Communications in **M**athematical **A**nalysis
Volume 22, Number 1, pp. 76–89 (2019) www.math-res-pub.org/cma
ISSN 1938-9787

Some New Stability, Boundedness, and Square Integrability Conditions for Third-Order Neutral Delay Differential Equations

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(Communicated by Michal Fečkan)

Abstract

In this paper, the authors establish some new sufficient conditions under which all solutions of a third order nonlinear neutral delay differential equation are stable, bounded, and square integrable. An example is also given to illustrate the results.

AMS Subject Classification: 34K12, 34K20, 34K40.

Keywords: Third order, neutral equation, stability, boundedness, square integrability.

1 Introduction

In this paper we consider third order neutral delay differential equations of the form

$$(q(t)(x''(t)+\beta(t)x''(t-r)))' + g(x(t),x'(t))x''(t) + f(x(t-r),x'(t-r)) + h(x(t-r)) = e(t), \tag{1.1}$$

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for all $t \geq t_1 \geq t_0 + r$, where $r > 0$ and the functions $f(x(t-r), x'(t-r))$, $g(x(t), x'(t))$, $h(x(t-r))$, and $q(t)$ are continuous on their respective domains, $q(t) > 0$, and $h(0) = 0$. In addition, it is also assumed that the derivatives $\beta'(t)$, $g_x(x,y) = \frac{\partial}{\partial x} g(x,y))$, $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$, $f_y(x,y) = \frac{\partial}{\partial y} f(x,y)$, and $h'(x)$ exist and are continuous.

Third order nonlinear equations in the form of (1.1) but with $q(t) \equiv 1$, $\beta(t) \equiv 0$, and $r = 0$, have received a lot of attention over the years, and much of what was known prior to the 1970s can be found in the monograph by Reissig, Sansone, and Conti [15]. For example, the asymptotic stability of (1.1) with $e \equiv 0$, $\beta \equiv 0$, and $r = 0$ has been previously discussed by several authors (see, for example, [13, 14, 12, 22]). Results on nonlinear third order equations with a delay can be found, for example, in [1, 8, 16, 17, 18, 19, 20] where boundedness and stability properties of solutions are examined. Third order neutral equations are discussed in [3] for the case $g \equiv 0 \equiv f$; also see [25]. The oscillation of solutions of third order nonlinear equations has also proved to be a popular research topic as can be seen, for example, from the results in [4, 5, 6, 7, 11, 21, 23, 25]. Physical applications of delay and neutral delay differential equations can be found in [2, 9, 10]. While it is clear that there are many possible special cases of equation (1.1), the types of results in this paper have not been previously obtained for an equation with such generality.

For convenience, we let

$$Z(t) = x''(t) + \beta(t)x''(t-r).$$

By a *solution* of (1.1) we mean a nontrivial function $x \in C^2([t_x, \infty), \mathbb{R})$ such that $q(t)Z(t) \in C^1([t_x, \infty), \mathbb{R})$ and which satisfies equation (1.1) on $[t_x, \infty)$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[t_x, \infty)$. Moreover, we tacitly assume that (1.1) possesses such solutions.

This paper is organized as follows. In Section 2, we give stability results for (1.1) with $e \equiv 0$. In Section 3 the boundedness of solutions is discussed. Finally, in Section 4, sufficient conditions for the square integrability of solutions and their first and second derivatives are given.

2 Stability

We will assume that there are positive constants a, b_0, b_1, q_0, q_1, q_2, μ, ρ, α, β_1, δ_0, δ_1, m, L, and K such that the following conditions on the functions in equation (1.1) are satisfied:

(i) $a + \mu \leq g(x,y) \leq a + \rho$, and $y g_x(x,y) \leq 0$;

(ii) $b_1 \geq \frac{f(x,y)}{y} \geq b_0$, $-K \leq f_x(x,y) \leq 0$, and $\left| f_y(x,y) \right| \leq L$;

(iii) $h'(x) \leq \delta_0$ for all x and $\frac{h(x)}{x} \geq \delta_1$ for $x \neq 0$;

(iv) $|\beta'(t)| \leq \alpha$, $0 \leq \beta(t) \leq \beta_1$, $q_0 \leq q(t) \leq q_1$, and $|q'(t)| \leq q_2$ for all $t \geq t_0$;

(v) $\int_0^\infty (|q'(s)| + |\beta'(s)|)\, ds < m < \infty$.

For ease of exposition, throughout this paper we will adopt the notation that

$$A = -ab_0 + q_1\delta_0 + 2 + \frac{1}{2}q_1\beta_1(\delta_0 + b_1), \; B = -\mu q_0 + \beta_1(\rho q_1 + 1) + 1, \text{ and } M = \delta_0 + K + L.$$

We begin with a stability result for equation (1.1) with $e \equiv 0$.

Theorem 2.1. *In addition to conditions (i)–(v), assume that there exist positive constants c_1, c_2, and ε such that*

(vi) $A + \frac{1}{2}[\beta_1(q_1\delta_0 + 2 + 2\beta_1 + 2\alpha) + \delta_0(q_1\alpha + \beta_1 q_2) + 2\alpha + 4\alpha\beta_1] = -c_1$,

(vii) $B + \frac{\beta_1}{2}[2 + 2\beta_1 + 2q_1\rho + q_1 b_1] + \varepsilon = -c_2$,

(viii) $a - \frac{q_1^2\delta_0}{2} > 0$.

Then zero solution of (1.1) is uniformly asymptotically stable provided

$$r < \min\left\{\frac{2c_1}{Ma + 2\mu_1}, \frac{2c_2}{Mq_1 + 2\mu_2}, \frac{2\varepsilon}{M\beta_1 q_1}\right\}$$

where

$$\mu_1 = \frac{1}{2}(q_1 + a + \beta_1 q_1)(\delta_0 + K) \quad \text{and} \quad \mu_2 = \frac{L}{2}(q_1 + a + \beta_1 q_1). \tag{2.1}$$

Proof. We will write equation (1.1) as the equivalent differential system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ [q(t)(z(t) + \beta(t)z(t-r))]' = -g(x(t), y(t))z(t) \\ \qquad\qquad -f(x(t), y(t)) - h(x(t)) + \Delta_1 + \Delta_2 + \Delta_3, \end{cases} \tag{2.2}$$

where

$$\Delta_1(t) = \int_{t-r}^{t} f_x(x(s), y(s))y(s)ds, \tag{2.3}$$

$$\Delta_2(t) = \int_{t-r}^{t} f_y(x(s), y(s))z(s)ds, \tag{2.4}$$

$$\Delta_3(t) = \int_{t-r}^{t} h'(x(s))y(s)ds. \tag{2.5}$$

By virtue of definition (2.2), we have

$$\begin{aligned} x'(t) + \beta(t)x'(t-r) &= y(t) + \beta(t)y(t-r) = Y(t), \\ x''(t) + \beta(t)x''(t-r) &= z(t) + \beta(t)z(t-r) = Z(t), \\ Y'(t) &= Z(t) + \beta'(t)y(t-r). \end{aligned} \tag{2.6}$$

The proof of this theorem depends on properties of the continuously differentiable function $W(t, x_t, y_t, z_t) = W$ defined by

$$W = Ve^{-\frac{1}{\theta}\int_0^t \omega(s)ds}, \tag{2.7}$$

where

$$\omega(t) = \left|\beta'(t)\right| + \left|q'(t)\right| \tag{2.8}$$

and

$$\begin{aligned} V = V(t, x_t, y_t, z_t) = a\int_0^x h(u)du + q(t)h(x)Y + Y^2 + \frac{1}{2}(ay + q(t)Z)^2 \\ + a\int_0^y [g(x,\xi) - a]\xi d\xi + q(t)\int_0^y f(x,\xi)d\xi + \lambda_1 \int_{t-r}^t y^2(s)ds \\ + \lambda_2 \int_{t-r}^t z^2(s)ds + \hat{\mu_1}\int_{-r}^0\int_{t+s}^t y^2(\xi)d\xi ds + \hat{\mu_2}\int_{-r}^0\int_{t+s}^t z^2(\xi)d\xi ds. \end{aligned} \tag{2.9}$$

Here, λ_1, λ_2, $\hat{\mu_1}$, $\hat{\mu_2}$, and θ are positive constants that will be determined later in the proof.

Noting that $h(0) = 0$, it is easy to check that

$$2\int_0^x h'(u)h(u)du = h^2(x),$$

and since $\frac{1}{2}(q(t)Z + ay)^2 \geq 0$, this and (iii) imply that

$$\begin{aligned} a\int_0^x h(u)du + q(t)\, h(x)Y \quad + \quad & Y^2 + \frac{1}{2}(q(t)Z + ay)^2 \\ \geq \quad & a\int_0^x h(u)du + (Y + \frac{q(t)}{2}h(x))^2 - \frac{q^2(t)}{4}h^2(x) \\ \geq \quad & a\int_0^x h(u)du - \frac{q^2(t)}{2}\int_0^x h'(u)h(u)du \\ \geq \quad & \int_0^x (a - \frac{q_1^2\delta_0}{2})h(u)du \\ \geq \quad & (a - \frac{q_1^2\delta_0}{2})\frac{\delta_1}{2}x^2. \end{aligned}$$

From (i)–(ii), it follows that

$$a\int_0^y [g(x,\xi) - a]\xi d\xi \geq \frac{\mu a}{2}y^2 \quad \text{and} \quad q(t)\int_0^y f(x,\xi)d\xi \geq \frac{q_0 b_0}{2}y^2.$$

We see that $V = V(t, x_t, y_t, z_t) \geq 0$ and $V = V(t, x_t, y_t, z_t) = 0$ if and only if $x = y = z = 0$, so the functional V is positive definite. Thus, there exists a sufficiently small positive constant K_0 such that

$$V \geq K_0(x^2(t) + y^2(t) + Z^2(t)). \tag{2.10}$$

By a straightforward calculation, the derivative of V along the trajectories of system (2.2) is

$$\begin{aligned} V'_{(2.2)} = U_1 + U_2 + U_3 + U_4 - \hat{\mu_1}\int_{t-r}^t y^2(\theta)d\theta - \hat{\mu_2}\int_{t-r}^t z^2(\theta)d\theta \\ + [2\beta(t)\beta'(t) - \lambda_1]y^2(t-r) - \lambda_2 z^2(t-r) \\ + ay(t)\int_0^y g_x(x,\xi)\xi d\xi + y(t)q(t)\int_0^y f_x(x,\xi)d\xi, \end{aligned}$$

where

$$\begin{aligned} U_1 &= q(t)h'(x)y^2(t) - af(x(t),y(t))y(t) + \lambda_1 y^2(t) + \hat{\mu_1} r y^2(t) + q'(t)\int_0^y f(x,\xi)d\xi \\ &\quad + (q(t)(a - g(x(t),y(t)) + \lambda_2 + \hat{\mu_2} r) z^2(t), \end{aligned}$$

$$\begin{aligned} U_2 &= 2y(t)z(t) + [q(t)\beta(t)h'(x) + 2\beta'(t)]\, y(t)y(t-r) \\ &\quad + 2\beta(t)y(t)z(t-r) + 2\beta(t)y(t-r)z(t) + 2\beta^2(t)y(t-r)z(t-r) \\ &\quad + \beta(t)q(t)\,[a - g(x(t),y(t))]\; z(t)z(t-r), \end{aligned}$$

and

$$\begin{aligned} U_3 &= -\beta(t)q(t)f(x(t),y(t))z(t-r) + [q(t)\beta'(t)\; + \beta(t)q'(t)]\; h(x)y(t-r) \\ &\quad + q'(t)h(x)y(t), \end{aligned}$$

$$U_4 = \Big(ay(t) + q(t)z(t) + \beta(t)q(t)z(t-r)\Big)\Big(\Delta_1(t) + \Delta_2(t) + \Delta_3(t)\Big).$$

From conditions (i)–(iv), we see that

$$ay(t)\int_0^y g_x(x,\xi)\xi d\xi + q(t)y(t)\int_0^y f_x(x,\xi)d\xi \le 0$$

and

$$U_1 \le \left(q_1\delta_0 - ab_0 + \lambda_1 + \frac{b_1}{2}\left|q'(t)\right| + \hat{\mu_1} r\right) y^2(t) + (-\mu q_0 + \lambda_2 + \hat{\mu_2} r) z^2(t).$$

Using Schwartz's inequality together with (i)–(iii), we obtain

$$\begin{aligned} U_2 \le\; & \frac{1}{2}\left[2 + 2\beta_1 + q_1\beta_1\delta_0 + 2\left|\beta'(t)\right|\right] y^2(t) + \frac{1}{2}\left[q_1\beta_1\delta_0 + 2\left|\beta'(t)\right| + 2\beta_1 + 2\beta_1^2\right] y^2(t-r) \\ & + \frac{1}{2}\left[2 + 2\beta_1 + \beta_1 q_1 \rho\right] z^2(t) + \frac{1}{2}\left[2\beta_1 + 2\beta_1^2 + 2\beta_1 q_1\rho\right] z^2(t-r). \end{aligned}$$

On the other hand, by (2.3)–(2.5),

$$\begin{aligned} q(t)z(t)\Big(\Delta_1(t) + \Delta_2(t) + \Delta_3(t)\Big) &\le \frac{Mq_1 r}{2} z^2(t) + \frac{q_1}{2}\Big(\delta_0 + K\Big)\int_{t-r}^t y^2(s)ds \\ &\quad + \frac{Lq_1}{2}\int_{t-r}^t z^2(s)ds, \end{aligned}$$

$$\begin{aligned} ay(t)\Big(\Delta_1(t) + \Delta_2(t) + \Delta_3(t)\Big) &\le \frac{aMr}{2} y^2(t) + \frac{a}{2}\Big(\delta_0 + K\Big)\int_{t-r}^t y^2(s)ds \\ &\quad + \frac{aL}{2}\int_{t-r}^t z^2(s)ds, \end{aligned}$$

and

$$\begin{aligned} \beta(t)q(t)z(t-r)\Big(\Delta_1(t) + \Delta_2(t) + \Delta_3(t)\Big) &\le \frac{\beta_1 q_1 Mr}{2} z^2(t-r) + \frac{\beta_1 q_1}{2}\Big(\delta_0 + K\Big)\int_{t-r}^t y^2(s)ds \\ &\quad + \frac{\beta_1 q_1 L}{2}\int_{t-r}^t z^2(s)ds. \end{aligned}$$

Hence, we have the estimate

$$
\begin{aligned}
U_4 \quad \leq \quad & \frac{Mq_1r}{2}z^2(t)+\frac{Mar}{2}y^2(t)+\frac{\beta_1q_1Mr}{2}z^2(t-r) \\
&+\frac{\delta_0+K}{2}(q_1+a+\beta_1q_1)\int_{t-r}^{t}y^2(s)ds, \\
&+\frac{L}{2}(q_1+a+\beta_1q_1)\int_{t-r}^{t}z^2(s)ds.
\end{aligned}
$$

Finally, condition (ii) and the facts that $h'(x)\leq\delta_0$ and $h(0)=0$ imply

$$
\begin{aligned}
U_3 \quad \leq \quad & \frac{1}{2}\left[\delta_0\left(q_1\left|\beta'(t)\right|\ +\beta_1\left|q'(t)\right|\right)+\left|q'(t)\right|\delta_0\right]\ x^2(t) \\
&+\frac{1}{2}\left[\beta(t)q(t)b_1+\left|q'(t)\right|\delta_0\right]y^2(t) \\
&+\frac{1}{2}\left(q_1\left|\beta'(t)\right|\ +\beta_1\left|q'(t)\right|\right)\delta_0\ y^2(t-r) \\
&+\frac{1}{2}\beta_1q_1b_1z^2(t-r).
\end{aligned}
$$

With some rearrangement of terms and using the above estimates, we obtain

$$
\begin{aligned}
V' \quad \leq \quad & \left(\frac{1}{2}\left[\delta_0q_1\left|\beta'(t)\right|\ +(\beta_1+1)\,\delta_0\left|q'(t)\right|\right]\right)x^2(t)+\left(\left|\beta'(t)\right|+\left(\frac{\delta_0}{2}+\frac{3b_1}{2}\right)\left|q'(t)\right|\right)y^2(t) \\
&+\left(-ab_0+q_1\delta_0+2+\frac{1}{2}q_1\beta_1\,(\delta_0+b_1)+\lambda_1+\left(\frac{Ma}{2}+\hat{\mu_1}\right)r\right)y^2(t) \\
&+\left[\frac{1}{2}\left[\beta_1\,(q_1\ \delta_0+2+2\beta_1+2\alpha)+\delta_0\,(q_1\alpha+\beta_1q_2)+2\alpha+4\alpha\beta_1\right]-\lambda_1\right]y^2(t-r) \\
&+\left[-\mu q_0+\rho\beta_1q_1+1+\beta_1+\lambda_2+\left(\hat{\mu_2}+\frac{Mq_1}{2}\right)r\right]z^2(t) \\
&+\left[\frac{\beta_1}{2}\left[2+2\beta_1+2q_1\rho+q_1b_1\right]-\lambda_2+\frac{\beta_1q_1M}{2}r\right]z^2(t-r) \\
&+\left(\frac{1}{2}\,(q_1+a+\beta_1q_1)\,(\delta_0+K)-\hat{\mu_1}\right)\int_{t-r}^{t}y^2(s)ds \\
&+\left(\frac{L}{2}(q_1+a+\beta_1q_1)-\hat{\mu_2}\right)\int_{t-r}^{t}z^2(s)ds.
\end{aligned}
$$

If we now choose

$$
\begin{aligned}
\hat{\mu_1}=\frac{1}{2}\,(q_1+a+\beta_1q_1)\,(\delta_0+K) \quad &= \quad \mu_1, \\
\hat{\mu_2}=\frac{L}{2}(q_1+a+\beta_1q_1) \quad &= \quad \mu_2, \\
\frac{1}{2}\left[\beta_1\,(q_1\delta_0+2+2\beta_1+2\alpha)+\delta_0\,(q_1\alpha+\beta_1q_2)+2\alpha+4\alpha\beta_1\right] \quad &= \quad \lambda_1, \\
\frac{\beta_1}{2}\left[2+2\beta_1+2q_1\rho+q_1b_1\right]+\varepsilon \quad &= \quad \lambda_2,
\end{aligned}
$$

then from conditions (vi)–(vii), we see that

$$\begin{aligned} V' &\le K_1\Big(\big|\beta'(t)\big|+\big|q'(t)\big|\Big)\Big(x^2(t)+y^2(t)\Big)+\Big(-c_1+\Big(\frac{Ma}{2}+\mu_1\Big)r\Big)y^2(t)\\ &\quad+\Big[-c_2+\Big(\mu_2+\frac{Mq_1}{2}\Big)r\Big]z^2(t)+\Big[-\varepsilon+\frac{\beta_1 q_1 M}{2}r\Big]z^2(t-r), \end{aligned}$$

where

$$K_1=\frac{1}{2}\max\{\delta_0 q_1,\,\delta_0(\beta_1+1),\,2,\,\delta_0+3b_1\}.$$

Taking

$$r<\min\left\{\frac{2c_1}{Ma+2\mu_1},\frac{2c_2}{Mq_1+2\mu_2},\frac{2\varepsilon}{M\beta_1 q_1}\right\},$$

we obtain

$$V'\le K_1\omega(t)\Big(x^2(t)+y^2(t)\Big)-K_2\Big(y^2(t)+z^2(t)\Big), \tag{2.11}$$

where

$$K_2 = \min\left\{c_1-\Big(\frac{Ma}{2}+\mu_1\Big)r, c_2-\Big(\mu_2+\frac{Mq_1}{2}\Big)r\right\}.$$

Hence, by (2.7), (2.10), and (2.11)

$$\begin{aligned} W' &= \Big(V'-\frac{1}{\theta}\omega(t)V\Big)e^{-\frac{1}{\theta}\int_0^t\omega(s)ds}\\ &\le \Big\{K_1\omega(t)\Big[x^2(t)+y^2(t)\Big]-K_2\Big[y^2(t)+z^2(t)\Big]\\ &\quad -\frac{K_0}{\theta}\omega(t)\Big[x^2(t)+y^2(t)+Z^2(t)\Big]\Big\}e^{-\frac{1}{\theta}\int_0^t\omega()ds}. \end{aligned}$$

By taking

$$\frac{K_0}{K_1}=\theta,$$

we obtain

$$W'\le -K_2\Big(y^2(t)+z^2(t)\Big)e^{-\frac{1}{\theta}\int_0^t\omega(s)ds}.$$

Since

$$\int_{t_0}^t\omega(s)ds<m,\ \text{for all } t\ge t_0,$$

we have

$$W'\le -K_3\Big(y^2(t)+z^2(t)\Big), \tag{2.12}$$

where $K_3=K_2e^{-\frac{mK_1}{K_0}}$.

From (2.12) we see that $W'\le 0$ and $W'=0$ on the set $M=\{(x,0,0)\}$. Now the largest invariant set contained in M is the origin, so by LaSalle's invariance principle, the zero solution of (2.2) is uniformly asymptotically stable. □

3 BOUNDEDNESS

Our main theorem in this section is for the forced equation (1.1). In this case our system becomes

$$\begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ [q(t)(z(t)+\beta(t)z(t-r))]' = -g(x(t),y(t))z(t) \\ \qquad\qquad -f(x(t),y(t)) - h(x(t)) + e(t) + \Delta_1 + \Delta_2 + \Delta_3. \end{cases} \tag{3.1}$$

Theorem 3.1. *In addition to the conditions of Theorem 2.1, assume that*

(I_1) $\int_0^\infty e(s)ds < \infty.$

Then there exists a positive constant D such that any solution of (3.1) satisfies

$$|x(t)| \leq D, \quad |y(t)| \leq D, \quad \textit{and} \quad |Z(t)| \leq D. \tag{3.2}$$

Proof. On differentiating (2.9) along the solutions of system (3.1) we obtain

$$V'_{(3.1)} \leq -K_2\left(y^2(t)+z^2(t)\right) + ae(t)y + e(t)q(t)Z, \quad \text{for all } t \geq t_1.$$

Applying conditions (I_1) and (iv) gives

$$V'_{(3.1)} \leq -K_2\left(y^2(t)+z^2(t)\right) + e(t)\left(a|y(t)| + q_1|Z(t)|\right).$$

Now, the inequality $|u| \leq u^2 + 1$ leads to

$$V'_{(3.1)} \leq -K_2\left(y^2(t)+z^2(t)\right) + K_4 e(t)(y^2 + Z^2 + 2), \text{ for all } t \geq t_1, \tag{3.3}$$

where $K_4 = \max\{a, q_1\}$.

In view of (2.10), the above estimates imply that

$$V'_{(3.1)} \leq -K_2\left(y^2(t)+z^2(t)\right) + \frac{K_4}{K_0} e(t)V + 2K_4 e(t). \tag{3.4}$$

Integrating both sides (3.4) from t_1 to t, we easily obtain

$$V(t) - V(t_1) \leq 2K_4 \int_{t_1}^t e(s)ds + \frac{K_4}{K_0}\int_{t_1}^t V(s)e(s)ds.$$

Condition (I_1) and an application of Gronwall's inequality shows that $V(t)$ is bounded, and the conclusion of the theorem follows immediately from (2.10). □

Remark 3.2. The uniform asymptotic stability from Theorem 2.1 together with the boundedness of all solutions from Theorem 3.1 ensure the global uniform asymptotic stability of the zero solution of the unforced equation.

4 SQUARE INTEGRABILITY

Our next result concerns the square integrability of solutions of equation (1.1).

Theorem 4.1. *If the conditions of Theorem 3.1 hold, then*

$$\int_{t_0}^{\infty}\Big(x''^2(s)+x'^2(s)+x^2(s)\Big)ds<\infty.$$

Proof. Define $H(t)$ by

$$H(t)=V(t)+\eta\int_{t_1}^{t}(z^2(s)+y^2(s))ds \quad \text{for all } t\geq t_1\geq t_0+r, \tag{4.1}$$

where $\eta>0$ is a constant to be specified later. Differentiating H and using (3.4) we obtain

$$H'(t)\leq(\eta-K_2)\Big(y^2(t)+z^2(t)\Big)+\left(\frac{K_4}{K_0}V+2K_4\right)e(t).$$

If we Choose $\eta<K_2$, then from the boundedness of V, we see that

$$H'(t)\leq K_5e(t), \tag{4.2}$$

for some $K_5>0$. Integrating (4.2) from t_1 to t, and using condition (I_1) we have that $H(t)$ is bounded. This implies the existence of positive constants κ_1 and κ_2 such that

$$\int_{t_1}^{\infty}y^2(s)ds\leq\kappa_1 \text{ and } \int_{t_1}^{\infty}z^2(s)ds\leq\kappa_2,$$

so

$$\int_{t_0}^{\infty}x'^2(s)ds<\infty \text{ and } \int_{t_0}^{\infty}x''^2(s)ds<\infty. \tag{4.3}$$

Next, we show that $\int_{t_1}^{\infty}x^2(s)ds<\infty$. Multiplying (1.1) by $x(t-r)$ gives

$$\begin{aligned}(q(t)(x''(t)+\beta(t)x''(t-r)))'\,x(t-r)+g(x(t),x'(t))x''(t)x(t-r)\\ +f(x(t-r),x'(t-r))x(t-r)+h(x(t-r))x(t-r)=e(t)x(t-r),\end{aligned} \tag{4.4}$$

and then integrating from t_1 to t, we have

$$\int_{t_1}^{t}h(x(s-r))x(s-r)ds=L_1(t)+L_2(t)+L_3(t)+L_4(t), \tag{4.5}$$

where

$$\begin{aligned}L_1(t)&=-\int_{t_1}^{t}(q(s)(x''(s)+\beta(s)x''(s-r)))'\,x(s-r)ds,\\ L_2(t)&=-\int_{t_1}^{t}g(x(s),x'(s))x''(s)x(s-r)ds,\\ L_3(t)&=-\int_{t_1}^{t}f(x(s-r),x'(s-r))x(s-r)ds,\\ L_4(t)&=\int_{t_1}^{t}e(s)x(s-r)ds.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
L_1(t) &= -\left[\left(q(t)\left(x''(t)+\beta(t)x''(t-r)\right)\right)x(t-r)-\left(q(t_1)\left(x''(t_1)+\beta(t_1)x''(t_1-r)\right)\right)x(t_1-r)\right] \\
&\quad + \int_{t_1}^{t} \left(q(s)\left(x''(s)+\beta(s)x''(s-r)\right)\right)x'(s-r)ds \\
&\le 2q_1(1+\beta_1)D^2 + \int_{t_1}^{t}\Big(q(s)x''(s)x'(s-r)+q(s)\beta(s)x''(s-r)x'(s-r)\Big)ds \\
&= 2q_1(1+\beta_1)D^2 + \frac{q_1}{2}\int_{t_1}^{t}\left[x''^2(s)+\beta_1 x''^2(s-r)+2x'^2(t-s)\right]ds \\
&\le l_1 < \infty
\end{aligned}$$

by (3.2) and (4.3).

In the same way, applying (i) and (4.3),

$$\begin{aligned}
L_2(t) &\le \int_{t_1}^{t}\left|g\left(x(s),x'(s)\right)x''(s)x(s-r)\right|ds \\
&\le \left\{\int_{t_1}^{t}\left[g\left(x(s),x'(s)\right)x''(s)\right]^2 ds\right\}^{\frac{1}{2}}\left\{\int_{t_1}^{t}x^2(s-r)ds\right\}^{\frac{1}{2}} \\
&\le \left\{(a+\rho)^2\int_{t_1}^{t}\left(x''(s)\right)^2 ds\right\}^{\frac{1}{2}}\left\{\int_{t_1}^{t}x^2(s-r)ds\right\}^{\frac{1}{2}} \\
&\le l_2\left\{\int_{t_1}^{t}x^2(s-r)ds\right\}^{\frac{1}{2}}.
\end{aligned}$$

Condition (ii) and (4.3) imply

$$\begin{aligned}
L_3(t) &\le \int_{t_1}^{t}\left|f(x(s-r),x'(s-r))x(s-r)\right|ds \\
&\le \left\{b_1^2\int_{t_1}^{t}\left(x'(s-r)\right)^2 ds\right\}^{\frac{1}{2}}\left\{\int_{t_1}^{t}x^2(s-r)ds\right\}^{\frac{1}{2}} \\
&\le l_3\left\{\int_{t_1}^{t}x^2(s-r)ds\right\}^{\frac{1}{2}}.
\end{aligned}$$

Finally,

$$L_4(t) \le \int_{t_1}^{t}|e(s)x(s-r)|ds \le D\int_{t_1}^{t}|e(s)|ds \le l_4 < \infty.$$

On the other hand, from condition (iii), we have

$$\int_{t_1}^{t}x(s-r)h(x(s-r))ds \ge \delta_1\int_{t_1}^{t}x^2(s-r)ds.$$

Therefore,

$$\delta_1\int_{t_1}^{t}x^2(s-r)ds \le l_1 + l_2\left\{\int_{t_1}^{t}x^2(s-r)ds\right\}^{\frac{1}{2}} + l_3\left\{\int_{t_1}^{t}x^2(s-r)ds\right\}^{\frac{1}{2}} + l_4.$$

If

$$\int_{t_1}^{t} x^2(s-r)ds \to \infty \text{ as } t \to \infty,$$

then dividing both sides of by $\left\{\int_{t_1}^{t} x^2(s-r)ds\right\}^{\frac{1}{2}}$ we immediately obtain a contradiction. This completes the proof of the theorem. □

We conclude our paper with an example to illustrate our theorems.

Example 4.2. Consider the third order nonlinear nonautonomous delay differential equation

$$\left(\ln(3+\cos t)\left(x''(t)+\frac{1}{54.93}\ln(2+\sin t)x''(t-r)\right)\right)'$$
$$+\left(46.4-\frac{1}{\pi}\arctan(x(t)x'(t))\right)x''(t)+13.5x'(t-r)-\frac{x'(t-r)}{\pi}\arctan(e^{x(t-r))}x'(t-r))$$
$$+x(t-r)+\frac{x(t-r)}{1+|x(t-r)|}=\frac{1}{1+t^2}. \quad (4.6)$$

Taking $a=2$, we see that

$$\mu = 43.9 \le g(x,y)-a = 45.5-\frac{1}{\pi}\arctan(xy) \le 44.9 = \rho,$$

and

$$yg_x(x,y) = -\frac{1}{\pi}\frac{y^2}{1+x^2y^2} \le 0.$$

Now

$$f(x,y) = 13.5y-\frac{y}{\pi}\arctan(e^x y),$$

so we see that $f(x,0)=f(0,0)=0$ for all x and

$$b_0 = 13 < \frac{f(x,y)}{y} = 13.5-\frac{1}{\pi}\arctan(e^x y) < 14 = b_1.$$

Moreover,

$$K = -\frac{1}{\pi} \le f_x(x,y) = -\frac{1}{\pi}\frac{y^2}{1+(e^x y)^2} \le 0$$

and

$$\left|f_y(x,y)\right| = \left|13.5-\frac{1}{\pi}\arctan(e^x y)-\frac{1}{\pi}\frac{e^x y}{1+(e^x y)^2}\right| \le 14.5 = L.$$

Since

$$h(x) = x+\frac{x}{1+|x|},$$

it is clear that $h(0)=0$ and

$$|h'(x)| = |1+\frac{1}{(1+|x|)^2}| \le 2 = \delta_0.$$

Since $0 \le \frac{1}{1+|x|} \le 1$ for all x, we have

$$\frac{h(x)}{x} \ge 1 = \delta_1 \quad \text{for all} \quad x \neq 0.$$

Also, we have

$$\begin{aligned}
q_0 &= \ln 2 \le q(t) = \ln(3+\cos t) \le 2\ln 2 = q_1, \\
\left|q'(t)\right| &= \left|\frac{\sin t}{3+\cos t}\right| \le \frac{1}{2} = q_2, \\
\int_{-\infty}^{+\infty} \left|q'(s)\right| ds &= \int_{-\infty}^{+\infty} \left|\frac{\sin s}{3+\cos s}\right| ds \le \int_{-\infty}^{+\infty} \frac{1}{3+\cos s} ds \\
&= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{1}{2+u^2} du = \frac{2}{\sqrt{2}} \tan^{-1}\left(\frac{\pi}{2\sqrt{2}}\right), \\
0 &\le \beta(t) = \frac{1}{54.93} \ln(2+\sin t) \le \frac{\ln 3}{54.93} = \beta_1, \\
\left|\beta'(t)\right| &= \frac{1}{54.93}\left|\frac{\cos t}{2+\sin t}\right| \le \frac{1}{54.93} = \alpha,
\end{aligned}$$

and

$$\int_0^t \left(\left|q'(s)\right| + \left|\beta'(s)\right|\right) ds < m.$$

for all $t \ge t_0$.

Choosing $\varepsilon = 10^{-1}$, we obtain

$$\begin{aligned}
-ab_0 + q_1\delta_0 + 2 + \frac{1}{2} q_1\beta_1(\delta_0 + b_1) + \frac{1}{2}[\beta_1(q_1\delta_0 + 2 + 2\beta_1 + 2\alpha) \\
+ \delta_0(q_1\alpha + \beta_1 q_2) + 2\alpha + 4\alpha\beta_1] = -20.903 = -c_1,
\end{aligned}$$

$$-\mu q_0 + \rho\beta_1 q_1 + 1 + \beta_1 + \frac{\beta_1}{2}[2 + 2\beta_1 + 2q_1\rho + q_1 b_1] + \varepsilon = -26.605 = -c_2,$$

and

$$a - \frac{q_1^2\delta_0}{2} = 7.8188 \times 10^{-2} > 0.$$

Hence, if $r < \min\left\{\frac{2c_1}{Ma+2\mu_1}, \frac{2c_2}{Mq_1+2\mu_2}, \frac{2\varepsilon}{M\beta_1 q_1}\right\} = \min\{1.0061, 0.73072, 0.42891\} = 0.42891$, then all the conditions of Theorems 2.1, 3.1, and 4.1 are satisfied, so for any solution $x(t)$ of equation (4.6), $x(t)$, $x'(t)$, and $x''(t)$ are bounded and square integrable. In addition, the zero solution of the unforced equation is globally uniformly asymptotically stable.

Conclusion

A non-linear neutral delay differential equation of the third order is considered. Using Lyapunov's direct method, we have derived new sufficient conditions for the global uniform asymptotic stability of the zero solution as well as the boundedness and square integrability of all solutions. The results here will be of interest to other researchers working on qualitative behavior of solutions of differential equations.

Acknowledgments

J. R. Graef's research was supported in part by a University of Tennessee at Chattanooga SimCenter – Center of Excellence in Applied Computational Science and Engineering (CEACSE) grant.

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