\
&= \Psi_k(t, V(2k+1-0, x, y))
\end{aligned} \tag{4.3}$$

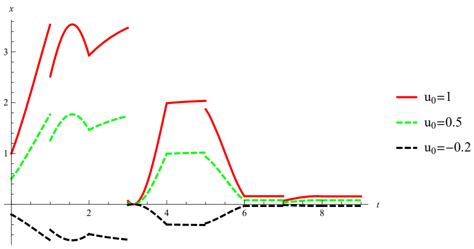

Figure 1. Graphs of the solutions of (4.4) for $t_0 = 0$ and different initial values.

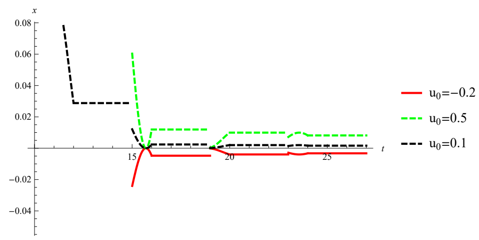

Figure 2. Graphs of the solutions of (4.4) for $t_0 = 10$ and different initial values.

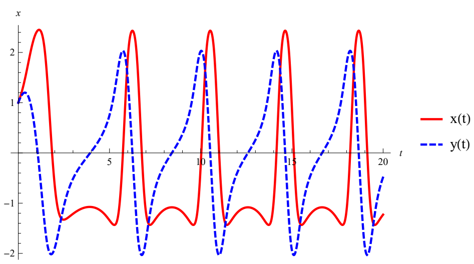

Figure 3. Graphs of the solutions of (4.1) with initial values $\phi_1(s) = \phi_2(s) = 1, s \in [-1,0]$.

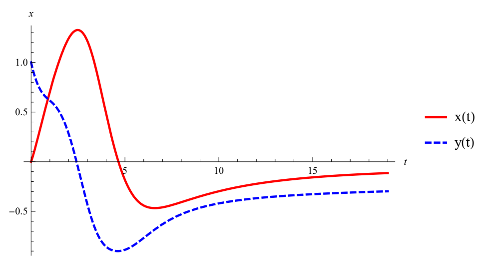

Figure 4. Graphs of the solutions of (4.1) with initial values $\phi_1(s) = \sin(s), \phi_2(s) = \cos(s), s \in [-1,0]$.

with $\Psi_k(t,u) = \sin^2(t)u$. Consider the IVP for the scalar differential equation with non-instantaneous impulses

$$\begin{aligned} &u' = 2e^{-t}u \; for \; t \in (t_0,\infty) \cap \cup_{k=0}^{\infty}(4k,4k+3], \\ &u(t) = \sin^2(t)u(4k+3-0) \; for \; t \in (4k+3,4k+4+2], k = 0,1,2,\dots, \\ &u(t_0) = u_0, \end{aligned} \tag{4.4}$$

Let there exist a nonnegative integer $p: \; t_0 \in (4p,4p+3]$. It has a solution

$$x(t) = \begin{cases} u_0 e^{2e^{-t_0}-2e^{-t}} & \text{if } t \in (t_0,4p+3], \\ \sin(t)u(4k+3-0) & \text{if } t \in (4k+3,4k+4], \; k = p,p+1,\dots, \\ \sin(4k+4)u(4k+3-0)e^{2e^{-4k+4}-2e^{-t}} & \text{if } t \in (4k+4,4k+7], \; k = p,p+1,\dots. \end{cases}$$

The solution of the scalar non-instantaneous equation (4.4) is practically stable (see the graphs on Figure 1 and Figure 2). According to Theorem 3.8, the solution of the system (4.1) is also practically stable see the graphs on Figure 3 and Figure 4).

EXAMPLE 3 (*time variable delay*). Consider the initial value problem (IVP) for a nonlinear system of non-instantaneous impulsive differential equations with state dependent delay (NIDDE)

$$\begin{aligned} & x'(t) = y(t)\frac{t}{1+t}\Big(x(t)+y^2(t)\Big)+e^{-t}y(\frac{t^2}{t+1}),\\ & y'(t) = -0.5x(t)\frac{t}{1+t}\Big(x^2(t)+y^2(t)\Big)+e^{-t}x(\frac{t^2}{t+1}) \text{ for } t\in(t_0,\infty)\cap\cup_{k=0}^{\infty}(2k,2k+1],\\ & x(t) = a\sin(t)x(2k+1-0),\\ & y(t) = b\sin(t)y(2k+1-0) \text{ for } t\in(2k+1,2k+2], k=0,1,2,\dots,\\ & x(s)=\phi_1(s),\quad y(s)=\phi_2(s)\quad s\in[-1,0], \end{aligned} \tag{4.5}$$

where $x,y\in\mathbb{R}$, $a,b\in(-1,1)$ are given constants.

In this case $s_k = 2k+1$, $t_k = 2k$ for $k=0,1,2,\dots$, $r=1$ and $\rho(t,x,y)\equiv\frac{t^2}{t+1}$ for any $t\geq 0$. The condition (H4) is satisfied since $t-1\leq\frac{t^2}{t+1}\leq t$ for all $t\geq 0$. Also, $\frac{t^2}{t+1}=t-\frac{t}{t+1}$, i.e., the delay of the argument is given by $\frac{t}{t+1}$ which is variable in time.

Let $V(t,x,y)=1.5(x^2+2y^2)$.

Let $t\in\cup_{k=0}^{\infty}(2k,2k+1]$ and $\psi=(\psi_1,\psi_2)\in PC_0$ be such that $V(t,\psi_1(0),\psi_2(0))>V(t+s,\psi_1(s),\psi_2(s))$ for $s\in[-1,0)$, i.e.

$$\psi_1^2(0)+2\psi_2^2(0)>\psi_1^2(s)+2\psi_2^2(s),\quad s\in[-1,0).$$

In this case $\psi_{1_{\rho(t,(\psi_1)_0,(\psi_2)_0)-t}}(s)=\psi_1(\frac{-t}{t+1})$ and $\psi_{2_{\rho(t,(\psi_1)_0,(\psi_2)_0)}}(s)=\psi_2(\frac{-t}{t+1})$. Note for all $t\geq 0$ the inequalities $-1\leq\frac{-t}{t+1}\leq 0$ hold, i.e., function ψ is well defined and $\psi_1^2(0)+2\psi_2^2(0)>2(\psi_{1_{\rho(t,(\psi_1)_0,(\psi_2)_0)-t}}(s))^2+2(\psi_{2_{\rho(t,(\psi_1)_0,(\psi_2)_0)-t}}(s))^2$.

Then similar to Example 2 and Eq. (4.2) and (4.3) we prove the validity of the conditions 2(ii) and 2(iii) of Theorem 3.8.

Applying the practical stability of the scalar comparison NIDDE (4.4) and to Theorem 3.8, we prove the practical stability of the solution of the system (4.5).

EXAMPLE 4 (*state dependent delay*). Consider the initial value problem (IVP) for a nonlinear system of non-instantaneous impulsive differential equations with state dependent delay (NIDDE)

$$\begin{aligned} & x'(t) = y(t)\frac{t}{1+t}\Big(x(t)+y^2(t)\Big)+e^{-t}y_{\rho(t,x(t),y(t))},\\ & y'(t) = -0.5x(t)\frac{t}{1+t}\Big(x^2(t)+y^2(t)\Big)+e^{-t}x_{\rho(t,x(t),y(t))} \text{ for } t\in(t_0,\infty)\cap\cup_{k=0}^{\infty}(2k,2k+1],\\ & x(t) = a\sin(t)x(2k+1-0),\quad y(t)=b\sin(t)\\ & y(2k+1-0) \text{ for } t\in(2k+1,2k+2], k=0,1,2,\dots,\\ & x(s)=\phi_1(s),\quad y(s)=\phi_2(s)\quad t\in[-r,0], \end{aligned} \tag{4.6}$$

where $x,y\in\mathbb{R}$, $r>0$ is a small constant, $a,b\in(-1,1)$ are given constants, and $\rho(t,x,y)=t-0.5(\sin^2(x)+\cos^2(y))$. In this case the delay of the argument is given by $0.5(\sin^2(x_t)+\cos^2(y_t))$ and it depends on the state. Then, $x_{\rho(t,x_t,y_t)}(s)=x(t-0.5(\sin^2(x(t+s))+\cos^2(y(t+s))))$ and $y_{\rho(t,x_t,y_t)}(s)=y(t-0.5(\sin^2(x(t+s))+\cos^2(y(t+s))))$ for $s\in[-1,0]$.

Let $V(t,x,y)=1.5(x^2+2y^2)$.

Let $t\in\cup_{k=0}^{\infty}(2k,2k+1]$ and $\psi=(\psi_1,\psi_2)\in PC_0$ be such that $V(t,\psi_1(0),\psi_2(0))>V(t+s,\psi_1(s),\psi_2(s))$ for $s\in[-r,0)$ or

$$\psi_1^2(0)+2\psi_2^2(0)>\psi_1^2(s)+2\psi_2^2(s),\quad s\in[-r,0).$$

In this case

$$\psi_{1_{\rho(t,(\psi_1)_0,(\psi_2)_0)-t}}(s) = \psi_1(-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s))),\ s \in [-1,0],$$

and

$$\psi_{2_{\rho(t,(\psi_1)_0,(\psi_2)_0)-t}}(s) = \psi_2(-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s))),\ s \in [-1,0],$$

The argument $-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s)) \in [-1,0]$ and therefore,

$$\begin{aligned}&(\psi_1(-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s))))^2 + 2(\psi_2(-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s))))^2 \\ &\le \psi_1^2(0) + 2\psi_2^2(0),\ s \in [-1,0].\end{aligned} \tag{4.7}$$

Then similarly to Example 2 by inequality (4.7) we get $D^+V(t,\psi_1(0),\psi_2(0),\psi_1,\psi_2) \le 2e^{-t}V(t,\psi_1(0),\psi_2(0))$ for any $t \in (2k,2k+1]$ and $V(t,a\sin(t)x,b\sin(t)y) \le \Psi_k(t,V(2k+1-0,x,y))$ for any $t \in (2k+1,2k+2]$ with $\Psi_k(t,u) = \sin^2(t)u$.

The zero solution of the comparison scalar NIDE (4.4) is practically stable (see the graphs on Figure 1 and Figure 2). According to Theorem 3.8 the solution of the system (4.6) is also practically stable.

Acknowledgments

M.F. is supported in part by the Slovak Research and Development Agency under the contract No. APVV-18-0308 and by the Slovak Grant Agency VEGA-SAV No. 2/0153/16 and No. 1/0078/17. S.H and K.I are partially supported by Fund NPD No. MU19FMI009.

References

[1] R. Agarwal, S. Hristova, D. O'Regan, Razumikhin method to delay differential equations with non-instantaneous impulses, *Diff. Eq. Appl.* **11**, 1 (2019), pp 143-165.

[2] R. Agarwal, D. O'Regan, S. Hristova, Stability by Lyapunov like functions of nonlinear differential equations with non-instantaneous impulses, *Appl. Math. Comput.* (2015). doi:10.1007 s12190-015-0961-z

[3] R. Agarwal, D. O'Regan, S. Hristova, Ulam type stability results for non- instantaneous impulsive differential equations with finite state dependent delay , *Dynamic Syst. Appl.* **28**, 1 (2019), pp 47-61.

[4] M. Fečkan, J.R. Wang, Y. Zhou, Periodic solutions for nonlinear evolution equations with non-instantaneous impulses, *Nonauton. Dyn. Syst.* **1** (2014), pp 93–101.

[5] C. Foley, M. C. Mackey., Dynamic hematological disease: a review, *J. Math. Biol.* **58** (2009), pp 285–322.

[6] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, Functional differential equations with state-dependent delays: theory and applications, in: A. Canada, P. Drabek, A. Fonda (Eds.), *Handbook of Differential Equations: Ordinary Differential Equations*, Vol. 3, Elsevier/North Holland, 2006, pp 435–545.

[7] E. Hernandez, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.* **141** (2013), pp 1641–1649.

[8] S. Hristova, K. Ivanova, Lipshitz stability of delay differential equations with non-instantaneous impulses, *Dyn. Syst, Appl.* **28**, 1 (2019), pp 167–181.

[9] T. Insperger, G. Stepan, J. Turi, State-dependent delay in regenerative turning processes,*Nonlinear Dyn.*, **47** (2007) pp 275–283.

[10] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[11] V. Lakshmikantham, S. Leela, A.A. Martynyuk, *Practical Stability of Nonlinear Systems*, World Scientific, Singapore, 1990.

[12] V. Lakshmikantham, Lizhi Wen, Binggen Zhang, *Theory of Differential Equations with Unbounded Delay*, Springer, 1994.

[13] J. La Salle, S. Lefschetz, *Stability by Liapunovs Direct Method*, Academic Press, Inc., New York, 1961.

[14] M. Mackey, Commodity price fluctuations: Price dependent delays and nonlinearities as explanatory factors, *J. Economic Th.* **48** (2) (1989), pp 497– 509.

[15] D.N. Pandey, S. Das, N. Sukavanam, Existence of solutions for a second order neutral differential equation with state dependent delay and not instantaneous impulses, *Intern. J. Nonlinear Sci.* **18**, 2 (2014), pp 145–155.

[16] M. Pierri, D. O'Regan, V. Rolnik, Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses, *Appl. Math. Comput.* **219** (2013), pp 6743–6749.

[17] A. Sood, S. K. Srivastava, On stability of differential systems with noninstantaneous impulses, *Math. Probl. Eng.* **2015**, 2015, Article ID 691687, 5 pages.

[18] H.-O. Walther, On a model for soft landing with state dependent delay, *J. Dyn. Diff. Eqns.***19** (2003), pp 593–622.