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Existence and Regularity of a Weak Solution to the Maxwell-Stokes Type System Containing p-curlcurl Equation

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Abstract

We consider the existence and regularity of a weak solution to the Maxwell-Stokes type system containing a p-curlcurl equation in a multiply-connected domain with holes. In this paper, we shows that the compatibility condition is necessary and sufficient for the existence of a weak solution to the Maxwell-Stokes type system, and that the unique solution has the $C^{1,\beta}$-regularity. The $C^{1,\beta}$-regularity is optimal.

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Keywords: Maxwell-Stokes type system, weak solution, p-curlcurl operator, regularity, multiply-connected domain with holes.

1 Introduction

In this paper, we consider the existence and regularity of a weak solution to the Maxwell and the Maxwell-Stokes type systems containing p-curlcurl equations in a bounded multiply-connected domain Ω with holes in $\mathbb{R}^3$.

In a bounded simply connected domain Ω in $\mathbb{R}^3$ without holes, Yin [18] considered the existence of a unique weak solution for the nonlinear Maxwell system, so-called, p-curlcurl system

$$\begin{cases} \operatorname{curl}[|\operatorname{curl}\boldsymbol{v}|^{p-2}\operatorname{curl}\boldsymbol{v}] = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma \end{cases} \tag{1.1}$$

where Γ denotes the $C^{2,\alpha}$ ($\alpha \in (0,1)$) boundary of Ω, $p > 1$, $\boldsymbol{n}$ the outer normal unit vector field on Γ, and $\boldsymbol{f}$ is a given vector field satisfying $\operatorname{div}\boldsymbol{f} = 0$ in Ω. If $\boldsymbol{f}$ is a C^α vector-valued

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function, then he showed the optimal $C^{1,\beta}$-regularity for some $\beta \in (0,1)$ of a weak solution in Yin [19]. See also Yin et al. [20].

The equation (1.1) is a steady-state approximation of Bean's critical state model for type II superconductors. For further physical background, see [20], Chapman [8] and Prigozhin [17].

Aramaki [4] extended the result of [19] on the $C^{1,\beta}$ regularity of a weak solution to a more general equation, in a simply connected domain without holes to the following system.

$$\begin{cases} \operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{v}|^2|)\operatorname{curl}\boldsymbol{v}] = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{v} = \boldsymbol{0} & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where the function $S(x,t) \in C^2(\Omega\times(0,\infty)) \cap C^0(\Omega\times[0,\infty))$ satisfies some structure conditions.

However, in a multiply-connected domain with holes, the systems (1.1) and (1.2) are not well posed. In fact, when the second Betti number of Ω is positive, if $\boldsymbol{v}$ is a weak solution of (1.1) or (1.2), then $\boldsymbol{v}+\boldsymbol{z}$, where $\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$ satisfies $\operatorname{curl}\boldsymbol{z} = \boldsymbol{0}, \operatorname{div}\boldsymbol{z} = 0$ in Ω and $\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ, is also a weak solution. Thus it is necessary to add some conditions to (1.1) and (1.2). Moreover, if $\boldsymbol{f}$ is not divergence free, it is natural to consider the existence of the potential π, that is, the Maxwell-Stokes type equation instead of the first equation of (1.2):

$$\operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v}] + \nabla\pi = \boldsymbol{f} \text{ in } \Omega. \tag{1.3}$$

The solvability of such systems depends on the nature of the nonlinearity of the equations and the type of the boundary conditions, and on the shape of the domain.

In this paper, we show a necessary and sufficient condition for the existence of the weak solution to the Maxwell-Stokes system (2.7a)-(2.7e) below in a multi-connected domain, and the optimal $C^{1,\beta}$-regularity of the weak solution.

The paper is organized as follows. In section 2, we give some preliminaries and state a theorem (Theorem 2.5) on the existence of a weak solution. In section 3, we give a proof of Theorem 2.5. We derive that the condition of Theorem 2.5 is necessary and sufficient for the existence of a weak solution. To do so, we reduce the problem to the Maxwell system without the potential. Section 4 is devoted to the regularity of the weak solution obtained in section 3.

2 Preliminaries

In this section, we introduce the shape of the domain and some spaces of vector-valued functions, and we state a theorem on the existence of a weak solution to the Maxwell-Stokes type problem.

Since we allow that Ω is a multiply-connected domain with holes, we assume that Ω has the following conditions as in Amrouche and Seloula [2] (cf. Amrouche and Seloula [1], Dautray and Lions [10] and Girault and Raviart [13]). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{r,\alpha}$ $(r \geq 1, 0 < \alpha \leq 1)$ with a boundary Γ and Ω be locally situated on one side of Γ. Moreover, we assume the following.

(O1) Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_I$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(O2) There exist n connected open surfaces Σ_j, $(j = 1, \dots, J)$, called cuts, contained in Ω such that

(a) Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$.

(b) $\partial\Sigma_j \subset \Gamma$ $(j = 1, \dots, J)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ.

(c) $\overline{\Sigma_j} \cap \overline{\Sigma_k} = \emptyset (j \neq k)$.

(d) The open set $\dot{\Omega} = \Omega \setminus (\cup_{j=1}^J \Sigma_j)$ is simply connected and pseudo $C^{1,1}$ class.

The number J is called the first Betti number which is equal to the number of handles of Ω, and I is called the second Betti number which is equal to the number of holes. If $J = 0$, then we say that Ω is simply connected, and if $I = 0$, then we say that Ω has no holes.

From now on we use the notations $L^p(\Omega)$ $(1 < p < \infty)$, $W^{m,p}(\Omega)$ $(m \geq 0, \text{integer})$, $W^{s,p}(\Gamma)$ $(s \in \mathbb{R}), C^{m,\alpha}(\Omega)$, and so on, for the standard L^p, Sobolev and Hölder spaces of functions, respectively. For any Banach space B, we denote $B \times B \times B$ by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. For the dual space $\boldsymbol{B}'$ of $\boldsymbol{B}$, we denote the duality bracket between $\boldsymbol{B}'$ and $\boldsymbol{B}$ by $\langle \cdot, \cdot \rangle_{\boldsymbol{B}',\boldsymbol{B}}$. For $1 < p < \infty$, we denote the conjugate exponent of p by p', that is, $(1/p) + (1/p') = 1$.

Define two spaces by

$$
\begin{aligned}
\mathbb{K}_N^p(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} = 0, \operatorname{curl} \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}, \\
\mathbb{K}_T^p(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} = 0, \operatorname{curl} \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.
\end{aligned}
$$

For any function $q \in W^{1,p}(\dot{\Omega})$, we write an extension of $\nabla q \in \boldsymbol{L}^p(\dot{\Omega})$ to $\boldsymbol{L}^p(\Omega)$ by $\widetilde{\nabla} q$. Let q_j^T $(j = 1, \dots J)$ be the unique solution in $W^{2,p}(\dot{\Omega})$ of the system

$$
\begin{cases}
-\Delta q_j^T = 0 & \text{in } \dot{\Omega}, \\
\boldsymbol{n} \cdot \nabla q_j^T = 0 & \text{on } \Gamma, \\
[q_j^T]_{\Sigma_k} = \text{const.}, \quad [\boldsymbol{n} \cdot \nabla q_j^T]_{\Sigma_k} = 0 & k = 1, \dots, J \\
\langle \boldsymbol{n} \cdot \nabla q_j^T, 1 \rangle_{\Sigma_k} = \delta_{jk} & k = 1, \dots, J,
\end{cases}
\tag{2.1}
$$

where $[q_j^T]_{\Sigma_k}$ is the jump of the function of q_j^T across Σ_k and

$$
\langle \boldsymbol{n} \cdot \nabla q_j^T, 1 \rangle_{\Sigma_k} = \int_{\Sigma_k} \boldsymbol{n} \cdot \nabla q_j^T \, dS,
$$

dS is the surface measure of Σ_k. Then according to [2, Corollary 4.1], $\{\widetilde{\nabla} q_j^T; j = 1, \dots, J\}$ is a basis of $\mathbb{K}_T^p(\Omega)$. We note that if Ω is of class $C^{2,\alpha}$, then $\widetilde{\nabla} q_j^T \in C^{1,\alpha}(\overline{\Omega})$.

On the other hand, let q_i^N $(i = 1, \dots I)$ be the unique solution in $W^{2,p}(\Omega)$ of the system

$$
\begin{cases}
-\Delta q_i^N = 0 & \text{in } \Omega, \\
q_i^N\big|_{\Gamma_0} = 0, \quad q_i^N\big|_{\Gamma_k} = \text{const.} & k = 1, \dots I, \\
\langle \boldsymbol{n} \cdot \nabla q_i^N, 1 \rangle_{\Gamma_k} = \delta_{ik} \ (k = 1, \dots I) & \langle \boldsymbol{n} \cdot \nabla q_i^N, 1 \rangle_{\Gamma_0} = -1,
\end{cases}
\tag{2.2}
$$

where

$$\langle \boldsymbol{n}\cdot\nabla q_i^N, 1\rangle_{\Gamma_k} = \int_{\Gamma_k} \boldsymbol{n}\cdot\nabla q_i^N dS,$$

dS is the surface measure of Γ_k. Then $\{\nabla q_i^N; i = 1,\dots I\}$ is a basis of $\mathbb{K}_N^p(\Omega)$ (cf. [2, Corollary 4.2]). We note that if Ω is of class $C^{2,\alpha}$, then $\nabla q_i^N \in \boldsymbol{C}^{1,\alpha}(\overline{\Omega})$. Thus we can see that $\dim\mathbb{K}_T^p(\Omega) = J$ and $\dim\mathbb{K}_N^p(\Omega) = I$.

We assume that a Carathéodory function $S(x,t)$ on $\Omega\times[0,\infty)$ satisfies that for a.e. $x\in\Omega$, $S(x,t)\in C^2((0,\infty))\cap C([0,\infty))$ and there exist a constant $1 < p < \infty$ and positive constants $0 < \lambda \le \Lambda < \infty$ such that for a.e. $x\in\Omega$ and $t > 0$

$$S(x,0) = 0 \text{ and } \lambda t^{(p-2)/2} \le S_t(x,t) \le \Lambda t^{(p-2)/2}, \tag{2.3a}$$

$$\lambda t^{(p-2)/2} \le S_t(x,t) + 2tS_{tt}(x,t) \le \Lambda t^{(p-2)/2}, \tag{2.3b}$$

$$S_{tt}(x,t) < 0 \text{ if } 1 < p < 2 \text{ and } S_{tt}(x,t) \ge 0 \text{ if } p \ge 2, \tag{2.3c}$$

where $S_t = \partial S/\partial t, S_{tt} = \partial^2 S/\partial t^2$. We note that from (2.3a), we have

$$\frac{2}{p}\lambda t^{p/2} \le S(x,t) \le \frac{2}{p}\Lambda t^{p/2} \text{ for } t \ge 0. \tag{2.4}$$

For a.e $x\in\Omega$, the function $G(t) = S(x,t^2)$ satisfies $G'(t) = 2tS_t(x,t^2)$ and

$$G''(t) = 2(S_t(x,t^2) + 2t^2S(x,t^2)) \ge 2\lambda t^{p-2} > 0 \text{ for } t > 0.$$

Hence, $G(t)$ is a strictly convex function in $[0,\infty)$.

Example 2.1. *When $S(x,t) = \nu(x)t^{p/2}$, where ν is a measurable function on Ω, and satisfies $0 < \nu_* \le \nu(x) \le \nu^* < \infty$, then it follows from elementary calculations that* (2.3a)-(2.3c) *hold.*

We give a lemma on a monotonic property of S_t.

Lemma 2.2. *There exists a constant $c > 0$ depending only on p and λ such that for all $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$,*

$$(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) \ge \begin{cases} c|\boldsymbol{a}-\boldsymbol{b}|^p & \text{if } p \ge 2, \\ c(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}|^2 & \text{if } 1 < p < 2. \end{cases}$$

In particular, S_t is strictly monotonic, that is,

$$(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) > 0 \text{ if } \boldsymbol{a}\ne\boldsymbol{b}.$$

For the proof, see Aramaki [6, Lemma 3.6].

We introduce some spaces of vector-valued functions. Define a space

$$\mathbb{X}^p(\Omega) = \{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega); \operatorname{curl}\boldsymbol{v}\in\boldsymbol{L}^p(\Omega), \operatorname{div}\boldsymbol{v}\in L^p(\Omega)\}$$

with the norm

$$\|\boldsymbol{v}\|_{\mathbb{X}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)}.$$

Then $\mathbb{X}^p(\Omega)$ is a Banach space. We note that if $\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)$ and $\operatorname{curl}\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)$, then the tangent trace $\boldsymbol{v}\times\boldsymbol{n}\in\boldsymbol{W}^{-1/p,p}(\Gamma)$ is well defined, and if $\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)$ and $\operatorname{div}\boldsymbol{v}\in L^p(\Omega)$, then the normal trace $\boldsymbol{v}\cdot\boldsymbol{n}\in W^{-1/p,p}(\Gamma)$ is well defined. Furthermore, we define two closed subspaces of $\mathbb{X}^p(\Omega)$ by

$$\mathbb{X}^p_N(\Omega)=\{\boldsymbol{v}\in\mathbb{X}^p(\Omega);\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}\text{ on }\Gamma\}.$$

and

$$\mathbb{V}^p_N(\Omega)=\{\boldsymbol{v}\in\mathbb{X}^p_N(\Omega);\operatorname{div}\boldsymbol{v}=0\text{ in }\Omega,\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0\text{ for }i=1,\dots,I\},$$

where $\langle\cdot,\cdot\rangle_{\Gamma_i}$ denote the duality bracket between $W^{-1/p,p}(\Gamma_i)$ and $W^{1-1/p',p'}(\Gamma_i)$. The following inequality is frequently used (cf. [2]). If we define

$$\mathbb{X}^{1,p}_N(\Omega)=\{\boldsymbol{v}\in\mathbb{X}^p(\Omega);\boldsymbol{v}\times\boldsymbol{n}\in\boldsymbol{W}^{1-1/p,p}(\Gamma)\},$$

then $\mathbb{X}^{1,p}_N(\Omega)\subset\boldsymbol{W}^{1,p}(\Omega)$, and there exists $C>0$ dependent only on p and Ω such that

$$\begin{aligned}&\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\\&\qquad\le C(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)}+\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}).\end{aligned}\tag{2.5}$$

Moreover, we can deduce the following (cf. [2, p. 40]). For any $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ,

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\le C(\|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)}+\sum_{i=1}^I|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|).\tag{2.6}$$

Thus we have

Lemma 2.3. *The space $\mathbb{V}^p_N(\Omega)$ is a reflexive, separable Banach space with the norm*

$$\|\boldsymbol{v}\|_{\mathbb{V}^p_N(\Omega)}=\|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$$

which is equivalent to the norm $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$.

We note that from the Sobolev embedding theorem, there exists a constant $C>0$ depending only on p and Ω such that for all $\boldsymbol{v}\in\mathbb{V}^p_N(\Omega)$,

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Gamma)}\le C\|\boldsymbol{v}\|_{\mathbb{V}^p_N(\Omega)},$$

where the second term of the left hand side denotes the norm of the trace of $\boldsymbol{v}$ on Γ.

In this paper, we consider the following Maxwell-Stokes type problem: to find $(\boldsymbol{u},\pi)$ such that

$$\operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}]+\nabla\pi=\boldsymbol{f}\text{ in }\Omega,\tag{2.7a}$$

$$\operatorname{div}\boldsymbol{u}=0\text{ in }\Omega,\tag{2.7b}$$

$$\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}\text{ on }\Gamma,\tag{2.7c}$$

$$\operatorname{div}_\Gamma(S_t(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}\times\boldsymbol{n})=g\text{ on }\Gamma,\tag{2.7d}$$

$$\langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0\text{ for }i=1,\dots I,\tag{2.7e}$$

where $\boldsymbol{f}$ and g are given functions such that $\boldsymbol{f}\in\boldsymbol{L}^{p'}(\Omega)$ and $g\in W^{-1/p',p'}(\Gamma)$, and $\operatorname{div}_\Gamma$ denotes the surface divergence (cf. Mitreau et al. [15]). When Ω has no holes, the last conditions (2.7e) are unnecessary.

We give the notion of a weak solution to the system (2.7a)-(2.7e).

Definition 2.4. *We say that* $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p'}(\Omega)/\mathbb{R}$ *is a weak solution of* (2.7a)-(2.7e), *if* $\boldsymbol{u} \in \mathbb{V}_N^p(\Omega)$ *and* $(\boldsymbol{u},\pi)$ *satisfies that*

$$\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\cdot\mathrm{curl}\,\boldsymbol{v}dx + \int_\Omega \nabla\pi\cdot\boldsymbol{v}dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx \textit{ for all } \boldsymbol{v}\in\mathbb{X}_N^p(\Omega) \tag{2.8}$$

and (2.7d) *holds in the distribution sense.*

We are in a position to state one of the main theorem.

Theorem 2.5. *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with a* $C^{1,1}$ *boundary* Γ *satisfying (O1) and (O2), and assume that a Carathéodory function* $S(x,t)$ *satisfies the structure conditions* (2.3a)-(2.3c). *Moreover, we assume that* $\boldsymbol{f}\in\boldsymbol{L}^{p'}(\Omega)$ *with* $\mathrm{div}\,\boldsymbol{f}\in L^{p'}(\Omega)$ *and* $g\in W^{-1/p',p'}(\Gamma)$. *Then there exists a weak solution* $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times W^{1,p'}(\Omega)/\mathbb{R}$ *of* (2.7a)-(2.7e), *if and only if* g *satisfies*

$$\langle g,1\rangle_{\Gamma_i} = 0 \textit{ for } i = 0,1,\ldots,I, \tag{2.9}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_i} = \langle\cdot,\cdot\rangle_{W^{-1/p',p'}(\Gamma_i),W^{1-1/p,p}(\Gamma_i)}$. *In this situation, the weak solution is unique and there exists a constant* $C>0$ *depneding only on* p,λ *and* Ω *such that*

$$\|\boldsymbol{u}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|^{p'}_{W^{1,p'}(\Omega)/\mathbb{R}} \le C(\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} + \|g\|^{p'}_{W^{-1/p',p'}(\Gamma)}). \tag{2.10}$$

3 Existence of a weak solution to the Maxwell-Stokes type problem

Before beginning the proof of Theorem 2.5, we consider the case without the potential. Let $\boldsymbol{F}\in\boldsymbol{L}^{p'}(\Omega)$. We consider the following problem: to find $\boldsymbol{w}\in\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{cases} \mathrm{curl}\,[S_t(x,|\mathrm{curl}\,\boldsymbol{w}|^2)\mathrm{curl}\,\boldsymbol{w}] = \boldsymbol{F} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle\boldsymbol{w}\cdot\boldsymbol{n},1\rangle_{\Gamma_i} = 0 & i = 1,\ldots I. \end{cases} \tag{3.1}$$

Definition 3.1. *We say that* $\boldsymbol{w}\in\boldsymbol{W}^{1,p}(\Omega)$ *is a weak solution of* (3.1), *if* $\boldsymbol{w}\in\mathbb{V}_N^p(\Omega)$ *and* $\boldsymbol{w}$ *satisfies that*

$$\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{w}|^2)\mathrm{curl}\,\boldsymbol{w}\cdot\mathrm{curl}\,\boldsymbol{v}dx = \int_\Omega \boldsymbol{F}\cdot\boldsymbol{v}dx \textit{ for all } \boldsymbol{v}\in\mathbb{X}_N^p(\Omega). \tag{3.2}$$

We can see that the compatibility conditions for the existence of weak solution to (3.1) are the following.

$$\mathrm{div}\,\boldsymbol{F} = 0 \text{ in } \Omega, \tag{3.3}$$

$$\langle\boldsymbol{F}\cdot\boldsymbol{n},1\rangle_{\Gamma_i} = 0 \text{ for } i = 0,1,\ldots,I, \tag{3.4}$$

Then we have the following proposition which has an important role for the proof of Theorem 2.5.

Proposition 3.2. *Assume that* $\boldsymbol{F} \in \boldsymbol{L}^{p'}(\Omega)$. *Then the compatibility conditions* (3.3) *and* (3.4) *are necessary and sufficient conditions for the existence of a weak solution* $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ *of* (3.1). *In this situation, the weak solution* $\boldsymbol{w}$ *is unique, and there exists a constant* $C > 0$ *depending only on* λ, p *and* Ω *such that*

$$||\boldsymbol{w}||^p_{\boldsymbol{W}^{1,p}(\Omega)} \le C||\boldsymbol{F}||^{p'}_{\boldsymbol{L}^{p'}(\Omega)}. \tag{3.5}$$

Proof. Step 1 (Necessity). Assume that there exists a weak solution $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ of (3.1). Since $\boldsymbol{C}_0^\infty(\Omega) \subset \mathbb{X}^p_N(\Omega)$, the first equations of (3.1) holds in the distribution sense. Hence (3.3) is clear. Then we can write

$$\begin{aligned}
\langle \boldsymbol{F}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} &= \langle \boldsymbol{n}\cdot \operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{w}|^2)\operatorname{curl}\boldsymbol{w}], 1\rangle_{\Gamma_i} \\
&= \langle \operatorname{div}_\Gamma(S_t(x, |\operatorname{curl}\boldsymbol{w}|^2)\operatorname{curl}\boldsymbol{w}\times\boldsymbol{n}), 1\rangle_{\Gamma_i} \\
&= -\langle S_t(x, |\operatorname{curl}\boldsymbol{w}|^2)\operatorname{curl}\boldsymbol{w}\times\boldsymbol{n}, \nabla 1\rangle_{\boldsymbol{W}^{-1/p',p'}(\Gamma_i),\boldsymbol{W}^{1-1/p,p}(\Gamma_i)} \\
&= 0
\end{aligned}$$

for $i = 0, 1, \ldots, I$. Thus (3.4) holds.

Step 2 (Sufficiency). We assume that (3.3) and (3.4) hold. Define a functional $I[\boldsymbol{v}]$ on $\mathbb{V}^p_N(\Omega)$ by

$$I[\boldsymbol{v}] = \frac{1}{2}\int_\Omega S(x, |\operatorname{curl}\boldsymbol{v}|^2)dx - \int_\Omega \boldsymbol{F}\cdot\boldsymbol{v}dx,$$

and put

$$I_* = \inf_{\boldsymbol{v}\in\mathbb{V}^p_N(\Omega)} I[\boldsymbol{v}].$$

Then we show that I has a unique minimizer $\boldsymbol{w} \in \mathbb{V}^p_N(\Omega)$, that is, $I_* = I[\boldsymbol{w}]$. Indeed, from (2.4), the Hölder and the Young inequalities, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\begin{aligned}
I[\boldsymbol{v}] &\ge \frac{\lambda}{p}\int_\Omega |\operatorname{curl}\boldsymbol{v}|^p dx - ||\boldsymbol{F}||_{\boldsymbol{L}^{p'}(\Omega)}||\boldsymbol{v}||_{\boldsymbol{L}^p(\Omega)} \\
&\ge \frac{\lambda}{p}||\boldsymbol{v}||^p_{\mathbb{V}^p_N(\Omega)} - \varepsilon||\boldsymbol{v}||^p_{\mathbb{V}^p_N(\Omega)} - C(\varepsilon)(||\boldsymbol{F}||^{p'}_{\boldsymbol{L}^{p'}(\Omega)}.
\end{aligned}$$

Choosing $\varepsilon > 0$ so that $\varepsilon = \lambda/2p$, we can see that I is coercive on $\mathbb{V}^p_N(\Omega)$. Since we know that $G(t) = S(x, t^2)$ is a strictly convex function, I is a strictly convex functional on $\mathbb{V}^p_N(\Omega)$. We show that I is lower semi-continuous on $\mathbb{V}^p_N(\Omega)$. Let $\boldsymbol{v}_j \to \boldsymbol{v}$ in $\mathbb{V}^p_N(\Omega)$. Then there exists a subsequence $\{\boldsymbol{v}_{j_k}\}$ of $\{\boldsymbol{v}_j\}$ such that

$$\lim_{k\to\infty}\int_\Omega S(x, |\operatorname{curl}\boldsymbol{v}_{j_k}|^2)dx = \liminf_{j\to\infty}\int_\Omega S(x, |\operatorname{curl}\boldsymbol{v}_j|^2)dx$$

and $\operatorname{curl}\boldsymbol{v}_{j_k} \to \operatorname{curl}\boldsymbol{v}$ a.e. in Ω. Since $S(x, t)$ is continuous with respect to t, $S(x, |\operatorname{curl}\boldsymbol{v}_{j_k}|^2) \to S(x, |\operatorname{curl}\boldsymbol{v}|^2)$ a.e. in Ω. Since $S(x, t) \ge 0$, it follows from Fatou lemma that

$$\int_\Omega S(x, |\operatorname{curl}\boldsymbol{v}|^2)dx \le \liminf_{k\to\infty}\int_\Omega S(x, |\operatorname{curl}\boldsymbol{v}_{j_k}|^2 dx = \liminf_{j\to\infty}\int_\Omega S(x, |\operatorname{curl}\boldsymbol{v}_j|^2)dx.$$

Therefore, we have $I[\boldsymbol{v}] \le \liminf_{j\to\infty} I[\boldsymbol{v}_j]$. Thus, we can see that I is lower semi-continuous. Since I is a lower semi-continuous, coercive and strictly convex functional on the reflexive

Banach space $\mathbb{V}^p_N(\Omega)$, I has a unique minimizer $\boldsymbol{w} \in \mathbb{V}^p_N(\Omega)$. For example, see Ekeland and Temam [12, Chapter 2, Proposition 1.2]. By the Euler-Lagrange equation,

$$\frac{d}{dt} I[\boldsymbol{w} + t\boldsymbol{v}]\bigg|_{t=0} = 0 \text{ for all } \boldsymbol{v} \in \mathbb{V}^p_N(\Omega).$$

This means that

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{w}|^2)\mathrm{curl}\, \boldsymbol{w} \cdot \mathrm{curl}\, \boldsymbol{v} dx = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v} dx \tag{3.6}$$

for all $\boldsymbol{v} \in \mathbb{V}^p_N(\Omega)$.

For any $\boldsymbol{z} \in \mathbb{X}^p_N(\Omega)$, we consider the following div-curl system.

$$\begin{cases} \mathrm{curl}\, \boldsymbol{\xi} = \mathrm{curl}\, \boldsymbol{z} & \text{in } \Omega, \\ \mathrm{div}\, \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{3.7}$$

It follows from Aramaki [3, Theorem 3.6] that (3.7) has a solution $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$. Define

$$\boldsymbol{v} = \boldsymbol{\xi} - \sum_{i=1}^{I} \langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \nabla q_i^N,$$

where $\{\nabla q_i^N\}_{i=1}^I$ is a basis of $\mathbb{K}^p_N(\Omega)$. Then we have that $\mathrm{curl}\, \boldsymbol{v} = \mathrm{curl}\, \boldsymbol{\xi} = \mathrm{curl}\, \boldsymbol{z}$ in Ω, $\mathrm{div}\, \boldsymbol{v} = 0$ in Ω, $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_k} = \langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_k} - \sum_{i=1}^{I} \langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \langle \nabla q_i^N \cdot \boldsymbol{n}, 1\rangle_{\Gamma_k} = 0 \text{ for } k = 1, \ldots, I.$$

Thus we have $\boldsymbol{v} \in \mathbb{V}^p_N(\Omega)$, and

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{w}|^2)\mathrm{curl}\, \boldsymbol{w} \cdot \mathrm{curl}\, \boldsymbol{v} dx = \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{w}|^2)\mathrm{curl}\, \boldsymbol{w} \cdot \mathrm{curl}\, \boldsymbol{z} dx.$$

From the compatibility conditions (3.3) and (3.4), it follows from [2, Lemma 4.1] that there exists $\boldsymbol{G} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that $\boldsymbol{F} = \mathrm{curl}\, \boldsymbol{G}$ in Ω. Hence we have

$$\begin{aligned} \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v} dx = \int_\Omega \mathrm{curl}\, \boldsymbol{G} \cdot \boldsymbol{v} dx = \int_\Omega \boldsymbol{G} \cdot \mathrm{curl}\, \boldsymbol{v} dx& \\ = \int_\Omega \boldsymbol{G} \cdot \mathrm{curl}\, \boldsymbol{z} dx = \int_\Omega \mathrm{curl}\, \boldsymbol{G} \cdot \boldsymbol{z} dx = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{z} dx.& \end{aligned}$$

Therefore, we have

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{w}|^2)\mathrm{curl}\, \boldsymbol{w} \cdot \boldsymbol{z} dx = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{z} dx \text{ for all } \boldsymbol{z} \in \mathbb{X}^p_N(\Omega).$$

That is, $\boldsymbol{w} \in \mathbb{V}^p_N(\Omega)$ is a weak solution of (3.1) in the sense of Definition 3.1.

Step 3 (Uniqueness). Let $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbb{V}^p_N(\Omega)$ be two weak solutions of (3.1). If we choose $\boldsymbol{v} = \boldsymbol{w}_1 - \boldsymbol{w}_2$ as a test function of (3.2), then we have

$$\int_\Omega (S_t(x, |\mathrm{curl}\, \boldsymbol{w}_1|^2)\mathrm{curl}\, \boldsymbol{w}_1 - S_t(x, |\mathrm{curl}\, \boldsymbol{w}_2|^2)\mathrm{curl}\, \boldsymbol{w}_2) \cdot \mathrm{curl}\, (\boldsymbol{w}_1 - \boldsymbol{w}_2) dx = 0.$$

By the strict monotonicity of S_t (Lemma 2.2), we have $\boldsymbol{w}_1 = \boldsymbol{w}_2$ in $\mathbb{V}^p_N(\Omega)$.

Step 4 (Estimate). If we choose $\boldsymbol{v} = \boldsymbol{w}$ as a test function of (3.2), then for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\lambda\|\mathrm{curl}\,\boldsymbol{w}\|^p_{\boldsymbol{L}^p(\Omega)} \le \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \le \varepsilon\|\boldsymbol{w}\|^p_{\boldsymbol{L}^p(\Omega)} + C(\varepsilon)\|\boldsymbol{F}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}.$$

Taking Lemma 2.3 into consideration, if we choose $\varepsilon > 0$ small enough, we get the estimate (3.5). □

Now, we give a proof of Theorem 2.5.

Proof of Theorem 2.5

Step 1 (Necessity). Assume that there exists a weak solution $(\boldsymbol{u}, \pi)$ of (2.7a)-(2.7e). Since $S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u} \in \boldsymbol{L}^{p'}(\Omega)$,

$$\mathrm{curl}\,[S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}] = \boldsymbol{f} - \nabla\pi \in \boldsymbol{L}^{p'}(\Omega)$$

and $\mathrm{div}_\Gamma(S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u} \times \boldsymbol{n}) = g \in W^{-1/p',p'}(\Gamma)$, we can see that $S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$ is well defined and from (2.7d),

$$\langle g, 1\rangle_{\Gamma_i} = -\langle S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u} \times \boldsymbol{n}, \nabla 1\rangle_{\boldsymbol{W}^{-1/p',p'}(\Gamma_i), \boldsymbol{W}^{1-1/p,p}(\Gamma_i)} = 0$$

for $i = 0, 1, \dots I$.

Step 2 (Sufficiency). We assume that (2.9) holds. We consider the following Neumann problem.

$$\begin{cases} \Delta\pi = \mathrm{div}\,\boldsymbol{f} & \text{in } \Omega, \\ \frac{\partial\pi}{\partial\boldsymbol{n}} = \boldsymbol{f}\cdot\boldsymbol{n} - g & \text{on } \Gamma. \end{cases} \tag{3.8}$$

Since $\langle g, 1\rangle_\Gamma = 0$, the compatibility condition for (3.8) holds. From Aramaki [5], the equation (3.8) has a unique weak solution $\pi \in W^{1,p'}(\Omega)$, up to an additive constant, and there exists a constant $C > 0$ depending only on p' and Ω such that

$$\|\pi\|_{W^{1,p'}(\Omega)} \le C(\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|g\|_{W^{-1/p',p'}(\Omega)}). \tag{3.9}$$

Define $\boldsymbol{F} = \boldsymbol{f} - \nabla\pi$. Then $\boldsymbol{F} \in \boldsymbol{L}^{p'}(\Omega)$, $\mathrm{div}\,\boldsymbol{F} = 0$ in Ω and

$$\langle \boldsymbol{F}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = \langle \boldsymbol{f}\cdot\boldsymbol{n} - \frac{\partial\pi}{\partial\boldsymbol{n}}, 1\rangle_{\Gamma_i} = \langle g, 1\rangle_{\Gamma_i} = 0$$

for $i = 0, 1, \dots I$. Thus, it follows from Proposition 3.2 that (3.1) has a unique weak solution $\boldsymbol{u} \in \mathbb{V}^p_N(\Omega)$, and there exists a constant $C > 0$ such that

$$\|\boldsymbol{u}\|^p_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|\boldsymbol{F}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}. \tag{3.10}$$

Therefore, we have

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\cdot\boldsymbol{v}dx = \int_\Omega \boldsymbol{F}\cdot\boldsymbol{v}dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx - \int_\Omega \nabla\pi\cdot\boldsymbol{v}dx$$

for all $\boldsymbol{v} \in \mathbb{X}^p_N(\Omega)$. (2.7d) follows from

$$\boldsymbol{f}\cdot\boldsymbol{n} = \frac{\partial\pi}{\partial\boldsymbol{n}} + \boldsymbol{n}\cdot\mathrm{curl}\,[S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}] = \frac{\partial\pi}{\partial\boldsymbol{n}} + \mathrm{div}_\Gamma(S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u} \times \boldsymbol{n})$$

and (3.8). Thus we can see that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,p'}(\Omega)/\mathbb{R}$ is a weak solution of (2.7a)-(2.7e).

Step 3 (Uniqueness). Let $(\boldsymbol{u}_1,\pi_1),(\boldsymbol{u}_2,\pi_2)$ be two weak solutions of (2.7a)-(2.7e). Then

$$\begin{cases} \Delta(\pi_1-\pi_2)=0 & \text{in } \Omega, \\ \frac{\partial}{\partial \boldsymbol{n}}(\pi_1-\pi_2)=0 & \text{on } \Gamma. \end{cases}$$

Hence $\pi_1-\pi_2=$ const.. If we choose $\boldsymbol{v}=\boldsymbol{u}_1-\boldsymbol{u}_2$ as a test function of (2.5), then we have

$$\int_\Omega (S_t(x,|\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1 - S_t(x,|\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2)\cdot \mathrm{curl}\,(\boldsymbol{u}_1-\boldsymbol{u}_2)dx=0.$$

By the strict monotonicity of S_t, we have $\boldsymbol{u}_1=\boldsymbol{u}_2$.

Step 4 (Estimate). From (3.9) and (3.10), there exists a constant $C>0$ depending only on p,λ and Ω such that

$$\|\boldsymbol{u}\|^p_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi\|^{p'}_{W^{1,p'}(\Omega)/\mathbb{R}} \le C(\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}+\|g\|^{p'}_{W^{-1/p',p'}(\Gamma)}).$$

This completes the proof of Theorem 2.5.

4 The Hölder regularity of the weak solution of (2.7a)-(2.7e)

In this section, we consider the Hölder regularity of the weak solution of (2.7a)-(2.7e). To do so, we assume that Ω is a bounded domain in $\mathbb{R}^3$ with a $C^{2,\alpha}$ boundary Γ $(0<\alpha<1)$, and a function $S(x,t)\in C^2(\Omega\times(0,\infty))\cap C^0(\Omega\times[0,\infty))$ satisfies (2.3a)-(2.3c) and that there exists a constant $C>0$ such that

$$|S_{tx}(x,t)|\le Ct^{(p-2)/2} \text{ for } t>0. \tag{4.1}$$

We have the following theorem.

Theorem 4.1. *Assume that Ω is a bounded domain in $\mathbb{R}^3$ with $C^{2,\alpha}$ boundary satisfying (O1) and (O2) for some $\alpha\in(0,1)$, and that a function $S(x,t)\in C^2(\Omega\times(0,\infty))\cap C^0(\Omega\times[0,\infty))$ satisfies the conditions* (2.3a)-(2.3c) *and* (4.1)*. Moreover, assume that $\boldsymbol{f}\in \boldsymbol{C}^\alpha(\overline{\Omega})$ with $\mathrm{div}\,\boldsymbol{f}\in C^\alpha(\overline{\Omega})$ and $g\in C^\alpha(\Gamma)$ satisfies* (2.6)*. Then the weak solution $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,p'}(\Omega)/\mathbb{R}$ of* (2.7a)-(2.7e) *belongs to $\boldsymbol{C}^{1,\beta}(\overline{\Omega})\times C^{1,\alpha}(\overline{\Omega})/\mathbb{R}$ for some $\beta\in(0,1)$, and there exists a constant $C>0$ depending only on the known data such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{C}^{1,\beta}(\overline{\Omega})}+\|\pi\|_{C^{1,\alpha}(\overline{\Omega})/\mathbb{R}}\le C. \tag{4.2}$$

In order to prove Theorem 4.1, we need the following lemma.

Lemma 4.2. *If we assume that $\boldsymbol{f}\in \boldsymbol{C}^\alpha(\overline{\Omega})$ with $\mathrm{div}\,\boldsymbol{f}\in C^\alpha(\overline{\Omega})$ and that $g\in C^\alpha(\Gamma)$ satisfies* (2.6)*, then the following Neumann problem*

$$\begin{cases} \Delta\pi=\mathrm{div}\,\boldsymbol{f} & \text{in } \Omega, \\ \frac{\partial\pi}{\partial\boldsymbol{n}}=\boldsymbol{f}\cdot\boldsymbol{n}-g & \text{on } \Gamma, \\ \int_\Omega \pi dx=0 \end{cases} \tag{4.3}$$

has a unique solution $\pi\in C^{1,\alpha}(\overline{\Omega})$, and there exists a constant $C>0$ depending only on α and Ω such that

$$\|\pi\|_{C^{1,\alpha}(\overline{\Omega})}\le C(\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}+\|g\|_{C^\alpha(\Gamma)}). \tag{4.4}$$

For the proof, see [5].

If we put $\boldsymbol{F} = \boldsymbol{f} - \nabla\pi \in \boldsymbol{C}^{\alpha}(\overline{\Omega})$, then $\operatorname{div}\boldsymbol{F} = 0$ in Ω and $\boldsymbol{F}\cdot\boldsymbol{n} = g$ on Γ, so $\langle \boldsymbol{F}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ for $i = 0, 1, \dots, I$. Hence $\boldsymbol{u}$ satisfies the following system.

$$\begin{cases} \operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & \text{for } i = 1, \dots, I. \end{cases} \tag{4.5}$$

We derive the following proposition.

Proposition 4.3. *If we assume that $\boldsymbol{F} \in \boldsymbol{C}^{\alpha}(\overline{\Omega})$ satisfies the compatibility conditions* (3.3) *and* (3.4)*, then the weak solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ of* (4.5)*, in fact, belongs to $\boldsymbol{C}^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$, and there exists a constant $C > 0$ depending only on known data such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{C}^{1,\beta}(\overline{\Omega})} \le C.$$

Proof. We consider the following div-curl system.

$$\begin{cases} \operatorname{curl}\boldsymbol{G} = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{G} = 0 & \text{in } \Omega, \\ \boldsymbol{n}\cdot\boldsymbol{G} = 0 & \text{on } \Gamma. \end{cases} \tag{4.6}$$

By hypotheses (3.4), it holds that $\langle \boldsymbol{F}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ for $i = 0, 1, \dots, I$. Therefore from Pan [16, Lemma 5.7 (ii)], the system (4.6) has a solution $\boldsymbol{G} \in \boldsymbol{C}^{1,\alpha}(\overline{\Omega})$, up to an additive element of $\mathbb{K}_T^p(\Omega)$, and there exists a constant $C = C(\Omega, \alpha)$ such that

$$\|\boldsymbol{G}\|_{\boldsymbol{C}^{1,\alpha}(\overline{\Omega})} \le C(\|\boldsymbol{F}\|_{\boldsymbol{C}^{\alpha}(\overline{\Omega})} + \|\boldsymbol{G}\|_{\boldsymbol{C}^{0}(\overline{\Omega})}). \tag{4.7}$$

Thus the weak solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfies

$$\begin{cases} \operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u} - \boldsymbol{G}] = \boldsymbol{0} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & i = 1, \dots I. \end{cases} \tag{4.8}$$

Since $S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u} \in L^{p'}(\Omega)$, $\boldsymbol{H} := S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u} - \boldsymbol{G} \in \boldsymbol{L}^{p'}(\Omega)$ and satisfies $\operatorname{curl}\boldsymbol{H} = \boldsymbol{0}$ in Ω. By the Helmholtz decomposition of $\boldsymbol{H}$ (cf. [2, Theorem 6.1]), we can write

$$\boldsymbol{H} = \boldsymbol{z} + \nabla\varphi + \operatorname{curl}\boldsymbol{w},$$

where $\boldsymbol{z} \in \mathbb{K}_T^{p'}(\Omega)$ is unique, $\varphi \in W^{1,p'}(\Omega)$ is unique up to an additive constant and $\boldsymbol{w} \in \boldsymbol{W}^{1,p'}(\Omega)$ satisfies $\operatorname{div}\boldsymbol{w} = 0$ in Ω and $\boldsymbol{w}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ, and is unique up to an additive element of $\mathbb{K}_N^{p'}(\Omega)$. However, since $\operatorname{curl}\boldsymbol{H} = \boldsymbol{0}$ in Ω, we have $\operatorname{curl}^2\boldsymbol{w} = \boldsymbol{0}$ in Ω. Therefore

$$\begin{aligned} 0 = \int_\Omega \operatorname{curl}^2\boldsymbol{w}\cdot\boldsymbol{w}\,dx &= \int_\Gamma (\boldsymbol{n}\times\operatorname{curl}\boldsymbol{w})\cdot\boldsymbol{w}\,dS + \int_\Omega |\operatorname{curl}\boldsymbol{w}|^2 dx \\ &= \int_\Gamma (\boldsymbol{w}\times\boldsymbol{n})\cdot\operatorname{curl}\boldsymbol{w}\,dS + \int_\Omega |\operatorname{curl}\boldsymbol{w}|^2 dx = \int_\Omega |\operatorname{curl}\boldsymbol{w}|^2 dx, \end{aligned}$$

where dS denotes the surface element of Γ. So we have $\operatorname{curl}\boldsymbol{w} = \boldsymbol{0}$ in Ω. Thus we can write

$$\boldsymbol{H} = z + \nabla\varphi,$$

and the following estimate holds.

$$\|z\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\varphi\|_{\boldsymbol{W}^{1,p'}(\Omega)/\mathbb{R}} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{C}^{\alpha}(\overline{\Omega})} + \|\boldsymbol{G}\|_{\boldsymbol{C}^{0}(\overline{\Omega})}).$$

Since Γ is $C^{2,\alpha}$ class, we note that $\mathbb{K}_T^{p'}(\Omega) \subset \boldsymbol{C}^{1,\alpha}(\overline{\Omega})$. Hence we can write

$$S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u} = \boldsymbol{G} + z + \nabla\varphi, \tag{4.9}$$

where $\boldsymbol{G}, z \in \boldsymbol{C}^{1,\alpha}(\overline{\Omega})$.

Define

$$\Phi(x,t) = \begin{cases} t(S_t(x,t))^2 & \text{for } x \in \Omega \text{ and } t > 0, \\ 0 & \text{for } x \in \Omega, t = 0. \end{cases}$$

Then $\Phi \in C^1(\Omega \times (0,\infty)) \cap C^0(\Omega \times [0,\infty))$, and from (2.3b), we see that Φ satisfies

$$\Phi_t(x,t) = S_t(x,t)(S_t(x,t) + 2tS_{tt}(x,t)) > 0 \text{ for } x \in \Omega, t > 0.$$

Hence, $\rho = \Phi(x,t)$ has an implicit function $t = \Psi(x,\rho) \in C^1(\Omega \times (0,\infty))$. Note that $\rho \approx t^{p-1}$. Define

$$f(x,\rho) = \frac{1}{S_t(x, \Psi(x,\rho))} \text{ for } \rho > 0.$$

Then we obtain the following properties whose proofs are given in [4, Lemma 2.1].

$$\Lambda^{-(p'-1)}\rho^{(p'-2)/2} \leq f(x,\rho) \leq \lambda^{-(p'-1)}\rho^{(p'-2)/2} \text{ for } \rho > 0, \tag{4.10}$$

$$|f_x(x,\rho)| \leq C\rho^{(p'-2)/2} \text{ for } \rho > 0, \tag{4.11}$$

and there exist constants c and C depending only on λ and Λ such that

$$c\rho^{(p'-2)/2} \leq f(x,\rho) + 2\rho f_\rho(x,\rho) \leq C\rho^{(p'-2)/2} \text{ for } \rho > 0. \tag{4.12}$$

For the proof, see [6]. By (4.9), we have

$$\Phi(x, |\operatorname{curl}\boldsymbol{u}|^2) = |\operatorname{curl}\boldsymbol{u}|^2(S_t(x, |\operatorname{curl}\boldsymbol{u}|^2))^2 = |\boldsymbol{G} + z + \nabla\varphi|^2,$$

$$|\operatorname{curl}\boldsymbol{u}|^2 = \Psi(x, |\boldsymbol{G} + z + \nabla\varphi|^2)$$

and

$$\operatorname{curl}\boldsymbol{u} = \frac{\boldsymbol{G} + z + \nabla\varphi}{S_t(x, |\operatorname{curl}\boldsymbol{u}|^2)} = f(x, |\boldsymbol{G} + z + \nabla\varphi|^2)(\boldsymbol{G} + z + \nabla\varphi).$$

Since $\operatorname{div}(\operatorname{curl}\boldsymbol{u}) = 0$ in Ω and $\boldsymbol{n} \cdot \operatorname{curl}\boldsymbol{u} = \operatorname{div}_\Gamma(\boldsymbol{u} \times \boldsymbol{n}) = 0$ on Γ, the function φ is a solution of the following system.

$$\begin{cases} \operatorname{div}[f(x, |\boldsymbol{G} + z + \nabla\varphi|^2)(\boldsymbol{G} + z + \nabla\varphi)] = 0 & \text{in } \Omega, \\ \boldsymbol{n} \cdot f(x, |\boldsymbol{G} + z + \nabla\varphi|^2)(\boldsymbol{G} + z + \nabla\varphi) = 0 & \text{on } \Gamma. \end{cases} \tag{4.13}$$

Since $f(x,|\boldsymbol{p}|^2)\boldsymbol{p}$ is of C^1 class with respect to $(x,\boldsymbol{p}) \in \Omega\times(\mathbb{R}^3\setminus\{0\})$ and of C^0 class with respect to $\boldsymbol{p}\in\mathbb{R}^3$ and $|f(x,|\boldsymbol{p}|^2)\boldsymbol{p}| \le C|\boldsymbol{p}|^{p'-1}$, we can see that $f(x,|\boldsymbol{p}|^2)\boldsymbol{p}$ is of C^γ class with respect to $\boldsymbol{p}$, where $\gamma=\min\{\alpha, p'-1\}$.

We want to show that if φ is a solution of (4.13), then $\varphi\in C^{1,\beta}(\overline{\Omega})$ for some $\beta\in(0,\gamma]$. Then $\boldsymbol{u}$ is a solution of

$$\begin{cases} \operatorname{curl}\boldsymbol{u} = f(x,|\boldsymbol{G}+\boldsymbol{z}+\nabla\varphi|^2)(\boldsymbol{G}+\boldsymbol{z}+\nabla\varphi) & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u}=0 & \text{in } \Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0} & \text{on } \Gamma,\\ \langle \boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0 & \text{for } i=1,\dots,I. \end{cases} \tag{4.14}$$

From the regularity theorem of Bolik and Wahl [7], we have $\boldsymbol{u}\in \boldsymbol{C}^{1,\beta}(\overline{\Omega})$, and there exists a constant $C>0$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{C}^{1,\beta}(\overline{\Omega})} \le C(\|f(x,|\boldsymbol{G}+\boldsymbol{z}+\nabla\varphi|^2)(\boldsymbol{G}+\boldsymbol{z}+\nabla\varphi)\|_{\boldsymbol{C}^{\beta}(\overline{\Omega})} + \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}). \tag{4.15}$$

Thus we can derive $\boldsymbol{C}^{1,\beta}$ regularity of a solution of (4.13).

In the sequel, we show $\varphi\in C^{1,\beta}(\overline{\Omega})$. We define $\boldsymbol{M}=\boldsymbol{G}+\boldsymbol{z}\in \boldsymbol{C}^{1,\alpha}(\overline{\Omega})$, then

$$\|\boldsymbol{M}\|_{\boldsymbol{C}^{1,\alpha}(\overline{\Omega})} \le \|\boldsymbol{G}\|_{\boldsymbol{C}^{1,\alpha}(\overline{\Omega})} + \|\boldsymbol{z}\|_{\boldsymbol{C}^{1,\alpha}(\overline{\Omega})}) \le C(\|\boldsymbol{F}\|_{\boldsymbol{C}^{\alpha}(\overline{\Omega})} + \|\boldsymbol{G}\|_{\boldsymbol{C}^{0}(\overline{\Omega})}).$$

Define $\boldsymbol{A}(x,\boldsymbol{p}) = f(x,|\boldsymbol{M}(x)+\boldsymbol{p}|^2)(\boldsymbol{M}(x)+\boldsymbol{p})$, $\boldsymbol{p}=(p_1,p_2,p_3)\in\mathbb{R}^3$. Then the equation (4.13) can be written by

$$\begin{cases} \operatorname{div}\boldsymbol{A}(x,\nabla\varphi)=0 & \text{in } \Omega,\\ \boldsymbol{n}\cdot\boldsymbol{A}(x,\nabla\varphi)=0 & \text{on } \Gamma. \end{cases} \tag{4.16}$$

We check the structure conditions of DiBenedetto [11]: there exist constants $c, C>0$ such that

(i) $\boldsymbol{A}(x,\boldsymbol{p})\cdot\boldsymbol{p} \ge c|\boldsymbol{p}|^{p'} - Cg_1(x)$,

(ii) $|\boldsymbol{A}(x,\boldsymbol{p})| \le C(|\boldsymbol{p}|^{p'-1}+g_2(x))$,

where $g_i\ge 0$ and $g_i\in L^\infty(\Omega)$.

Indeed, from (4.10), for any $\varepsilon>0$, there exists a constant $C(\varepsilon)>0$ such that

$$\begin{aligned} \boldsymbol{A}(x,\boldsymbol{p})\cdot\boldsymbol{p} &= f(x,|\boldsymbol{M}(x)+\boldsymbol{p}|^2)(\boldsymbol{M}(x)+\boldsymbol{p})\cdot(\boldsymbol{M}(x)+\boldsymbol{p})\\ &\quad - f(x,|\boldsymbol{M}(x)+\boldsymbol{p}|^2)(\boldsymbol{M}(x)+\boldsymbol{p})\cdot\boldsymbol{M}(x)\\ &\ge \Lambda^{-(p'-1)}|\boldsymbol{M}(x)+\boldsymbol{p}|^{p'} - \lambda^{-(p'-1)}|\boldsymbol{M}(x)+\boldsymbol{p}|^{p'-1}|\boldsymbol{M}(x)|\\ &\ge \Lambda^{-(p'-1)}|\boldsymbol{M}(x)+\boldsymbol{p}|^{p'} - \varepsilon|\boldsymbol{M}(x)+\boldsymbol{p}|^{p'} - C(\varepsilon)|\boldsymbol{M}(x)|^{p'}. \end{aligned}$$

If we choose $\varepsilon>0$ so that $\varepsilon<\Lambda^{-(p'-1)}$, we have

$$\boldsymbol{A}(x,\boldsymbol{p})\cdot\boldsymbol{p} \ge c_1|\boldsymbol{M}(x)+\boldsymbol{p}|^{p'} - C_1|\boldsymbol{M}(x)|^{p'}.$$

On the other hand, since

$$|\boldsymbol{p}|^{p'} = |\boldsymbol{M}(x)+\boldsymbol{p}-\boldsymbol{M}(x)|^{p'} \le 2^{p'-1}(|\boldsymbol{M}(x)+\boldsymbol{p}|^{p'} + |\boldsymbol{M}(x)|^{p'}),$$

we have $\boldsymbol{A}(x,\boldsymbol{p})\cdot\boldsymbol{p} \ge c|\boldsymbol{p}|^{p'} - C|\boldsymbol{M}(x)|^{p'}$. Since $\boldsymbol{M} \in C^{1,\alpha}(\overline{\Omega})$, $|\boldsymbol{M}(x)|^{p'} \in L^\infty(\Omega)$, so (i) holds. Furthermore, since

$$|\boldsymbol{A}(x,\boldsymbol{p})| \le C_2|\boldsymbol{M}(x)+\boldsymbol{p}|^{p'-1} \le C(|\boldsymbol{p}|^{p'-1} + |\boldsymbol{M}(x)|^{p'-1}),$$

(ii) holds. From [6, Proposition 4.1], we have the following global boundedness of a weak solution φ.

$$\sup_\Omega |\varphi| \le (\|\varphi\|_{L^{p'}(\Omega)} + \||\boldsymbol{M}|^{p'}\|^{1/p'}_{L^\infty(\Omega)}).$$

Therefore, φ is a bounded weak solution of (4.16). According to [11, Theorem 1.3], we can see that $\varphi \in C^\delta(\overline{\Omega})$ for some $\delta \in (0,1)$.

Next, we show that the gradient of φ is Hölder continuous. We follow the idea of Lieberman [14]. We first consider the case where $\boldsymbol{M}(x) = \boldsymbol{k} = \text{const.}$. Put $\phi(x) = \boldsymbol{k}\cdot x$, and $\psi = \phi + \varphi$. Define

$$\boldsymbol{B}(x,\boldsymbol{p}) = (B_1(x,\boldsymbol{p}), B_2(x,\boldsymbol{p}), B_3(x,\boldsymbol{p})) = f(x,|\boldsymbol{p}|^2)\boldsymbol{p}.$$

Then (4.16) is written into the following form.

$$\begin{cases} \operatorname{div}\boldsymbol{B}(x,\nabla\psi) = 0 & \text{in } \Omega, \\ \boldsymbol{n}\cdot\boldsymbol{B}(x,\nabla\psi) = 0 & \text{on } \Gamma. \end{cases} \tag{4.17}$$

We see that

$$\sum_{i,j=1}^{3} \frac{\partial B_i}{\partial p_j}\xi_i\xi_j = f(x,|\boldsymbol{p}|^2)|\xi|^2 + 2f_\rho(x,|\boldsymbol{p}|^2)(\boldsymbol{p}\cdot\xi)^2.$$

Here $f_\rho(x,|\boldsymbol{p}|^2)$ may change the sign. When $f_\rho(x,|\boldsymbol{p}|^2) \ge 0$, it follows from (4.10) that

$$\sum_{i,j=1}^{3} \frac{\partial B_i}{\partial p_j}\xi_i\xi_j \ge f(x,|\boldsymbol{p}|^2)|\xi|^2 \ge \Lambda^{-(p'-1)}|\boldsymbol{p}|^{p'-2}|\xi|^2.$$

When $f_\rho(x,|\boldsymbol{p}|^2) < 0$, it follows from (4.12) that

$$\begin{aligned} \sum_{i,j=1}^{3} \frac{\partial B_i}{\partial p_j}\xi_i\xi_j &\ge f(x,|\boldsymbol{p}|^2)|\xi|^2 + 2f_\rho(x,|\boldsymbol{p}|^2)|\boldsymbol{p}|^2|\xi|^2 \\ &= (f(x,|\boldsymbol{p}|^2) + 2f_\rho(x,|\boldsymbol{p}|^2)|\boldsymbol{p}|^2)|\xi|^2 \\ &\ge c|\boldsymbol{p}|^{p'-2}|\xi|^2. \end{aligned}$$

Thus there exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{3} \frac{\partial B_i}{\partial p_j}\xi_i\xi_j \ge c|\boldsymbol{p}|^{p'-2}|\xi|^2. \tag{4.18}$$

On the other hand,

$$\left|\frac{\partial B_i}{\partial p_j}\right| \le f(x,|\boldsymbol{p}|^2) + 2|f_\rho(x,|\boldsymbol{p}|^2)||\boldsymbol{p}|^2.$$

If we use the relations $f(x,\rho) \le \lambda^{-(p'-1)}\rho^{(p'-2)/2}$ by (4.10) and

$$\begin{aligned}
\rho|f_\rho(x,\rho)| &= \frac{\Phi(x,t)|S_{tt}(x,t)|}{(S_t(x,t))^2\Phi_t(x,t)} \\
&= \frac{\Phi(x,t)|S_{tt}(x,t)|}{(S_t(x,t))^2S_t(x,t)(S_t(x,t)+2tS_{tt}(x,t))} \\
&\le C\frac{t^{p-1}t^{(p-4)/2}}{t^{3(p-2)/2}t^{(p-2)/2}} \\
&\le C_1t^{-(p-2)/2} \\
&\le C_2\rho^{-(p-2)/(2(p-1))} \\
&\le C_3\rho^{(p'-2)/2},
\end{aligned}$$

we have

$$\left|\frac{\partial B_i}{\partial p_j}\right| \le C|\boldsymbol{p}|^{p'-2}. \tag{4.19}$$

Finally, from (4.11), we have

$$|\boldsymbol{B}(x,\boldsymbol{p}) - \boldsymbol{B}(y,\boldsymbol{p})| \le C|x-y||\boldsymbol{p}|^{p'-1}.$$

Thus we see that the structure conditions of [14] hold. Therefore, when $\boldsymbol{M}(x) = \boldsymbol{k} =$ const., if we apply [14, Theorem 2], there exist $\beta \in (0,1)$ and a constant C dependent on $\lambda, \Lambda, p, \sup_{\overline{\Omega}}|\varphi|$ and Ω such that $\varphi \in C^{1,\beta}(\overline{\Omega})$, and

$$\|\varphi\|_{C^{1,\beta}(\overline{\Omega})} \le C. \tag{4.20}$$

Next we use the perturbation method to verify the regularity of weak solution φ of (4.16). Fix $x_0 \in \Omega$ and choose a ball $B_{R_0}(x_0)$ with center x_0 and radius $R_0 > 0$ such that $B_{R_0}(x_0) \subset \Omega$. For any $0 < R \le R_0$, we consider the following equation

$$\begin{cases} \operatorname{div}[f(x_0,|\boldsymbol{F}(x_0)+\nabla\overline{\varphi}|^2)(\boldsymbol{F}(x_0)+\nabla\overline{\varphi})] = 0 & \text{in } B_R(x_0), \\ \overline{\varphi} = \varphi & \text{on } \partial B_R(x_0), \end{cases} \tag{4.21}$$

where $\partial B_R(x_0)$ denotes the boundary of $B_R(x_0)$. By [14, Lemma 5], equation (4.21) has a unique weak solution $\overline{\varphi}$ in $W^{1,q}(B_R(x_0))$. Moreover, we can derive that $\overline{\varphi} \in C^{1,\beta}(\overline{B_R(x_0)})$ and $\overline{\varphi}$ satisfies

$$\|\overline{\varphi}\|_{C^{1,\beta}(\overline{B_R(x_0)})} \le C.$$

Using this fact and the perturbation method, the weak solution $\varphi \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$ for some $\beta \in (0,1)$ and for any $\Omega' \Subset \Omega$, there exists a constant C depending only on the known data and $\operatorname{dist}(\Omega', \partial\Omega)$ such that

$$\|\varphi\|_{C^{1,\beta}(\overline{\Omega'})} \le C.$$

Here we use a variant of the perturbation method developed by Choe [9, pp. 36-38] (cf. [4, Appendix B]). Finally $C^{1,\beta}$ regularity near the boundary Γ follows from [14, Lemma 6] and the perturbation method. □

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