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# Convergence to Attractors of Nonexpansive Set-Valued Mappings

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**Abstract**

In our previous work we have shown that if for any initial point there exists a trajectory of a nonexpansive set-valued mapping attracted by a given set, then this property is stable under small perturbations of the mapping. In the present paper we obtain several extensions of this result.

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## 1 Introduction and preliminaries

During more than fifty-five years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 4, 6, 7, 8, 9, 10, 12, 13, 15, 18, 19] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [3, 5, 16, 17, 18, 19].

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In our previous work [14] we have shown that if for any initial point there exists a trajectory of a nonexpansive set-valued mapping attracted by a given set, then this property is stable under small perturbations of the mapping. More precisely, we have proved in [14] two assertions. In the first assertion we have shown that for a given positive number $\delta$, if the perturbations are small enough, then for any initial state there exists a trajectory which is attracted by a $\delta$-neighborhood of the attractor. In the second assertion we have shown that under the same assumptions, for any initial state there exists a trajectory with a subsequence which is attracted by the attractor. In the present paper we obtain several extensions of these assertions.

Let $(X,\rho)$ be a metric space. For each $x \in X$ and each nonempty set $A \subset X$, define

$$\rho(x,A) := \inf\{\rho(x,y) : \ y \in A\}.$$

For each pair of nonempty sets $A, B \subset X$, define

$$H(A,B) := \max\{\sup_{x\in A}\rho(x,B), \ \sup_{y\in B}\rho(y,A)\}.$$

Let a set-valued mapping $T : X \to 2^X \setminus \{\emptyset\}$ satisfy

$$H(T(x),T(y)) \le \rho(x,y) \text{ for all } x,y \in X. \tag{1.1}$$

Assume that a sequence $\{\epsilon_i\}_{i=0}^{\infty} \subset (0,\infty)$ satisfies

$$\sum_{i=0}^{\infty} \epsilon_i < \infty \tag{1.2}$$

and that for each integer $i \ge 0$, a set-valued mapping $T_i : X \to 2^X \setminus \{\emptyset\}$ satisfies

$$H(T_i(x),T(x)) \le \epsilon_i, \ \ x \in X. \tag{1.3}$$

Let $F$ be a nonempty subset of $X$.

The following theorem is the main result of [14].

**Theorem 1.1.** *Assume that for each $x \in X$, there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, $x_{i+1} \in T(x_i)$ for all integers $i \ge 0$, and*

$$\lim_{i\to\infty} \rho(x_i,F) = 0.$$

*Then the following two assertions hold.*

*1. Let $\delta > 0$. For each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, for each integer $i \ge 0$,*

$$x_{i+1} \in T_i(x_i),$$

*and*

$$\rho(x_i,F) \le \delta \textit{ for all sufficiently large integers } i \ge 0.$$

*2. For each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, for each integer $i \ge 0$,*

$$x_{i+1} \in T_i(x_i),$$

*and*

$$\liminf_{i\to\infty} \rho(x_i,F) = 0.$$

## 2 First result

The following result was obtained in [11]. Here we present an alternative proof.

**Theorem 2.1.** *Assume that*

$$T(x) \cap F \neq \emptyset \text{ for all } x \in F. \tag{2.1}$$

*The following two properties are equivalent:*

*(a) for each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, $x_{i+1} \in T(x_i)$ for all integers $i \geq 0$, and*

$$\liminf_{i \to \infty} \rho(x_i, F) = 0;$$

*(b) for each $\delta > 0$ and each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, for each integer $i \geq 0$,*

$$x_{i+1} \in T(x_i),$$

*and*

$$\rho(x_i, F) \leq \delta \text{ for all sufficiently large integers } i \geq 0.$$

*Proof.* Assume first that (a) is true. To show that (b) holds, let $\delta > 0$ and $x \in X$ be given. In view of (a), there exist a natural number $t_0$ and a finite sequence $\{x_i\}_{i=0}^{t_0} \subset X$ such that

$$x_0 = x, \tag{2.2}$$

$$x_{i+1} \in T(x_i),\ i = 0, 1, \ldots, t_0 - 1, \tag{2.3}$$

and

$$\rho(x_{t_0}, F) < \delta/2. \tag{2.4}$$

Using induction, we now define $x_t \in X$ for all integers $t > t_0$ such that for each integer $t \geq t_0$,

$$x_{t+1} \in T(x_t) \tag{2.5}$$

and

$$\rho(x_t, F) < \delta/2. \tag{2.6}$$

Assume that $\tau \geq t_0$ is an integer, $x_t \in X$, $t = 0, \ldots, \tau$, are defined such that (2.5) is true for each $t = 0, \ldots, \tau - 1$ and that (2.6) is true for each $t = t_0, \ldots, \tau$. In particular, we have

$$\rho(x_\tau, F) < \delta/2. \tag{2.7}$$

In view of (2.7), there exists

$$z_1 \in F \tag{2.8}$$

such that

$$\rho(x_F, z_1) < \delta/2. \tag{2.9}$$

By (1.4) and (2.8), there exists

$$z_2 \in T(z_1) \cap F. \tag{2.10}$$

In view of (1.1),

$$H(T(x_\tau), T(z_1)) \leq \rho(x_\tau, z_1). \tag{2.11}$$

By (2.9)–(2.11),

$$\rho(z_2, T(x_\tau)) \le H(T(x_\tau), T(z_1)) < \delta/2.$$

This implies that there exists a point

$$x_{\tau+1} \in T(x_\tau)$$

such that

$$\rho(z_2, x_{\tau+1}) < \delta/2.$$

When combined with (2.10), this implies that

$$\rho(x_{\tau+1}, F) < \delta/2.$$

Thus the assumption made for $\tau$ also holds for $\tau+1$. This means that we have constructed by induction a sequence $\{x_t\}_{t=0}^\infty$ which satisfies (2.5) for all integers $t \ge 0$ and satisfies (2.6) for all integers $t \ge \tau_0$. In other words, property (b) holds.

Assume now that (b) is true. In order to show that property (a) holds, let $x \in X$ be given. By property (b), there exists a finite sequence $\{x_i\}_{i=0}^{S_1} \subset X$ such that

$$x_0 = x, \tag{2.12}$$

$$x_{i+1} \in T(x_i) \tag{2.13}$$

for all $i = 0, \ldots, S_1 - 1$, and

$$\rho(x_{S_1}, F) < 2^{-1}. \tag{2.14}$$

Assume that $q \ge 1$ is an integer and that we defined natural numbers

$$S_1 < \cdots < S_q$$

and a finite sequence $\{x_t\}_{t=0}^{S_q} \subset X$ such that (2.12) holds and for each $i = 0, \ldots, S_q - 1$, (2.13) holds, and

$$\rho(x_{S_k}, F) < 2^{-k} \tag{2.15}$$

for all $k = 1, \ldots, q$. (Note that for $q = 1$ our assumption holds in view of (2.12)–(2.14)). Property (b) implies that there exist an integer $S_{q+1} > S_q$ and points $x_t$, $t = S_q + 1, \ldots, S_{q+1}$, such that (2.13) holds for $t = S_q, \ldots, S_{q+1} - 1$, and

$$\rho(x_{S_{q+1}}, F) < 2^{-q-1}.$$

Thus the assumption made for $q$ also holds for $q+1$. This means that we have constructed by induction a strictly increasing sequence of natural numbers $\{S_i\}_{i=1}^\infty$ and a sequence of points $\{x_t\}_{t=0}^\infty \subset X$ such that (2.12) holds, (2.13) holds for all $i = 1, 2, \ldots$, and (2.15) holds for $k = 1, 2, \ldots$. By (2.15),

$$\liminf_{t\to\infty} \rho(x_t, F) = 0.$$

In other words, property (a) holds. This completes the proof of Theorem 2.1. □

# 3 Second result

**Theorem 3.1.** *Assume that*

$$z \in T(z) \text{ for all } z \in F \tag{3.1}$$

*and that the following property holds:*

*(a) for each $x \in X$. there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, $x_{i+1} \in T(x_i)$ for all integers $i \geq 0$, and*

$$\liminf_{i \to \infty} \rho(x_i, F) = 0.$$

*Then for each $\delta > 0$ and each $x \in X$, there exist a point $z \in F$ and a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, for each integer $i \geq 0$,*

$$x_{i+1} \in T(x_i),$$

*and*

$$\rho(x_i, z) < \delta \text{ for all sufficiently large integers } i \geq 0.$$

*Proof.* Let $\delta > 0$ and $x \in X$ be given. In view of (a), there exist a natural number $t_0$ and a finite sequence $\{x_i\}_{i=0}^{t_0} \subset X$ such that

$$x_0 = x, \tag{3.2}$$

$$x_{i+1} \in T(x_i),\ i = 0, 1, \ldots, t_0 - 1, \tag{3.3}$$

and

$$\rho(x_{t_0}, F) < \delta/2. \tag{3.4}$$

In view of (3.4), there exists a point

$$z \in F \tag{3.5}$$

such that

$$\rho(x_{t_0}, z) < \delta/2. \tag{3.6}$$

We are now going to define by induction points $x_t \in X$ for all integers $t > t_0$ such that for each integer $t \geq t_0$,

$$x_{t+1} \in T(x_t) \tag{3.7}$$

and

$$\rho(x_t, z) < \delta/2. \tag{3.8}$$

To this end, assume that $\tau \geq t_0$ is an integer, $x_t \in X$, $t = 0, \ldots, \tau$, are defined such that (3.2) is true, (3.7) holds for each $t = 0, \ldots, \tau - 1$, and (3.8) is true for each integer $t \in [t_0, \tau]$. (Note that in view of (3.3) and (3.6), our assumption does hold for $\tau = t_0$.) In particular, we have

$$\rho(x_\tau, z) < \delta/2. \tag{3.9}$$

In view of (3.1) and (3.5),

$$z \in T(z). \tag{3.10}$$

It follows from (1.1) and (3.9) that

$$H(T(x_\tau), T(z)) \leq \rho(x_\tau, z) < \delta/2. \tag{3.11}$$

By (3.10) and (3.11),

$$\rho(z, T(x_\tau)) < \delta/2.$$

Therefore there exists a point

$$x_{\tau+1} \in T(x_\tau)$$

such that

$$\rho(z, x_{\tau+1}) < \delta/2.$$

Thus our assumption concerning $\tau$ also holds for $\tau + 1$. This means that we have constructed by induction a sequence $\{x_t\}_{t=0}^{\infty} \subset X$ which satisfies (3.2), (3.7) for all integers $t \geq 0$, and (3.8) for all integers $t \geq t_0$. Theorem 3.1 is proved. □

## 4 Auxiliary result

In the sequel we use the following result which was obtained in [14].

**Lemma 4.1.** *Let $q \geq 0$ be an integer and let a sequence $\{x_i\}_{i=q}^{\infty} \subset X$ satisfy*

$$x_{i+1} \in T(x_i)$$

*for each integer $i \geq q$. Then there exists a sequence $\{y_i\}_{i=q}^{\infty} \subset X$ such that*

$$y_q = x_q,$$

$$y_{i+1} \in T_i(y_i) \text{ for all integers } i \geq q$$

*and for all integers $j \geq q + 1$, we have*

$$\rho(y_j, x_j) \leq \sum_{i=q}^{j-1} 2\epsilon_i.$$

## 5 Third result

**Theorem 5.1.** *Assume that for each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, $x_{i+1} \in T(x_i)$ for all integers $i \geq 0$ and*

$$\liminf_{i \to \infty} \rho(x_i, F) = 0.$$

*Then for each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, for each integer $i \geq 0$,*

$$x_{i+1} \in T_i(x_i),$$

*and*

$$\liminf_{i \to \infty} \rho(x_i, F) = 0.$$

Theorem 5.1 is the main result of [11].

## 6 Fourth result

**Theorem 6.1.** *Assume that for each $\delta > 0$ and each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, for each integer $i \geq 0$,*

$$x_{i+1} \in T(x_i),$$

*and*

$$\rho(x_i, F) < \delta \text{ for all sufficiently large integers } i \geq 0.$$

*Let $\delta > 0$ and each $x \in X$. Then there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, for each integer $i \geq 0$,*

$$x_{i+1} \in T_i(x_i),$$

*and*

$$\rho(x_i, F) < \delta \text{ for all sufficiently large integers } i \geq 0.$$

*Proof.* There exists a natural number $q$ such that

$$\sum_{i=q}^{\infty} \epsilon_i < 4^{-1}\delta. \tag{6.1}$$

There exists a finite sequence $\{x_i\}_{i=0}^{q} \subset X$ such that

$$x_0 = x \tag{6.2}$$

and

$$x_{i+1} \in T_i(x_i),\ i = 0, 1, \ldots, q-1. \tag{6.3}$$

By assumption, there exists a sequence $\{u_i\}_{i=q}^{\infty} \subset X$ such that

$$u_q = x_q, \tag{6.4}$$

$$u_{i+1} \in T(u_i),\ i = q, q+1, \ldots, \tag{6.5}$$

and

$$\rho(u_i, F) < \delta/2 \text{ for all sufficiently large integers } i. \tag{6.6}$$

By (6.6), there exists an integer

$$p > q \tag{6.7}$$

such that

$$\rho(u_i, F) < \delta/2 \text{ for all integers } i \geq p. \tag{6.8}$$

Lemma 4, (6.4) and (6.5) imply that there exists a sequence of points $\{x_i\}_{i=q}^{\infty} \subset X$ such that

$$x_{i+1} \in T_i(x_i) \text{ for all integers } i \geq q,$$

and for all integers $j > q$,

$$\rho(u_j, x_j) \leq \sum_{i=q}^{j-1} 2\epsilon_i. \tag{6.9}$$

By (6.1), (6.8) and (6.9), for all integers $j \geq p$,

$$\rho(x_j, F) \leq \rho(x_j, u_j) + \rho(u_j, F)$$
$$\leq \sum_{i=q}^{j-1} 2\epsilon_i + \delta/2 < \delta.$$

This completes the proof of Theorem 6.1. □

## 7 Fifth result

**Theorem 7.1.** *Assume that for each $\delta > 0$ and each $x \in X$, there exist a point $z \in F$ and a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, for each integer $i \geq 0$,*

$$x_{i+1} \in T(x_i)$$

*and*

$$\rho(x_i, z) < \delta \text{ for all sufficiently large integers } i \geq 0.$$

*Let $\delta > 0$ and $x \in X$. Then there exist a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ and a point $z \in F$ such that $x_0 = x$, for each integer $i \geq 0$,*

$$x_{i+1} \in T_i(x_i),$$

*and*

$$\rho(x_i, z) < \delta \text{ for all sufficiently large integers } i \geq 0.$$

*Proof.* There exists a natural number $q$ such that

$$\sum_{i=q}^{\infty} \epsilon_i < 4^{-1}\delta. \tag{7.1}$$

There also exists a finite sequence $\{x_i\}_{i=0}^{q} \subset X$ such that

$$x_0 = x \tag{7.2}$$

and

$$x_{i+1} \in T_i(x_i),\ i = 0, 1, \ldots, q-1. \tag{7.3}$$

By assumption, there exist a sequence $\{u_i\}_{i=q}^{\infty} \subset X$ and a point $z \in F$ such that

$$u_q = x_q, \tag{7.4}$$

$$u_{i+1} \in T(u_i),\ i = q, q+1, \ldots, \tag{7.5}$$

and

$$\rho(u_i, z) < \delta/2 \text{ for all sufficiently large integers } i \geq 0. \tag{7.6}$$

By (7.6), there exists an integer

$$p > q \tag{7.7}$$

such that

$$\rho(u_i, z) < \delta/2 \text{ for all integers } i \geq p. \tag{7.8}$$

Lemma 4, (7.4) and (7.5) imply that there exists a sequence $\{x_i\}_{i=q}^{\infty} \subset X$ such that

$$x_{i+1} \in T_i(x_i) \text{ for all integers } i \geq q, \tag{7.9}$$

and for all integers $j > q$,

$$\rho(u_j, x_j) \leq \sum_{i=q}^{j-1} 2\epsilon_i. \tag{7.10}$$

By (7.1), (7.7), (7.8) and (7.10), we have for all integers $j \geq p$,

$$\rho(x_j, z) \leq \rho(x_j, u_j) + \rho(u_j, z)$$

$$< \sum_{i=q}^{j-1} 2\epsilon_i + \delta/2 < \delta.$$

This completes the proof of Theorem 7.1. □

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