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Multiple Solutions for Semilinear $\Delta_\gamma-$differential Equations in $\mathbb{R}^N$ with Sign-changing Potential

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Abstract

In this paper, we study the existence of infinitely many nontrivial solutions of the semilinear Δ_γ differential equations in $\mathbb{R}^N$

$$-\Delta_\gamma u + b(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \quad u \in S^2_\gamma(\mathbb{R}^N),$$

where Δ_γ is the subelliptic operator of the type

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \quad \partial_{x_j} := \frac{\partial}{\partial x_j}, \quad \gamma = (\gamma_1, \gamma_2, ..., \gamma_N),$$

and the potential b is allowed to be sign-changing, and the primitive of the nonlinearity f is of superquadratic growth near infinity in u and allowed to be sign-changing.

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1 Introduction

In the last years, the semilinear Schrödinger equation

$$-\Delta u + b(x)u = f(x,u), \quad x \in \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

has been studied by many authors. The Schrödinger equation has found a great deal of interest last years because not only it is important in applications but it provides a good

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model for developing mathematical methods.With the aid of variational methods, the existence and multiplicity of nontrivial solutions for the problem (1.1) have been extensively investigated in the literature over the past several decades. See, e.g., [1–6, 10, 11, 19–22, 28] and references quoted in them.

In this paper, we study the existence and multiplicity of nontrivial weak solutions to the following the problem

$$-\Delta_\gamma u + b(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \quad u \in S^2_\gamma(\mathbb{R}^N), \tag{1.2}$$

where Δ_γ is a subelliptic operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \quad \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N) : \mathbb{R}^N \longrightarrow \mathbb{R}^N.$$

The $\Delta_\gamma-$operator was considered by A. E. Kogoj and E. Lanconelli in [9]. This operator has the same form as in [7], however the functions $\gamma(x)$ in [9] are more generalized than those considered in [7]. The $\Delta_\gamma-$operator contains many degenerate elliptic operators such as the Grushin–type operator

$$G_\alpha := \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha \geq 0,$$

where (x,y) denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ (see [8]), and the operator of the form

$$P_{\alpha,\beta} := \Delta_x + \Delta_y + |x|^{2\alpha}|y|^{2\beta}\Delta_z, \quad (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where α, β are nonnegative real numbers (see [23, 27]). Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [26, 27] (see also some recent results in [9, 12–17, 23–25]).

To study the problem (1.2), we make the following assumptions:

(B1) $b \in C(\mathbb{R}^N, \mathbb{R})$ is bounded from below.

(B2) There exists a constant $d_0 > 0$ such that

$$\lim_{|y| \to +\infty} \operatorname{meas}\{x \in \mathbb{R}^N : |x-y| \leq d_0, b(x) \leq M\} = 0, \quad \forall M > 0,$$

where $\operatorname{meas}\{\cdot\}$ denotes the Lebesgue measure of a set in $\mathbb{R}^N$.

While the nonlinearities $f : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ and its primitive $F(x,\xi) = \int_0^\xi f(x,\tau)d\tau$ are such that

(B3) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist $c_1 > 0$ and $2 < p < 2^*_\gamma$ such that

$$|f(x,\xi)| \leq c_1\left(|\xi| + |\xi|^{p-1}\right), \quad \text{for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R},$$

where $2^*_\gamma := \frac{2\widetilde{N}}{\widetilde{N}-2}$ (where $\widetilde{N}$ is defined by formula (2.1)).

(B4) $F(x,\xi) \geq 0$ for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}$ and

$$\lim_{|\xi|\to\infty} \frac{F(x,\xi)}{|\xi|^2} = +\infty, \quad \text{uniformly in } x \in \mathbb{R}^N.$$

(B5) There exists $\theta \geq 1$ such that

$$\theta\mathcal{F}(x,\xi) \geq \mathcal{F}(x,\tau\xi), \qquad \text{for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R} \text{ and } \tau \in [0,1]$$

where $\mathcal{F}(x,\xi) = \xi f(x,\xi) - 2F(x,\xi)$.

(B6) $f(x,-\xi) = -f(x,\xi)$, for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}$.

(B7) $f(x,\xi) = o(|\xi|)$, as $|\xi| \to 0$, uniformly in $x \in \mathbb{R}^N$.

(B8) There are constants $\mu > 2$ and $r_1 > 0$ such that

$$\mu F(x,\xi) \leq \xi f(x,\xi), \qquad \text{for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R},\ |\xi| \geq r_1.$$

Now, we are ready to state the main results of this paper.

Theorem 1.1. *Assume that b and f satisfy* (B1)–(B5) *and* (B6). *Then the problem* (1.2) *possesses infinitely many nontrivial solutions.*

Theorem 1.2. *Assume that b and f satisfy* (B1)–(B4) *and* (B6)–(B8). *Then the problem* (1.2) *has a ground state solution* u_0*, that is* $\Phi(u_0) = \inf_{u\in\mathcal{M}} \Phi(u)$*, where* $\mathcal{M} = \{u \in S^2_{\gamma,b(x)}(\mathbb{R}^N) : u \neq 0, \Phi'(u)(u) = 0\}$.

Remark 1.3. Our result is not covered by those in [17]. For example, when

$$b(x) = \begin{cases} 2n|x| - 2n(n-1) + c_0 & \text{if } n-1 \leq |x| \leq (2n-1)/2, \\ -2n|x| + 2n^2 + c_0 & \text{if } (2n-1)/2 \leq |x| \leq n, \end{cases}$$

for $n \in \mathbb{N}$ and $c_0 \in \mathbb{R}$ and

$$f(x,\xi) = a(x)\xi\ln(1+|\xi|), \quad \forall\, (x,\xi) \in \mathbb{R}^N \times \mathbb{R},$$

where $a(x)$ is a continuous bounded function with positive lower bound, it is easy to check that $f(x,\xi), b(x)$ satisfy (B1)–(B6) but do not satisfy the conditions (A3) in [17] where (A3): there are constants $\mu > 2$ and $r_1 > 0$ such that

$$\mu F(x,\xi) \leq \xi f(x,\xi), \qquad \text{for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R},\ |\xi| \geq r_1,$$

and not satisfy the conditions (B_1) in [17] where (B_1): $b \in L^1_{loc}(\mathbb{R}^N)$ and

$$\mu_0 = \operatorname*{essinf}_{x\in\mathbb{R}^N} b(x) := \sup\left\{\mu \in \mathbb{R} : \text{meas}\{x \in \mathbb{R}^N, b(x) < \mu\} = 0\right\} > 0.$$

Remark 1.4. By using (B1) we know that there exists $c_0 > 0$ such that $b_1(x) = b(x) + c_0 \geq 1$ for any $x \in \mathbb{R}^N$. Let $f_1(x,\xi) = f(x,\xi) + c_0\xi$ for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}$. Then it is easy to verify that the study of (1.2) is equivalent to investigate the problem

$$-\Delta_\gamma u + b(x)u = f_1(x,u) \quad \text{in } \mathbb{R}^N, \quad u \in S^2_\gamma(\mathbb{R}^N). \tag{1.3}$$

Hence, from now on, we assume that $b(x) \geq 1$ for any $x \in \mathbb{R}^N$ in (B1).

The paper is organized as follows. In Section 2 for convenience of the readers, we recall some function spaces, embedding theorems and associated functional settings. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2 Preliminary results

We recall the functional setting in [9, 14]. We consider the operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \quad \partial_{x_j} := \frac{\partial}{\partial x_j},\ j = 1, 2, \ldots, N.$$

Here, the functions $\gamma_j : \mathbb{R}^N \to \mathbb{R}$ are assumed to be continuous, different from zero and of class C^1 in $\mathbb{R}^N \backslash \Pi$, where

$$\Pi := \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0 \right\}.$$

Moreover, we assume the following properties:
i) There exists a semigroup of dilations $\{\delta_t\}_{t>0}$ such that

$$\begin{aligned} \delta_t : \mathbb{R}^N &\longrightarrow \mathbb{R}^N \\ (x_1, \ldots, x_N) &\longmapsto \delta_t(x_1, \ldots, x_N) = (t^{\varepsilon_1} x_1, \ldots, t^{\varepsilon_N} x_N), \end{aligned}$$

where $1 = \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_N$, such that γ_j is δ_t-homogeneous of degree $\varepsilon_j - 1$, i. e.,

$$\gamma_j(\delta_t(x)) = t^{\varepsilon_j - 1} \gamma_j(x), \quad \forall\, x \in \mathbb{R}^N, \quad \forall\, t > 0,\ j = 1, \ldots, N.$$

The number

$$\widetilde{N} := \sum_{j=1}^{N} \varepsilon_j \tag{2.1}$$

is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.
ii)

$$\gamma_1 = 1,\ \gamma_j(x) = \gamma_j(x_1, x_2, \ldots, x_{j-1}),\ j = 2, \ldots, N.$$

iii) There exist a constant $\rho \ge 0$ such that

$$0 \le x_k \partial_{x_k} \gamma_j(x) \le \rho \gamma_j(x), \quad \forall\, k \in \{1, 2, \ldots, j-1\}, \quad \forall j = 2, \ldots, N,$$

and for every $x \in \overline{\mathbb{R}}_+^N := \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N :\ x_j \ge 0,\ \forall j = 1, 2, \ldots, N\right\}$.
iv) Equalities $\gamma_j(x) = \gamma_j(x^*)$ $(j = 1, 2, \ldots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where

$$x^* = (|x_1|, \ldots, |x_N|) \quad \text{if} \quad x = (x_1, x_2, \ldots, x_N).$$

Definition 2.1. By $S_\gamma^p(\mathbb{R}^N)$ $(1 \le p < +\infty)$ we will denote the set of all functions $u \in L^p(\mathbb{R}^N)$ such that $\gamma_j \partial_{x_j} u \in L^p(\mathbb{R}^N)$ for all $j = 1, \ldots, N$. We define the norm in this space as follows

$$\|u\|_{S_\gamma^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(|u|^p + \left|\nabla_\gamma u\right|^p\right) \mathrm{d}x \right)^{\frac{1}{p}},$$

where $\nabla_\gamma u = (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \ldots, \gamma_N \partial_{x_N} u)$.

If $p = 2$ we can also define the scalar product in $S_\gamma^2(\mathbb{R}^N)$ as follows

$$(u, v)_{S_\gamma^2(\mathbb{R}^N)} = (u, v)_{L^2(\mathbb{R}^N)} + (\nabla_\gamma u, \nabla_\gamma v)_{L^2(\mathbb{R}^N)}.$$

Define

$$S^2_{\gamma,b(x)}(\mathbb{R}^N)=\left\{u\in S^2_\gamma(\mathbb{R}^N):\int\limits_{\mathbb{R}^N}\left(\left|\nabla_\gamma u\right|^2+b(x)u^2\right)\mathrm{d}x<+\infty\right\}$$

with $b(x)$ satisfying conditions (B1),(B2) then $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ is a Hilbert space with the norm

$$\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}=\left(\int\limits_{\mathbb{R}^N}\left(\left|\nabla_\gamma u\right|^2+b(x)u^2\right)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Proposition 2.2. *Assume that b satisfy* (B1) *and* (B2). *Then the embedding map from $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ is compact for $2\le q<2^*_\gamma$.*

Proof. The proof of this proposition is similar to the one of Lemma 2.2 in [17]. We omit the details. □

Definition 2.3. Let $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$ be a real Banach space with its dual space $\mathbb{X}^*$ and $\Phi\in C^1(\mathbb{X},\mathbb{R})$. For $c\in\mathbb{R}$ we say that Φ satises the $(C)_c$ condition if for any sequence $\{x_m\}_{n=1}^\infty\subset\mathbb{X}$ with

$$\Phi(x_m)\to c\ \text{ and }\ (1+\|x_m\|_{\mathbb{X}})\left\|\Phi'(x_m)\right\|_{\mathbb{X}^*}\to 0,$$

then there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ that converges strongly in $\mathbb{X}$. If Φ satisfies the $(C)_c$ condition for all $c>0$ then we say that Φ satisfies the Cerami condition.

Define the Euler–Lagrange functional associated with the problem (1.2) as follows

$$\Phi(u)=\frac{1}{2}\int\limits_{\mathbb{R}^N}\left(\left|\nabla_\gamma u\right|^2+b(x)u^2\right)\mathrm{d}x-\int\limits_{\mathbb{R}^N}F(x,u)\mathrm{d}x. \tag{2.2}$$

From Lemma 2.3 in [17] and f satisfies (B3), $b(x)$ satisfies (B1), we have Φ is well defined on $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ and $\Phi\in C^1(S^2_{\gamma,b(x)}(\mathbb{R}^N),\mathbb{R})$ with

$$\Phi'(u)(v)=\int\limits_{\mathbb{R}^N}\left(\nabla_\gamma u\cdot\nabla_\gamma v+b(x)uv\right)\mathrm{d}x-\int\limits_{\mathbb{R}^N}f(x,u)\,v\mathrm{d}x$$

for all $v\in S^2_{\gamma,b(x)}(\mathbb{R}^N)$. One can also check that the critical points of Φ are weak solutions of problem (1.2).

The following variant fountain theorem was established in [29].

Lemma 2.4 (see [29])**.** *Let $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$ be a Banach space, $\mathbb{X}=\overline{\oplus_{j\in\mathbb{N}}\mathbb{X}_j}$, with $\dim\mathbb{X}_j<\infty$ for any $j\in\mathbb{N}$. Set $\mathbb{Y}_k=\oplus_{j=1}^k\mathbb{X}_j$ and $\mathbb{Z}_k=\overline{\oplus_{j=k}^\infty\mathbb{X}_j}$. Let $\Phi_\lambda:\mathbb{X}\to\mathbb{R}$ a family of $C^1(\mathbb{X},\mathbb{R})$ functionals defined by*

$$\Phi_\lambda(u)=A(u)-\lambda B(u),\quad \lambda\in[1,2].$$

Assume that Φ_λ satisfies the following assumptions:

(i) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda\in[1,2]$, $\Phi_\lambda(-u)=\Phi_\lambda(u)$ for all $(\lambda,u)\in[1,2]\times\mathbb{X}$;

(ii) $B(u) \geq 0$ *for all* $u \in \mathbb{X}$*, and* $A(u) \to \infty$ *or* $B(u) \to \infty$ *as* $\|u\|_{\mathbb{X}} \to \infty$*;*

(iii) There exist $r_k > \rho_k$ *such that*

$$\beta_k(\lambda) = \max_{u \in \mathbb{Y}_k, \|u\|_{\mathbb{X}} = r_k} \Phi_\lambda(u) < \alpha_k(\lambda) = \inf_{u \in \mathbb{Z}_k, \|u\|_{\mathbb{X}} = \rho_k} \Phi_\lambda(u), \quad \forall \lambda \in [1,2].$$

Then

$$\alpha_k(\lambda) \leq \xi_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)) \quad \forall \lambda \in [1,2],$$

where $B_k = \{u \in \mathbb{Y}_k : \ \|u\|_{\mathbb{X}} \leq r_k\}$ *and*

$$\Gamma_k = \{\gamma \in C(B_k, \mathbb{X}) : \gamma \text{ is odd}, \ \gamma = Id \ \text{ on } \partial B_k\}.$$

Moreover, for a.e. $\lambda \in [1,2]$*, there exists a sequence* $\{u_m^k(\lambda)\}_{m \in \mathbb{N}} \subset \mathbb{X}$ *such that*

$$\sup_{m \in \mathbb{N}} \|u_m^k(\lambda)\| < \infty, \ \ \Phi'_\lambda(u_m^k(\lambda)) \to 0, \quad \Phi_\lambda(u_m^k(\lambda)) \to \xi_k(\lambda) \quad \text{as } m \to \infty.$$

3 Proof of Theorems

In order to apply Lemma 2.4 to prove our main result, we define the functionals A, B and Φ_λ on our working space $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ by

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(\left|\nabla_\gamma u\right|^2 + b(x)u^2 \right) \mathrm{d}x, \quad B(u) = \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x,$$

and

$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(\left|\nabla_\gamma u\right|^2 + b(x)u^2 \right) \mathrm{d}x - \lambda \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x$$

for all $\lambda \in [1,2]$ and $u \in S^2_{\gamma,b(x)}(\mathbb{R}^N)$. Then we have $\Phi_\lambda \in C^1(S^2_{\gamma,b(x)}(\mathbb{R}^N), \mathbb{R})$ for all $\lambda \in [1,2]$. Let $\{e_j\}_{j=1}^\infty$ be a total orthonormal basis of $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ and define $\mathbb{X}_j = \mathbb{R}e_j$. Note that $\Phi_1 = \Phi$, where Φ is the functional defined in (2.2).

We further need the following lemmas.

Lemma 3.1. *Assume that* (B1)–(B3) *are satisfied. Then there exist* $k_1 \in \mathbb{N}$ *and a sequence* $\{\rho_k\}_{k=1}^\infty$ *such that* $\rho_k \to \infty$ *as* $k \to \infty$ *and*

$$\alpha_k(\lambda) = \inf_{u \in \mathbb{Z}_k, \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = \rho_k} \Phi_\lambda(u), \quad \forall k \geq k_1,$$

where $\mathbb{Z}_k = \overline{\oplus_{j=k}^\infty \mathbb{X}_j}$ *for all* $k \in \mathbb{N}$.

Proof. Let us define

$$b_2(k) = \sup_{u \in \mathbb{Z}_k, \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1} \|u\|_{L^2(\mathbb{R}^N)}, \quad b_p(k) = \sup_{u \in \mathbb{Z}_k, \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1} \|u\|_{L^p(\mathbb{R}^N)}.$$

We aim to prove that

$$b_2(k) \to 0, \quad b_p(k) \to 0 \quad \text{as } k \to \infty. \tag{3.1}$$

It is clear that $b_2(k)$ and $b_p(k)$ are decreasing with respect to k so there exist $b_2, b_p \geq 0$ such that $b_2(k) \to b_2$ and $b_p(k) \to b_p$ as $k \to \infty$. For any $k \geq 0$, there exists $u_k \in \mathbb{Z}_k$ such that $\|u_k\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1$ and $\|u_k\|_{L^2(\mathbb{R}^N)} \geq \frac{b_2(k)}{2}$, hence we can assume that $u_k \rightharpoonup u$ in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$. From definition of $\mathbb{Z}_k$, we have $u = 0$. Since $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$ by Proposition 2.2, we have $u_k \to 0$ in $L^2(\mathbb{R}^N)$, which implies that $b_2 = 0$. Similarly we can prove $b_p = 0$. Then, for any $u \in \mathbb{Z}_k$ and $\lambda \in [1,2]$, we can see that

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}{2} - 2\int\limits_{\mathbb{R}^N} F(x,u)\mathrm{d}x \\
&\geq \frac{\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}{2} - 2c_1\left(\|u\|^2_{L^2(\mathbb{R}^N)} + \|u\|^p_{L^p(\mathbb{R}^N)}\right) \\
&\geq \frac{\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}{2} - 2c_1\left(b_2^2(k)\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} + b_p^p(k)\|u\|^p_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}\right).
\end{aligned}$$

By using (3.1), we can find $k_1 \in \mathbb{N}$ such that

$$2c_1 b_2^2(k) \leq \frac{1}{4}, \quad \forall k \geq k_1.$$

For each $k \geq k_1$, we choose

$$\rho_k := (16c_1 b_p^p(k))^{\frac{1}{2-p}}.$$

Let us note that

$$\rho_k \to \infty \quad \text{as } k \to \infty, \tag{3.2}$$

since $p > 2$. Then we deduce that

$$\alpha_k(\lambda) := \inf_{u\in\mathbb{Z}_k, \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}=\rho_k} \Phi_\lambda(u) \geq \frac{1}{8}\rho_k^2 > 0$$

for any $k \geq k_1$. □

Lemma 3.2. *Assume that* (B1)–(B4) *hold. Then for the positive integer k_1 and the sequence ρ_k obtained in Lemma 3.1, there exists $r_k > \rho_k$ for any $k \geq k_1$ such that*

$$\beta_k(\lambda) = \max_{u\in\mathbb{Y}_k, \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}=r_k} \Phi_\lambda(u) < 0,$$

where $\mathbb{Y}_k = \oplus_{j=1}^k \mathbb{X}_j$ for all $k \in \mathbb{N}$.

Proof. Firstly we prove that for any finite dimensional subspace $F \subset S^2_{\gamma,b(x)}(\mathbb{R}^N)$ there exists a constant $\delta > 0$ such that

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : |u(x)| \geq \delta\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}\right\} \geq \delta, \quad \forall u \in F \setminus \{0\}. \tag{3.3}$$

We argue by contradiction and we suppose that for any $n \in \mathbb{N}$ there exists $0 \neq u_n \in F$ such that

$$\text{meas}\left\{x \in \mathbb{R}^N: \ |u_n(x)| \geq \frac{1}{n}\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}\right\} < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Let $v_n := \frac{u_n}{\|u_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}} \in F$ for all $n \in \mathbb{N}$. Then $\|v_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1$ for all $n \in \mathbb{N}$ and

$$\text{meas}\left\{x \in \mathbb{R}^N : |v_n(x)| \geq \frac{1}{n}\right\} < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Up to a subsequence, we may assume that $v_n \to v$ in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$ for some $v \in F$ since F is a finite dimensional space. Clearly $\|v\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = 1$. By using Proposition 2.2 and the fact that all norms are equivalent on F, we deduce that

$$\|v_n - v\|_{L^2(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Since $v \neq 0$, there exists $\delta_0 > 0$ such that

$$\text{meas}\left\{x \in \mathbb{R}^N: \ |v(x)| \geq \delta_0\right\} \geq \delta_0. \tag{3.6}$$

Set

$$\Lambda_0 := \left\{x \in \mathbb{R}^N: \ |v(x)| \geq \delta_0\right\}$$

and for all $n \in \mathbb{N}$,

$$\Lambda_n := \left\{x \in \mathbb{R}^N: \ |v_n(x)| \geq \frac{1}{n}\right\}, \quad \Lambda_n^c := \mathbb{R}^N \setminus \Lambda_n.$$

Taking into account (3.4) and (3.6), we obtain

$$\text{meas}\,(\Lambda_n \cap \Lambda_0) \geq \text{meas}\,\Lambda_0 - \text{meas}\,\Lambda_n^c \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}.$$

for n large enough. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n - v|^2 \mathrm{d}x &\geq \int_{\Lambda_n \cap \Lambda_0} |v_n - v|^2 \mathrm{d}x \\
&\geq \int_{\Lambda_n \cap \Lambda_0} (|v| - |v_n|)^2 \mathrm{d}x \\
&\geq \left(\delta_0 - \frac{1}{n}\right)^2 |\Lambda_n \cap \Lambda_0| \\
&\geq \frac{\delta_0^3}{8} > 0
\end{aligned}$$

which contradicts (3.5).

Now, by using that $\mathbb{Y}_k$ is finite dimensional and (3.3), we can find $\delta_k > 0$ such that

$$\text{meas}\left\{x \in \mathbb{R}^N : |u(x)| \geq \delta_k \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}\right\} \geq \delta_k, \quad \forall u \in \mathbb{Y}_k \setminus \{0\}. \tag{3.7}$$

By (B4), for any $k \in \mathbb{N}$ there exists a constant $R_k > 0$ such that

$$F(x,u) \geq \frac{|u|^2}{\delta_k^3}, \quad \forall\, x \in \mathbb{R}^N \text{ and } |u| \geq R_k.$$

Set

$$A_u^k = \left\{ x \in \mathbb{R}^N : \ |u(x)| \geq \delta_k \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \right\}$$

and let us observe that, by (3.7), $\operatorname{meas}(A_u^k) \geq \delta_k$ for any $u \in \mathbb{Y}_k \setminus \{0\}$. Then for any $u \in \mathbb{Y}_k$ such that $\|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \geq \frac{R_k}{\delta_k}$, we have

$$\begin{aligned}
\Phi_\lambda(u) &\leq \frac{1}{2}\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \\
&\leq \frac{1}{2}\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \int_{A_u^k} \frac{|u|^2}{\delta_k^3}\mathrm{d}x \\
&\leq \frac{1}{2}\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = -\frac{1}{2}\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}.
\end{aligned}$$

Choosing $r_k > \max\{\rho_k, \frac{R_k}{\delta_k}\}$ for all $k \geq k_1$, we have

$$\beta_k(\lambda) = \max_{u \in \mathbb{Y}_k, \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = r_k} \Phi_\lambda(u) \leq -\frac{1}{2} r_k^p < 0, \quad \forall\, k \geq k_1.$$

□

From (B3) and Proposition 2.2 we can see that Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Moreover, by (B6), Φ_λ is even. Then condition (i) in Lemma 2.4 is satisfied. Condition (ii) is clearly true, while (iii) follows by Lemma 3.1 and Lemma 3.2. Then, by Lemma 2.4, for any $k \geq k_1$ and $\lambda \in [1,2]$ there exists a sequence $\{u_m^k(\lambda)\}_{n\in\mathbb{N}} \subset S^2_{\gamma,b(x)}(\mathbb{R}^N)$ such that

$$\sup_{m\in\mathbb{N}} \|u_m^k(\lambda)\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} < \infty, \quad \Phi'_\lambda(u_m^k(\lambda)) \to 0, \quad \Phi_\lambda(u_m^k(\lambda)) \to \xi_k(\lambda) \quad \text{ as } m \to \infty$$

where

$$\xi_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u))$$

with

$$\Gamma_k = \left\{ \gamma \in C(B_k, S^2_{\gamma,b(x)}(\mathbb{R}^N)) : \ \gamma \text{ is odd}, \gamma = Id \text{ on } \partial B_k \right\},$$

$$B_k = \left\{ u \in \mathbb{Y}_k : \ \|u\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \leq r_k \right\}.$$

In particular, from the proof of Lemma 3.1, we deduce that for any $k \geq k_1$ and $\lambda \in [1,2]$

$$\frac{1}{8}\rho_k^2 =: c_k \leq \xi_k(\lambda) \leq d_k := \max_{u \in B_k} \Phi_1(u), \tag{3.8}$$

and $c_k \to \infty$ as $k \to \infty$ by (3.2). As a consequence, for any $k \geq k_1$, we can choose $\lambda_n \to 1$ (depending on k) and get the corresponding sequences satisfying

$$\sup_{m\in\mathbb{N}} \|u_m^k(\lambda_n)\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} < \infty, \quad \Phi'_{\lambda_n}(u_m^k(\lambda_n)) \to 0 \quad \text{as } m \to \infty. \tag{3.9}$$

Now, we prove that for any $k \geq k_1$, $\{u_m^k(\lambda_n)\}_{m\in\mathbb{N}}$ admits a strongly convergent subsequence $\{u_n^k\}_{n\in\mathbb{N}}$, and that such subsequence is bounded.

Lemma 3.3. *For each λ_n given above, the sequence $\{u_m^k(\lambda_n)\}_{m\in\mathbb{N}}$ has a strong convergent subsequence.*

Proof. By (3.9) we may assume, without loss of generality, that as $m \to \infty$,

$$u_m^k(\lambda_n) \rightharpoonup u_n^k \quad \text{in } S^2_{\gamma,b(x)}(\mathbb{R}^N)$$

for some $u_n^k \in S^2_{\gamma,b(x)}(\mathbb{R}^N)$. By Proposition 2.2 we have

$$u_m^k(\lambda_n) \to u_n^k \qquad \text{in } L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N). \tag{3.10}$$

By (B3) and Hölder inequality it follows that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N} f(x,u_m^k(\lambda_n))(u_m^k(\lambda_n)-u_n^k)\,dx\Big| \\ &\leq c_1\|u_m^k(\lambda_n)\|_{L^2(\mathbb{R}^N)}\|u_m^k(\lambda_n)-u_n^k\|_{L^2(\mathbb{R}^N)} + c_1\|u_m^k(\lambda_n)\|^{p-1}_{L^p(\mathbb{R}^N)}\|u_m^k(\lambda_n)-u_n^k\|_{L^p(\mathbb{R}^N)}\end{aligned}$$

so, by using (3.10), we obtain

$$\lim_{m\to\infty}\int_{\mathbb{R}^N} f(x,u_m^k(\lambda_n))(u_m^k(\lambda_n)-u_n^k)\,dx = 0.$$

Since $\Phi'_{\lambda_n}(u_m^k(\lambda_n)) \to 0$ as $m \to \infty$, and

$$\langle \Phi'_\lambda(u), v\rangle = \langle A'(u), v\rangle - \lambda\langle B'(u), v\rangle,$$

we deduce that

$$\langle A'(u_m^k(\lambda_n)), u_m^k(\lambda_n) - u_n^k\rangle \to 0 \quad \text{as} \quad m \to \infty.$$

Then we have

$$u_m^k(\lambda_n) \to u_n^k \quad \text{in } S^2_{\gamma,b(x)}(\mathbb{R}^N) \quad \text{as} \quad m \to \infty.$$

□

Therefore, without loss of generality, we may assume that

$$\lim_{m\to\infty} u_m^k(\lambda_n) = u_n^k, \quad \forall n \in \mathbb{N},\ k \geq k_1.$$

As a consequence, we obtain

$$\Phi'_{\lambda_n}(u_n^k) = 0,\ \Phi_{\lambda_n}(u_n^k) \in [c_k, d_k], \quad \forall n \in \mathbb{N},\ k \geq k_1. \tag{3.11}$$

Lemma 3.4. *For any $k \geq k_1$, the sequence $\{u_n^k\}_{n\in\mathbb{N}}$ is bounded.*

Proof. For simplicity we set $u_n = u_n^k$. We suppose by contradiction that, up to a subsequence,

$$\|u_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \to \infty \quad \text{as } n \to \infty. \tag{3.12}$$

Let $w_n = u_n/\|u_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}$ for any $n \in \mathbb{N}$. Then, up to subsequence, we may assume that

$$\begin{aligned} w_n &\rightharpoonup w \quad \text{in } S^2_{\gamma,b(x)}(\mathbb{R}^N), \\ w_n &\to w \quad \text{in } L^2(\mathbb{R}^N)\cap L^p(\mathbb{R}^N), \\ w_n &\to w \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.13}$$

Now we distinguish two cases.

Case $w = 0$. We can say that for any $n \in \mathbb{N}$ there exists $t_n \in [0,1]$ such that

$$\Phi_{\lambda_n}(t_n u_n) = \max_{t\in[0,1]} \Phi_{\lambda_n}(tu_n). \tag{3.14}$$

Since (3.12) holds, for any $j \in \mathbb{N}$, we can choose $r_j = (4j)^{1/2} w_n$ such that

$$r_j \|u_n\|^{-1}_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \in (0,1) \tag{3.15}$$

provided n is large enough. By (3.13), $F(\cdot,0) = 0$ and the continuity of F, we can see that

$$F(x, r_j w_n) \to F(x, r_j w) = 0 \quad \text{a.e. } x \in \mathbb{R}^N \tag{3.16}$$

as $n \to \infty$ for any $j \in \mathbb{N}$. Then, taking into account (3.13), (3.16), (B3), (B4) and by using the Dominated Convergence Theorem we deduce that

$$F(x, r_j w_n) \to 0 \quad \text{in } L^1(\mathbb{R}^N) \tag{3.17}$$

as $n \to \infty$ for any $j \in \mathbb{N}$. Then (3.14), (3.15) and (3.17) yield

$$\Phi_{\lambda_n}(t_n u_n) \geq \Phi_{\lambda_n}(r_j w_n) \geq 2j - \lambda_n \int_{\mathbb{R}^N} F(x, r_j w_n)\mathrm{d}x \geq j$$

provided n is large enough and for any $j \in \mathbb{N}$. As a consequence

$$\Phi_{\lambda_n}(t_n u_n) \to \infty \quad \text{as } n \to \infty. \tag{3.18}$$

Since $\Phi_{\lambda_n}(0) = 0$ and $\Phi_{\lambda_n}(u_n) \in [c_k, d_k]$, we deduce that $t_n \in (0,1)$ for n large enough. Thus, by (3.14) we have

$$\langle \Phi'_{\lambda_n}(t_n u_n), t_n u_n \rangle = t_n \frac{d}{dt}\Big|_{t=t_n} \Phi_{\lambda_n}(tu_n) = 0. \tag{3.19}$$

Taking into account (B5) and (3.19), we obtain

$$\begin{aligned}
\frac{1}{\theta}\Phi_{\lambda_n}(t_n u_n) &= \frac{1}{\theta}\Big(\Phi_{\lambda_n}(t_n u_n) - \frac{1}{2}\langle \Phi'_{\lambda_n}(t_n u_n), t_n u_n\rangle\Big) \\
&= \frac{\lambda_n}{2\theta}\int\limits_{\mathbb{R}^N} \mathcal{F}(x, t_n u_n)\mathrm{d}x \\
&\leq \frac{\lambda_n}{2}\int\limits_{\mathbb{R}^N} \mathcal{F}(x, u_n)\mathrm{d}x \\
&= \Phi_{\lambda_n}(u_n) - \frac{1}{2}\langle \Phi'_{\lambda_n}(u_n), u_n\rangle = \Phi_{\lambda_n}(u_n)
\end{aligned}$$

which contradicts (3.11) and (3.18).

Case $w \not\equiv 0$. Thus the set $\Omega := \{x \in \mathbb{R}^N : w(x) \neq 0\}$ has positive Lebesgue measure. By using (3.12) and that $w \not\equiv 0$, we have

$$u_n(x) \to \infty \quad \text{a.e. } x \in \Omega \quad \text{as } n \to \infty. \tag{3.20}$$

Putting together (3.13), (3.20), and (B4), and by applying Fatou's Lemma, we can easily deduce that

$$\begin{aligned}
\frac{1}{2} - \frac{\Phi_{\lambda_n}(u_n)}{\|u_n\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}} &= \lambda_n \int\limits_{\mathbb{R}^N} \frac{F(x, u_n(x))}{\|u_n\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}}\mathrm{d}x \\
&\geq \lambda_n \int\limits_{\Omega} |w_n|^2 \frac{F(x, u_n(x))}{|u_n|^2}\mathrm{d}x \to \infty \text{ as } n \to \infty
\end{aligned}$$

which gives a contradiction because of (3.11).

Then, we have proved that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$. □

Lemma 3.5. *Let* (B1)–(B3) *and* (B8) *be satisfied. Then* Φ *satisfies the* $(C)_c$ *condition for all* $c > 0$ *on* $S^2_{\gamma,b(x)}(\mathbb{R}^N)$.

Proof. The proof of this lemma is similar to the one of Lemma 3.1 in [17]. We omit the details. □

Theorem 3.6. *Assume that b and f satisfy* (B1)–(B4), (B6) *and* (B7). *Then the problem* (1.2) *possesses infinitely many nontrivial solutions.*

Proof. The proof of this theorem is similar to the one of Theorem 1.1 in [17]. We omit the details. □

Proof of Theorem 1.1. Taking into account Lemma 3.4 and (3.11), for each $k \geq k_1$, we can use similar arguments to those in the proof of Lemma 3.3, to show that the sequence $\{u_n^k\}_{n\in\mathbb{N}}$ admits a strong convergent subsequence with the limit u^k being just a critical point of $\Phi_1 = \Phi$. Clearly, $\Phi(u^k) \in [c_k, d_k]$ for all $k \geq k_1$. Since $c_k \to \infty$ as $k \to \infty$ in (3.8), we deduce the existence of infinitely many nontrivial critical points of Φ. As a consequence, we have that (1.2) possesses infinitely many nontrivial weak solutions. □

Proof of Theorem 1.2. From (B5), we have

$$\Phi(u) = \frac{1}{2}\|u\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} - \int\limits_{\mathbb{R}^N} F(x,u)\mathrm{d}x$$
$$= \int\limits_{\mathbb{R}^N} \left[\frac{1}{2}f(x,u)u - F(x,u)\right]\mathrm{d}x \geq 0,$$

and so $m = \inf\limits_{u\in\mathcal{M}} \Phi(u) \geq 0$. From Theorem 3.6 we choose a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ such that $\Phi(u_n) \to m$, as $n \to \infty$, and $\|\Phi'(u_n)\|_{(S^2_{\gamma,b(x)}(\mathbb{R}^N))^*}(1 + \|u_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}) = 0$ by Lemma 3.5, we have there exists $u_0 \in S^2_{\gamma,b(x)}(\mathbb{R}^N)$ such that $u_n \to u_0$ in $S^2_{\gamma,b(x)}(\mathbb{R}^N)$. Since $\Phi \in C^1(S^2_{\gamma,b(x)}(\mathbb{R}^N),\mathbb{R})$, one has

$$\Phi(u_0) = \lim_{n\to\infty}\Phi(u_n) = m, \quad \Phi'(u_0) = \lim_{n\to\infty}\Phi'(u_n).$$

Hence, we obtain that u_0 is also a critical point of Φ and $\Phi(u_0) = \inf\limits_{u\in\mathcal{M}}\Phi(u)$. Furthermore, under assumptions (B7) and (B3), we have

$$|f(x,u_n)| \leq \varepsilon|u_n| + C_\varepsilon|u_n|^{p-1}, \quad \forall\varepsilon > 0 \tag{3.21}$$

for $2 < p < 2^*_\gamma$. By (3.21) and Proposition 2.2, we have

$$\begin{aligned}\|u_n\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} &= \int\limits_{\mathbb{R}^N} f(x,u_n)u_n\mathrm{d}x \\ &\leq \varepsilon\|u_n\|^2_{L^2(\mathbb{R}^N)} + C_\varepsilon\|u_n\|^p_{L^p(\mathbb{R}^N)} \\ &\leq C_1\varepsilon\|u_n\|^2_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} + C_\varepsilon C_2\|u_n\|^p_{S^2_{\gamma,b(x)}(\mathbb{R}^N)}.\end{aligned} \tag{3.22}$$

For sufficiently small $\varepsilon > 0$, (3.22) implies that there exists a constant $\omega > 0$ such that

$$\|u_0\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} = \lim_{n\to\infty}\|u_n\|_{S^2_{\gamma,b(x)}(\mathbb{R}^N)} \geq \omega > 0.$$

Thus $u_0 \neq 0$. □

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