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Laplace Transform, Gronwall Inequality and Delay Differential Equations for General Conformable Fractional Derivative

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Abstract

In the present paper, the general conformable fractional derivative (GCFD) is considered and a corresponding Laplace transform is defined. Gronwall inequality is proved to show the exponential boundedness of a solution and using the Laplace transform the solution is found for certain classes of delay differential equations with GCFD.

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1 Introduction

The recently introduced conformable fractional derivative (CFD) [6] is defined as

$$\mathrm{D}_\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for $\alpha \in (0,1]$ and $t > 0$. One can see that unlike the usual fractional derivatives such as Riemann-Liouville or Caputo derivative (see e.g. [12]), that are defined using an integral,

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CFD has a local nature. It was shown in [6] and later in [1] that it has many properties analogous to those of classic integer-order derivative. For instance, $\mathrm{D}_\alpha(fg) = g\mathrm{D}_\alpha f + f\mathrm{D}_\alpha g$.

Also in [1], the Laplace transform was defined and Gronwall inequality was proved to provide a tool for studying stability and find solutions of equations involving CFD. In [17], the authors defined so-called general conformable fractional derivative (GCFD) as a Gâteaux derivative in the direction of a fractional conformable function. They also gave physical and geometrical interpretation of GCFD. In this paper, we find explicit formulas for solutions of certain classes of delay differential equations with GCFD. To reach our aim we need to define a general conformable Laplace transform and prove some of its properties. A Gronwall-type inequality is proved to show that the Laplace transform can be applied. In the delayed equations we suppose the commutativity of the matrix coefficients. Similar problems were studied for differential [8, 9] as well as for difference equations [7, 10], or with variable delays [13].

The present paper is organized as follows. In Section 2, we recall basic definitions of general conformable fractional calculus and define the general conformable Laplace transform. Properties of this Laplace transform are proved in Section 3. Here we also present important examples. Section 4 is devoted to Gronwall inequality and its corollary. In final section, we consider certain classes of Cauchy problems for delay differential equations with GCFD, multiple delays and linear parts given by pairwise permutable matrices, and we derive the closed-form formulas for solutions.

Throughout the paper, we denote $\mathbb{N}$ the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2 Preliminary results

First we recall the definition of general conformable fractional derivative using the notion of fractional conformable function as it was established in [17].

Definition 2.1. Let $t_0 \in \mathbb{R}$. Continuous real function $\psi\colon [t_0, \infty) \times (0, 1] \to \mathbb{R}$ satisfying $\psi(\cdot, 1) \equiv 1$ and

$$\psi(\cdot, p) \neq \psi(\cdot, q) \text{ whenever } p, q \in (0, 1],\ p \neq q,$$

and the constant function $\psi(\cdot, \cdot) \equiv 1$ are called fractional conformable functions.

Definition 2.2. Let ψ be a fractional conformable function and $p \in (0, 1]$. The general conformable fractional derivative (GCFD) is defined as

$$\mathrm{D}^p_\psi f(u) := \lim_{\varepsilon \to 0} \frac{f(u + \varepsilon\psi(u, p)) - f(u)}{\varepsilon}.$$

If f is differentiable at $u > 0$ and $p \in (0, 1]$, then

$$\mathrm{D}^p_\psi f(u) = f'(u)\psi(u, p). \tag{2.1}$$

Next we define the corresponding integral operator (see [17]).

Definition 2.3. Let $u \geq a \geq t_0$, $f\colon (a, u] \to \mathbb{R}$ be a given function and ψ be a fractional conformable function. The p-fractional integral of f is defined as

$$\mathrm{I}^{p,\psi}_a f(u) = \int_a^u f(t)\,\mathrm{d}_{p,\psi}t := \int_a^u \frac{f(t)}{\psi(t, p)}\,\mathrm{d}t$$

if the right-hand side exists.

Definition 2.4. Let $f\colon (t_0,\infty] \to \mathbb{R}$ be a given function and ψ be a fractional conformable function positive on $[t_0,\infty)\times(0,1]$ and satisfying

$$\int_{t_0}^{\infty} \mathrm{d}_{p,\psi} t = \infty, \quad \forall p \in (0,1]. \tag{2.2}$$

The general conformable Laplace transform of f is defined as

$$\mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s) := \int_{t_0}^{\infty} \mathrm{e}^{-s\eta(t)} f(t)\,\mathrm{d}_{p,\psi} t$$

for such $s \in \mathbb{R}$ that the right-hand side exists, where

$$\eta(t) := \int_{t_0}^{t} \mathrm{d}_{p,\psi}\tilde{t}.$$

Sometimes we will shortly denote $F_{p,\psi}^{t_0}(s) = \mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s)$.

Remark 2.5. Since η is increasing on $[t_0,\infty)$, denoting $\omega\colon [0,\infty) \to [t_0,\infty)$ the inverse function to η, we get

$$\mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s) = \mathcal{L}\{f(\omega(t))\}(s)$$

where there is the usual Laplace transform [15] on the right-hand side. Note that it is defined if $f \circ \omega$ is exponentially bounded, i.e., there exist positive constants c_1, c_2 such that $|f(\omega(t))| \le c_1 \mathrm{e}^{c_2 t}$ for all $t \ge 0$, or in other words

$$|f(t)| \le c_1 \mathrm{e}^{c_2 \eta(t)}, \quad \forall t \ge t_0. \tag{2.3}$$

Clearly, then $\mathcal{L}_{p,\psi}^{t_0}\{f(t)\}$ is defined on (c_2,∞). This can be also seen from

$$\left|\mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s)\right| \le c_1 \int_{t_0}^{\infty} \mathrm{e}^{(c_2-s)\eta(t)} \eta'(t)\,\mathrm{d}t = c_1 \int_{0}^{\infty} \mathrm{e}^{(c_2-s)\tilde{t}}\,\mathrm{d}\tilde{t}. \tag{2.4}$$

Remark 2.6. In the paper, we often work with the general conformable Laplace transform of a vector or matrix function of the form $f(t)w$ or $f(t)B$, where f is a scalar function, $w = (w_i)_i$ is a constant vector with coordinates w_i (i.e., the outer index i means that the i-th coordinate of the left-hand side is inside the bracket) and $B = (B_{ij})_{ij}$ is a constant matrix with elements B_{ij}. These are understood in the following sense:

$$\begin{aligned}\mathcal{L}_{p,\psi}^{t_0}\{f(t)w\}(s) &= \mathcal{L}_{p,\psi}^{t_0}\{(f(t)w_i)_i\}(s) = \left(\mathcal{L}_{p,\psi}^{t_0}\{f(t)w_i\}(s)\right)_i \\ &= \left(\mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s)w_i\right)_i = \mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s)\,(w_i)_i = \left(\mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s)\right)w\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{p,\psi}^{t_0}\{f(t)B\}(s) &= \mathcal{L}_{p,\psi}^{t_0}\{(f(t)B_{ij})_{ij}\}(s) = \left(\mathcal{L}_{p,\psi}^{t_0}\{f(t)B_{ij}\}(s)\right)_{ij} \\ &= \left(\mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s)B_{ij}\right)_{ij} = \mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s)\left(B_{ij}\right)_{ij} = \mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s)B.\end{aligned}$$

3 Properties of the general conformable Laplace transform

In this section, we collect and prove some properties of the general conformable Laplace transform. One can compare them to the ones of the conformable Laplace transform proved in [11] (see also [1]) or of the classic Laplace transform [15].

Lemma 3.1. *If $\mathcal{L}^{t_0}_{p,\psi}\{f_1(t)\}$ and $\mathcal{L}^{t_0}_{p,\psi}\{f_2(t)\}$ exist on (s_1,∞) and (s_2,∞), respectively, then for any $\alpha_1,\alpha_2 \in \mathbb{R}$,*

$$\mathcal{L}^{t_0}_{p,\psi}\{\alpha_1 f_1(t)+\alpha_2 f_2(t)\} = \alpha_1\mathcal{L}^{t_0}_{p,\psi}\{f_1(t)\}+\alpha_2\mathcal{L}^{t_0}_{p,\psi}\{f_2(t)\}$$

on $(\max\{s_1,s_2\},\infty)$.

Proof. The statement follows from the linearity of the Riemann integral. □

Lemma 3.2. *Let $f\colon [t_0,\infty)\to\mathbb{R}$ fulfill (2.3) for some $c_1,c_2>0$ and $\mathcal{L}^{t_0}_{p,\psi}\{f(t)\}$ exist. Then the following holds true:*

(i) function $F^{t_0}_{p,\psi}$ is analytic on (c_2,∞);

(ii) $\mathcal{L}^{t_0}_{p,\psi}\{\eta(t)f(t)\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}F^{t_0}_{p,\psi}(s)$ for all $s>c_2$;

(iii) if f is differentiable, then $\mathcal{L}^{t_0}_{p,\psi}\{\mathrm{D}^p_\psi f(t)\}$ exists on (c_2,∞) and it holds

$$\mathcal{L}^{t_0}_{p,\psi}\{\mathrm{D}^p_\psi f(t)\}(s) = sF^{t_0}_{p,\psi}(s)-f(t_0^+), \quad s>c_2$$

where $f(t_0^+)=\lim_{t\to t_0^+} f(t)$;

(iv) $\lim_{s\to\infty} F^{t_0}_{p,\psi}(s)=0$;

(v) for $s>c+c_2$,

$$\mathcal{L}^{t_0}_{p,\psi}\{\mathrm{e}^{c\eta(t)}f(t)\}(s) = F^{t_0}_{p,\psi}(s-c);$$

(vi) if $\lim_{t\to t_0^+}\frac{f(t)}{\eta(t)}$ exists, then

$$\mathcal{L}^{t_0}_{p,\psi}\left\{\frac{f(t)}{\eta(t)}\right\}(s) = \int_s^\infty F^{t_0}_{p,\psi}(u)\,\mathrm{d}u$$

for all $s>c_2$;

Proof. One can prove statements (ii) and (v) directly from the definition of the general conformable Laplace transform.

Statement (i) follows by Remark 2.5 from the analyticity of the classic Laplace transform [15, Theorem 3.1].

Statement (iii) follows from Definition 2.4, equation (2.1) and integration per partes.

From estimation (2.4) we have

$$\left|\mathcal{L}^{t_0}_{p,\psi}\{f(t)\}(s)\right| \le \frac{c_1}{s-c_2} \to 0$$

as $s\to\infty$, and statement (iv) follows.

To prove statement (vi), it is enough to use Remark 2.5 and an analogous result for the classic Laplace transform [15, Theorem 1.37]. □

Lemma 3.3. *Let $a > 0$, $f\colon [0,\infty) \to \mathbb{R}$ satisfy $|f(t)| \le c_1 \mathrm{e}^{c_2 t}$ for some $c_1, c_2 > 0$ and all $t \ge 0$, and $\mathcal{L}^{t_0}_{p,\psi}\{f(a\eta(t))\}$ exist. Then*

$$\mathcal{L}^{t_0}_{p,\psi}\{f(a\eta(t))\}(s) = \frac{1}{a}\mathcal{L}^{t_0}_{p,\psi}\{f(\eta(t))\}\left(\frac{s}{a}\right)$$

for all $s > c_2 a$.

Proof. The result follows by taking substitution $a\eta(t) = \eta(u)$. □

Next, we calculate the general conformable Laplace transform of certain functions.

Example 3.4. Suppose that ψ is a positive fractional conformable function satisfying (2.2).

(i) $\mathcal{L}^{t_0}_{p,\psi}\{1\}(s) = \mathcal{L}\{1\}(s) = \frac{1}{s}$ for any $s > 0$.

(ii) $\mathcal{L}^{t_0}_{p,\psi}\{t\}(s) = \mathcal{L}\{\omega(t)\}(s)$ for any $s > c_2$ for some $c_2 > 0$ such that there exists $c_1 > 0$ such that $t \le c_1 \mathrm{e}^{c_2\eta(t)}$ for all $t \ge t_0$. In particular, if $\psi(t,p) = (t-t_0)^{1-p}$ (known as conformable derivative [1, 6]), then

$$\mathcal{L}^{t_0}_{p,\psi}\{t\}(s) = \mathcal{L}\left\{t_0 + (pt)^{\frac{1}{p}}\right\}(s) = \frac{t_0}{s} + \frac{p^{\frac{1}{p}}\Gamma\left(1+\frac{1}{p}\right)}{s^{1+\frac{1}{p}}}$$

for all $s > 0$, since now $\eta(t) = \frac{(t-t_0)^p}{p}$. For any $c_2 > 0$ and any fixed $p \in (0,1]$, the condition

$$t \le c_1 \mathrm{e}^{c_2 \frac{(t-t_0)^p}{p}}, \quad \forall t \ge t_0$$

holds with $c_1 = \max_{t\ge t_0} t\,\mathrm{e}^{-c_2\frac{(t-t_0)^p}{p}}$.

(iii) $\mathcal{L}^{t_0}_{p,\psi}\{\mathrm{e}^{c\eta(t)}\}(s) = \mathcal{L}^{t_0}_{p,\psi}\{1\}(s-c) = \frac{1}{s-c}$ for any $s > c$.

(iv) $\mathcal{L}^{t_0}_{p,\psi}\{\sin(c\eta(t))\}(s) = \mathcal{L}\{\sin(ct)\}(s) = \frac{c}{c^2+s^2}$ for any $s > 0$.

The next lemma is a generalization of a result from [4, Theorem 3] on the conformable Laplace transform of a convolution.

Lemma 3.5. *Let $f, g\colon [0,\infty) \to \mathbb{R}$ satisfy (2.3) with some $c_1^f, c_2^f > 0$ and $c_1^g, c_2^g > 0$, respectively. Suppose that $\mathcal{L}^{t_0}_{p,\psi}\{f(\eta(t))\}$ and $\mathcal{L}^{t_0}_{p,\psi}\{g(\eta(t))\}$ exist. Then*

$$\mathcal{L}^{t_0}_{p,\psi}\{(f*g)(\eta(t))\}(s) = \mathcal{L}^{t_0}_{p,\psi}\{f(\eta(t))\}(s)\,\mathcal{L}^{t_0}_{p,\psi}\{g(\eta(t))\}(s)$$

for all $s > \max\{c_2^f, c_2^g\}$.

Proof. Using Remark 2.5 and the convolution theorem for classic Laplace transform [15, Theorem 2.39] we get

$$\begin{aligned}\mathcal{L}^{t_0}_{p,\psi}\{f(\eta(t))\}(s)\,\mathcal{L}^{t_0}_{p,\psi}\{g(\eta(t))\}(s) &= \mathcal{L}\{f(t)\}(s)\,\mathcal{L}\{g(t)\}(s)\\ &= \mathcal{L}\{(f*g)(t)\}(s) = \mathcal{L}^{t_0}_{p,\psi}\{(f*g)(\eta(t))\}(s).\end{aligned}$$ □

Note that to get the statement as in [4, Theorem 3], one needs to denote

$$(f *_{p,\psi} g)(t) = \int_{t_0}^{t} f(\eta(t) - \eta(\tilde{t}))g(\eta(\tilde{t}))\, \mathrm{d}_{p,\psi}\tilde{t}$$

for $t \geq t_0$. Then one obtains

$$\mathcal{L}_{p,\psi}^{t_0}\{(f *_{p,\psi} g)(t)\}(s) = \mathcal{L}_{p,\psi}^{t_0}\{(f * g)(\eta(t))\}(s),$$

since

$$(f * g)(\eta(t)) = \int_0^{\eta(t)} f(\eta(t) - \tilde{t})g(\tilde{t})\, \mathrm{d}\tilde{t} = \int_{t_0}^{t} f(\eta(t) - \eta(\tilde{t}))g(\eta(\tilde{t}))\, \mathrm{d}_{p,\psi}\tilde{t}$$

for $t \geq t_0$.

Immediately, we obtain a simple generalization:

Corollary 3.6. *Let* $2 \leq n \in \mathbb{N}$, $f_i\colon [0,\infty) \to \mathbb{R}$ *satisfy* (2.3) *with some* $c_1^i, c_2^i > 0$ *for* $i = 1, \ldots, n$. *If* $\mathcal{L}_{p,\psi}^{t_0}\{f_i(\eta(t))\}$ *exists for each* $i = 1, \ldots, n$*, then*

$$\mathcal{L}_{p,\psi}^{t_0}\{(f_1 * \cdots * f_n)(\eta(t))\}(s) = \prod_{i=1}^{n} \mathcal{L}_{p,\psi}^{t_0}\{f_i(\eta(t))\}(s)$$

for all $s > \max_{i=1,\ldots,n} c_2^i$.

Proof. The proof is done by a mathematical induction with respect to n. The case $n = 2$ follows by Lemma 3.5. If the statement holds for $n = k$, then by the same lemma,

$$\begin{aligned}\mathcal{L}_{p,\psi}^{t_0}\{(f_1 * \cdots * f_{k+1})(\eta(t))\}(s) &= \mathcal{L}_{p,\psi}^{t_0}\{((f_1 * \cdots * f_k) * f_{k+1})(\eta(t))\}(s) \\ = \mathcal{L}_{p,\psi}^{t_0}\{(f_1 * \cdots * f_k)(\eta(t))\}(s)\mathcal{L}_{p,\psi}^{t_0}\{f_{k+1}(\eta(t))\}(s) &= \prod_{i=1}^{k+1} \mathcal{L}_{p,\psi}^{t_0}\{f_i(\eta(t))\}(s)\end{aligned}$$

what was to be proved. □

We proceed with analogues to the initial-value theorem [15, Theorem 2.34] and the final-value theorem [15, Theorem 2.36].

Lemma 3.7. *Let a differentiable function* $f\colon [t_0,\infty) \to \mathbb{R}$ *and its derivative* f' *fulfill* (2.3) *for some* $c_1, c_2 > 0$, $c_1', c_2' > 0$*, respectively, and* $\mathcal{L}_{p,\psi}^{t_0}\{f(t)\}$ *exist. Then*

$$f(t_0^+) = \lim_{t \to t_0^+} f(t) = \lim_{s \to \infty} sF_{p,\psi}^{t_0}(s).$$

Proof. Combining Lemma 3.2.iii with Lemma 3.2.iv for $\mathcal{L}_{p,\psi}^{t_0}\{\mathrm{D}_\psi^p f(t)\}$ instead of $F_{p,\psi}^{t_0}$, we get

$$sF_{p,\psi}^{t_0}(s) - f(t_0^+) = \mathcal{L}_{p,\psi}^{t_0}\{\mathrm{D}_\psi^p f(t)\}(s) \to 0$$

as $s \to \infty$. Hence the statement follows. □

Lemma 3.8. *Let a differentiable function $f\colon [t_0,\infty)\to\mathbb{R}$ fulfill* (2.3) *for some $c_1,c_2>0$ and $\mathcal{L}^{t_0}_{p,\psi}\{f(t)\}$ exist. If $\lim_{t\to\infty} f(t)$ exists, then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF^{t_0}_{p,\psi}(s).$$

Proof. First note that f is bounded, due to the existence of the limit. So, it fulfills (2.3) with $c_2=0$. Consequently, $\mathcal{L}^{t_0}_{p,\psi}\{f(t)\}$ exists on $(0,\infty)$. Similarly to the proof of Lemma 3.7, we get

$$\lim_{s\to 0^+} sF^{t_0}_{p,\psi}(s) - f(t_0^+) = \lim_{s\to 0^+} \mathcal{L}^{t_0}_{p,\psi}\{\mathrm{D}^p_\psi f(t)\}(s) = \lim_{s\to 0^+} \int_{t_0}^{\infty} \mathrm{e}^{-s\eta(t)} f'(t)\,\mathrm{d}t$$

where the last identity follows from (2.1). Now since η is increasing, the integral on the right-hand side is uniformly convergent on $[0,c)$ for some $c>0$ due to the well-known Abel's criterion [5, Problem 1.5.36]. Consequently, the order of the limit and the integral can be changed to obtain

$$\lim_{s\to 0^+} \int_{t_0}^{\infty} \mathrm{e}^{-s\eta(t)} f'(t)\,\mathrm{d}t = \int_{t_0}^{\infty} f'(t)\,\mathrm{d}t = \lim_{t\to\infty} f(t) - f(t_0^+).$$

That completes the proof. □

Now we investigate the general conformable Laplace transform of a function with a retarded argument. This will be useful in Section 5.

Lemma 3.9. *Let $\tau>0$, $f\colon[-\tau,\infty)\to\mathbb{R}$ satisfy $|f(t)|\le c_1\,\mathrm{e}^{c_2(t+\tau)}$ for some $c_1,c_2>0$ and all $t\ge-\tau$, and $\mathcal{L}^{t_0}_{p,\psi}\{f(\eta(t))\}$ exist. Then*

$$\mathcal{L}^{t_0}_{p,\psi}\{f(\eta(t)-\tau)\}(s) = \mathrm{e}^{-s\tau}\left(\int_{-\tau}^{0} \mathrm{e}^{-st} f(t)\,\mathrm{d}t + \mathcal{L}^{t_0}_{p,\psi}\{f(\eta(t))\}(s)\right)$$

for all $s>c_2$.

Proof. By Remark 2.5 we obtain

$$\mathcal{L}^{t_0}_{p,\psi}\{f(\eta(t)-\tau)\}(s) = \mathcal{L}\{f(t-\tau)\}(s) = \mathrm{e}^{-s\tau}\left(\int_{-\tau}^{0} \mathrm{e}^{-st} f(t)\,\mathrm{d}t + \mathcal{L}\{f(t)\}(s)\right)$$

and the proof is finished by the same remark. □

From now on, we shall denote $\sigma\colon\mathbb{R}\to\mathbb{R}$ the Heaviside step function defined as

$$\sigma(t) = \begin{cases} 0, & t<0,\\ 1, & t\ge 0.\end{cases}$$

We apply the latter lemma in the following examples.

Example 3.10. Suppose that ψ is a positive fractional conformable function satisfying (2.2).

(i) $\mathcal{L}^{t_0}_{p,\psi}\{\sigma(\eta(t)-\tau)\}(s) = \frac{\mathrm{e}^{-s\tau}}{s}$ for any $s>0$ and $\tau>0$, since $\mathcal{L}^{t_0}_{p,\psi}\{\sigma(\eta(t))\}(s) = \mathcal{L}^{t_0}_{p,\psi}\{1\}(s) = \frac{1}{s}$ by Example 3.4.i.

(ii) For any $n \in \mathbb{N}$, $\tau > 0$, it holds

$$\mathcal{L}_{p,\psi}^{t_0}\left\{\frac{(\eta(t)-n\tau)^{n-1}}{(n-1)!}\,\sigma(\eta(t)-n\tau)\right\}(s)=\left(\frac{\mathrm{e}^{-s\tau}}{s}\right)^{n}$$

for any $s > 0$. This can be proved directly using Lemma 3.9 with $n\tau$ instead of τ, and

$$\mathcal{L}_{p,\psi}^{t_0}\left\{\frac{(\eta(t))^{n-1}}{(n-1)!}\,\sigma(\eta(t))\right\}(s)=\mathcal{L}\left\{\frac{t^{n-1}}{(n-1)!}\right\}(s)=\frac{1}{s^{n}}.$$

(iii) For any $n \in \mathbb{N}$, $\tau_1,\dots,\tau_n > 0$, $k_1,\dots,k_n \in \mathbb{N}_0$ such that $k_1+\dots+k_n > 0$ it holds

$$\mathcal{L}_{p,\psi}^{t_0}\left\{\frac{\left(\eta(t)-\sum_{m=1}^{n}k_m\tau_m\right)^{\sum_{m=1}^{n}k_m-1}}{\left(\sum_{m=1}^{n}k_m-1\right)!}\,\sigma\left(\eta(t)-\sum_{m=1}^{n}k_m\tau_m\right)\right\}(s)=\prod_{m=1}^{n}\left(\frac{\mathrm{e}^{-s\tau_m}}{s}\right)^{k_m}$$

for any $s > 0$. Indeed, it can be shown as in the preceding example. Nevertheless, since

$$\begin{aligned}
&\mathcal{L}_{p,\psi}^{t_0}\left\{\frac{\left(\eta(t)-\sum_{m=1}^{n}k_m\tau_m\right)^{\sum_{m=1}^{n}k_m-1}}{\left(\sum_{m=1}^{n}k_m-1\right)!}\,\sigma\left(\eta(t)-\sum_{m=1}^{n}k_m\tau_m\right)\right\}(s)\\
&=\mathcal{L}\left\{\frac{\left(t-\sum_{m=1}^{n}k_m\tau_m\right)^{\sum_{m=1}^{n}k_m-1}}{\left(\sum_{m=1}^{n}k_m-1\right)!}\,\sigma\left(t-\sum_{m=1}^{n}k_m\tau_m\right)\right\}(s)
\end{aligned}$$

by Remark 2.5, the statement follows also from [14, Lemma 2.3].

(iv) For any $n \in \mathbb{N}$, $\tau_1,\dots,\tau_n > 0$, $k_1,\dots,k_n \in \mathbb{N}_0$ such that $k_1+\dots+k_n > 0$ it holds

$$\mathcal{L}_{p,\psi}^{t_0}\left\{\frac{\left(\eta(t)-\sum_{m=1}^{n}k_m\tau_m\right)^{\sum_{m=1}^{n}k_m}}{\left(\sum_{m=1}^{n}k_m\right)!}\,\sigma\left(\eta(t)-\sum_{m=1}^{n}k_m\tau_m\right)\right\}(s)=\frac{1}{s}\prod_{m=1}^{n}\left(\frac{\mathrm{e}^{-s\tau_m}}{s}\right)^{k_m}$$

for any $s > 0$. Indeed, from the right-hand side using Examples 3.10.i, 3.10.iii and Lemma 3.5, we have

$$\begin{aligned}
\frac{1}{s}\prod_{m=1}^{n}\left(\frac{\mathrm{e}^{-s\tau_m}}{s}\right)^{k_m}&=\mathcal{L}_{p,\psi}^{t_0}\{\sigma(\eta(t))\}(s)\\
&\times\mathcal{L}_{p,\psi}^{t_0}\left\{\frac{\left(\eta(t)-\sum_{m=1}^{n}k_m\tau_m\right)^{\sum_{m=1}^{n}k_m-1}}{\left(\sum_{m=1}^{n}k_m-1\right)!}\,\sigma\left(\eta(t)-\sum_{m=1}^{n}k_m\tau_m\right)\right\}(s)\\
&=\mathcal{L}_{p,\psi}^{t_0}\left\{\left(\frac{\left(\cdot-\sum_{m=1}^{n}k_m\tau_m\right)^{\sum_{m=1}^{n}k_m-1}}{\left(\sum_{m=1}^{n}k_m-1\right)!}\,\sigma\left(\cdot-\sum_{m=1}^{n}k_m\tau_m\right)*\sigma\right)(\eta(t))\right\}(s)
\end{aligned}$$

for any $s > 0$. Then the statement follows from

$$\int_0^{\eta(t)} \frac{\left(\eta(t) - \sum_{m=1}^n k_m\tau_m - q\right)^{\sum_{m=1}^n k_m - 1}}{\left(\sum_{m=1}^n k_m - 1\right)!} \sigma\left(\eta(t) - \sum_{m=1}^n k_m\tau_m - q\right)\sigma(q)\,\mathrm{d}q$$
$$= \int_0^{\eta(t)-\sum_{m=1}^n k_m\tau_m} \frac{\left(\eta(t) - \sum_{m=1}^n k_m\tau_m - q\right)^{\sum_{m=1}^n k_m - 1}}{\left(\sum_{m=1}^n k_m - 1\right)!}\,\mathrm{d}q\,\sigma\left(\eta(t) - \sum_{m=1}^n k_m\tau_m\right)$$
$$= \frac{\left(\eta(t) - \sum_{m=1}^n k_m\tau_m\right)^{\sum_{m=1}^n k_m}}{\left(\sum_{m=1}^n k_m\right)!} \sigma\left(\eta(t) - \sum_{m=1}^n k_m\tau_m\right).$$

Another kind of a delay is considered in the following statement.

Lemma 3.11. *Let $\tau > 0$, $f\colon [t_0 - \tau, \infty) \to \mathbb{R}$ satisfy* (2.3) *for some $c_1, c_2 > 0$ and all $t \geq t_0$, and $\mathcal{L}^{t_0}_{p,\psi}\{f(t)\}$ exist. Let us assume that*

$$\frac{g(t+\tau)}{g(t)} = C(s,\tau), \quad \forall t \geq t_0 \tag{3.1}$$

where $g(t) = (\psi(t,p)\,\mathrm{e}^{s\eta(t)})^{-1}$, i.e., assume that the ratio $\frac{g(t+\tau)}{g(t)}$ is independent of t. Then

$$\mathcal{L}^{t_0}_{p,\psi}\{f(t-\tau)\}(s) = \int_{t_0-\tau}^{t_0} \frac{\mathrm{e}^{-s\eta(t+\tau)} f(t)}{\psi(t+\tau,p)}\,\mathrm{d}t + C(s,\tau)F^{t_0}_{p,\psi}(s)$$

for all $s > c_2$.

Proof. First notice that denoting $\tilde{c}_1 = \max\{c_1, \max_{t\in[t_0-\tau,t_0]}|f(t)|\}$ and using the fact that η is increasing we have

$$|f(t-\tau)| \leq \begin{cases} \tilde{c}_1 \leq \tilde{c}_1\,\mathrm{e}^{c_2\eta(t)}, & t \in [t_0, t_0+\tau), \\ c_1\,\mathrm{e}^{c_2\eta(t-\tau)} \leq \tilde{c}_1\,\mathrm{e}^{c_2\eta(t)}, & t \in [t_0+\tau, \infty). \end{cases}$$

So $\mathcal{L}^{t_0}_{p,\psi}\{f(t-\tau)\}$ exists on (c_2, ∞). Then we compute

$$\mathcal{L}^{t_0}_{p,\psi}\{f(t-\tau)\}(s) = \int_{t_0-\tau}^{t_0} \frac{\mathrm{e}^{-s\eta(t+\tau)} f(t)}{\psi(t+\tau,p)}\,\mathrm{d}t + \int_{t_0}^{\infty} \frac{\mathrm{e}^{-s\eta(t)} f(t)}{\psi(t,p)} C(s,\tau)\,\mathrm{d}t$$
$$= \int_{t_0-\tau}^{t_0} \frac{\mathrm{e}^{-s\eta(t+\tau)} f(t)}{\psi(t+\tau,p)}\,\mathrm{d}t + C(s,\tau)F^{t_0}_{p,\psi}(s)$$

for all $s > c_2$. □

Example 3.12. Here we verify condition (3.1) for several types of ψ:

(i) if $\psi(t,p) \equiv 1$ (GCFD coincides with the classic derivative), then $\eta(t) = t - t_0$, $g(t) = \mathrm{e}^{-s(t-t_0)}$ and
$$\frac{g(t+\tau)}{g(t)} = \mathrm{e}^{-s\tau} = C(s,\tau);$$

(ii) if $\psi(t,p)=p$, then $\eta(t)=\frac{t-t_0}{p}$, $g(t)=\frac{1}{p}\,\mathrm{e}^{-\frac{s}{p}(t-t_0)}$ and

$$\frac{g(t+\tau)}{g(t)}=\mathrm{e}^{-\frac{s\tau}{p}}=C(s,\tau);$$

(iii) if $\psi(t,p)=(t-t_0)^{1-p}$ (conformable derivative), then $\eta(t)=\frac{(t-t_0)^p}{p}$, $g(t)=(t-t_0)^{p-1}\,\mathrm{e}^{-\frac{s}{p}(t-t_0)^p}$ and

$$\frac{g(t+\tau)}{g(t)}=\left(\frac{t-t_0}{t+\tau-t_0}\right)^{1-p}\mathrm{e}^{\frac{s}{p}((t-t_0)^p-(t+\tau-t_0)^p)},$$

i.e., condition (3.1) is not satisfied in the case of the conformable derivative as the left-hand side varies with t;

(iv) if $\psi(t+\tau,p)=\psi(t,p)$ for all $t\geq t_0$, $p\in(0,1]$ (ψ is τ–periodic in t), then

$$\eta(t+\tau)-\eta(t)=\int_t^{t+\tau}\mathrm{d}_{p,\psi}\tilde{t}=\int_t^{t_0+n\tau}\mathrm{d}_{p,\psi}\tilde{t}+\int_{t_0+n\tau}^{t+\tau}\mathrm{d}_{p,\psi}\tilde{t}$$

$$=\int_{t+\tau}^{t_0+(n+1)\tau}\mathrm{d}_{p,\psi}\tilde{t}+\int_{t_0+n\tau}^{t+\tau}\mathrm{d}_{p,\psi}\tilde{t}=\int_{t_0+n\tau}^{t_0+(n+1)\tau}\mathrm{d}_{p,\psi}\tilde{t}=\int_{t_0}^{t_0+\tau}\mathrm{d}_{p,\psi}\tilde{t}=\eta(t_0+\tau),\quad t\geq t_0$$

for some $n\in\mathbb{N}_0$ such that $t\leq t_0+n\tau<t+\tau$. Therefore,

$$\frac{g(t+\tau)}{g(t)}=\mathrm{e}^{-s(\eta(t+\tau)-\eta(t))}=\mathrm{e}^{-s\eta(t_0+\tau)}=C(s,\tau),\quad\forall t\geq t_0.$$

4 General conformable fractional Gronwall inequality

Here we state and prove Gronwall inequality for general conformable fractional calculus and its corollary.

Lemma 4.1. *Let $x,\alpha,\beta\in C([t_0,\infty),\mathbb{R})$, $\beta(t)\geq 0$ for all $t\in[t_0,\infty)$, and ψ be a positive fractional conformable function. If*

$$x(t)\leq\alpha(t)+\int_{t_0}^t\beta(s)x(s)\,\mathrm{d}_{p,\psi}s,\quad\forall t\geq t_0,\tag{4.1}$$

then

$$x(t)\leq\alpha(t)+\int_{t_0}^t\alpha(s)\beta(s)\,\mathrm{e}^{\int_s^t\beta(q)\,\mathrm{d}_{p,\psi}q}\,\mathrm{d}_{p,\psi}s,\quad\forall t\geq t_0.\tag{4.2}$$

Proof. Using the classic Gronwall inequality [3, Lemma 1.6] we get

$$x(t)\leq\alpha(t)+\int_{t_0}^t\alpha(s)\frac{\beta(s)}{\psi(s,p)}\,\mathrm{e}^{\int_s^t\frac{\beta(q)}{\psi(q,p)}\,\mathrm{d}_{p,\psi}q}\,\mathrm{d}_{p,\psi}s,$$

which is (4.2). □

Clearly, Lemma 4.1 remains valid if one takes $[t_0,b]$ for some $b>t_0$ instead of $[t_0,\infty)$. Immediately, we obtain the following corollary for particular functions α,β.

Corollary 4.2. *Under the assumptions of Lemma 4.1, if α is nondecreasing and β is constant, then inequality* (4.1) *implies*

$$x(t) \le \alpha(t)\,\mathrm{e}^{\beta\eta(t)}, \quad \forall t \ge t_0.$$

In particular, if α is constant, then

$$x(t) \le \alpha\,\mathrm{e}^{\beta\eta(t)}, \quad \forall t \ge t_0.$$

Proof. From (4.2) we have

$$\begin{aligned} x(t) &\le \alpha(t)\left(1+\beta\int_{t_0}^{t}\mathrm{e}^{\beta(\eta(t)-\eta(s))}\,\mathrm{d}_{p,\psi}s\right) = \alpha(t)\left(1+\beta\mathrm{e}^{\beta\eta(t)}\int_{t_0}^{t}\mathrm{e}^{-\beta\eta(s)}\,\mathrm{d}_{p,\psi}s\right)\\ &= \alpha(t)\left(1+\beta\mathrm{e}^{\beta\eta(t)}\int_{0}^{\eta(t)}\mathrm{e}^{-\beta s}\,\mathrm{d}s\right) = \alpha(t)\,\mathrm{e}^{\beta\eta(t)} \end{aligned}$$

for all $t \ge t_0$. □

5 Application to delay equations

In this section, we consider Cauchy problems for linear delay differential equations with general conformable fractional derivative, multiple delays and linear parts given by pairwise permutable matrices. We always consider $n \in \mathbb{N}$ constant delays $0 < \tau_1, \ldots, \tau_n$, and denote $\tau := \max\{\tau_1, \ldots, \tau_n\}$. Next, we denote $|\cdot|$ the norm of a vector without any respect to its dimension, and $\|\cdot\|$ the corresponding induced matrix norm.

First we consider the initial-function problem

$$\mathrm{D}^p_\psi x(\eta(t)) = Ax(\eta(t)) + B_1x(\eta(t)-\tau_1) + \cdots + B_nx(\eta(t)-\tau_n) + f(\eta(t)), \quad t \ge t_0 \tag{5.1}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0] \tag{5.2}$$

for a given function $\varphi \in C([-\tau,0],\mathbb{R}^N)$. For brevity we set $C_\varphi := \max_{t\in[-\tau,0]}|\varphi(t)|$. We shall look for a differentiable function $x\colon [-\tau,\infty) \to \mathbb{R}^N$ satisfying the above problem assuming that A, B_1, …, B_n are pairwise permutable $N\times N$ matrices. First we show that x is exponentially bounded.

Lemma 5.1. *If $f\colon [0,\infty) \to \mathbb{R}^N$ is such that $|f(t)| \le c_1\,\mathrm{e}^{c_2t}$ for some $c_1, c_2 > 0$ and all $t \ge 0$, then the solution x of* (5.1)*,* (5.2) *satisfies $|x(t)| \le d_1\,\mathrm{e}^{d_2t}$ for some $d_1, d_2 > 0$ and all $t \ge 0$.*

Proof. Applying the operator $\mathrm{I}^{p,\psi}_{t_0}$ to equation (5.1) and using the identity (see [17, Theorem 11])

$$\mathrm{I}^{p,\psi}_a\mathrm{D}^p_\psi f(u) = f(u) - f(a)$$

for any $u \ge a \ge t_0$, we get

$$\begin{aligned} x(\eta(t)) = x(\eta(t_0)) &+ A\int_{t_0}^{t}x(\eta(\tilde t))\,\mathrm{d}_{p,\psi}\tilde t\\ +\sum_{i=1}^{n}B_i\int_{t_0}^{t}x(\eta(\tilde t)-\tau_i)\,\mathrm{d}_{p,\psi}\tilde t &+ \int_{t_0}^{t}f(\eta(\tilde t))\,\mathrm{d}_{p,\psi}\tilde t, \quad t \ge t_0, \end{aligned}$$

or equivalently

$$x(t)=\varphi(0)+A\int_0^t x(\tilde{t})\,\mathrm{d}\tilde{t}+\sum_{i=1}^n B_i\int_0^t x(\tilde{t}-\tau_i)\,\mathrm{d}\tilde{t}+\int_0^t f(\tilde{t})\,\mathrm{d}\tilde{t},\quad t\ge 0.$$

For the norm we have

$$|x(t)|\le C_\varphi+\|A\|\int_0^t |x(\tilde{t})|\,\mathrm{d}\tilde{t}+\sum_{i=1}^n \|B_i\|\int_0^t |x(\tilde{t}-\tau_i)|\,\mathrm{d}\tilde{t}+\frac{c_1}{c_2}\,\mathrm{e}^{c_2 t}=:z(t),\quad t\ge 0.$$

Note that z is increasing. One can see that for $0\le\tilde{t}\le t$,

$$|x(\tilde{t}-\tau_i)|\le \max_{\zeta\in[0,\tau_i]}|x(\zeta-\tau_i)|+\max_{\zeta\in[\tau_i,\tilde{t}+\tau_i]}|x(\zeta-\tau_i)|\le C_\varphi+z(\tilde{t})\le 2z(\tilde{t})$$

for each $i=1,\dots,n$. Hence

$$z(t)\le C_\varphi+\frac{c_1}{c_2}\,\mathrm{e}^{c_2 t}+\left(\|A\|+2\sum_{i=1}^n\|B_i\|\right)\int_0^t z(\tilde{t})\,\mathrm{d}\tilde{t},\quad t\ge 0.$$

Consequently, by the Gronwall lemma,

$$|x(t)|\le z(t)\le\left(C_\varphi+\frac{c_1}{c_2}\,\mathrm{e}^{c_2 t}\right)\mathrm{e}^{(\|A\|+2\sum_{i=1}^n\|B_i\|)t},\quad t\ge 0 \tag{5.3}$$

and the statement is obvious. □

Clearly, if $|x(t)|\le c_1\,\mathrm{e}^{c_2 t}$ for some $c_1,c_2>0$ and all $t\ge 0$, then

$$|x(t)|\le\begin{cases}\tilde{c}_1\le\tilde{c}_1\,\mathrm{e}^{c_2(t+\tau_i)}, & t\in[-\tau_i,0),\\ c_1\,\mathrm{e}^{c_2 t}\le\tilde{c}_1\,\mathrm{e}^{c_2 t}, & t\in[0,\infty)\end{cases}$$

where $\tilde{c}_1=\max\{c_1,C_\varphi\}$. Thus the general conformable Laplace transform can be applied to (5.1) to obtain the following result.

Theorem 5.2. *Let $n\in\mathbb{N}$, $0<\tau_1,\dots,\tau_n\in\mathbb{R}$, A, B_1, $\dots$, B_n be pairwise permutable $N\times N$ matrices, i.e., $AB_i=B_iA$ and $B_iB_j=B_jB_i$ for each $i,j\in\{1,\dots,n\}$, $\varphi\in C([-\tau,0],\mathbb{R}^N)$, and $f\colon[0,\infty)\to\mathbb{R}^N$ be a given function such that $|f(t)|\le c_1\,\mathrm{e}^{c_2 t}$ for some $c_1,c_2>0$ and all $t\ge 0$. If $p\in(0,1]$ and ψ is a positive fractional conformable function satisfying* (2.2)*, then the solution of the Cauchy problem* (5.1), (5.2) *has the form*

$$x(t)=\begin{cases}\varphi(t), & -\tau\le t<0,\\ \mathcal{B}(t)\varphi(0)+\sum_{j=1}^n B_j\int_0^{\tau_j}\mathcal{B}(t-s)\varphi(s-\tau_j)\,\mathrm{d}s & \\ \quad+\int_0^t\mathcal{B}(t-s)f(s)\,\mathrm{d}s, & 0\le t\end{cases} \tag{5.4}$$

where

$$\mathcal{B}(t)=\mathrm{e}^{At}\sum_{\substack{\sum_{m=1}^n k_m\tau_m\le t\\ k_1,\dots,k_n\ge 0}}\frac{(t-\sum_{m=1}^n k_m\tau_m)^{\sum_{m=1}^n k_m}}{k_1!\dots k_n!}\prod_{m=1}^n\widetilde{B}_m^{k_m}$$

for any $t\in\mathbb{R}$, and $\widetilde{B}_m=B_m\,\mathrm{e}^{-A\tau_m}$ for each $m=1,\dots,n$.

Proof. By Lemma 5.1 and the preceding discussion, the assumptions of Lemma 3.2.iii and Lemma 3.9 are satisfied, and after applying the general conformable Laplace transform to equation (5.1), we get

$$s\mathcal{L}_{p,\psi}^{t_0}\{x(\eta(t))\}(s)-x(\eta(t_0))=A\mathcal{L}_{p,\psi}^{t_0}\{x(\eta(t))\}(s)$$
$$+\sum_{i=1}^{n}B_i\,\mathrm{e}^{-s\tau_i}\left(\int_{-\tau_i}^{0}\mathrm{e}^{-st}\,x(t)\,\mathrm{d}t+\mathcal{L}_{p,\psi}^{t_0}\{x(\eta(t))\}(s)\right)+\mathcal{L}_{p,\psi}^{t_0}\{f(\eta(t))\}(s)$$

for $s>d_2=\|A\|+2\sum_{i=1}^{n}\|B_i\|+c_2$ (see (5.3)) or, using the classic Laplace transform,

$$s\mathcal{L}\{x(t)\}(s)-\varphi(0)=A\mathcal{L}\{x(t)\}(s)$$
$$+\sum_{i=1}^{n}B_i\left(\int_{0}^{\tau_i}\mathrm{e}^{-st}\,\varphi(t-\tau_i)\,\mathrm{d}t+\mathrm{e}^{-s\tau_i}\,\mathcal{L}\{x(t)\}(s)\right)+\mathcal{L}\{f(t)\}(s).$$

This is precisely the Laplace transform of the equation

$$x'(t)=Ax(t)+B_1x(t-\tau_1)+\cdots+B_nx(t-\tau_n)+f(t),\quad t\geq 0 \tag{5.5}$$

along with (5.2), which is known (cf. [14, Theorem 3.3]) to have the solution (5.4). □

Remark 5.3. Equation (5.1) can be directly converted to (5.5). Indeed, since x is differentiable, from (2.1) we have

$$\mathrm{D}_{\psi}^{p}x(\eta(t))=\psi(t,p)\frac{\mathrm{d}}{\mathrm{d}t}x(\eta(t))=x'(\eta(t)).$$

Therefore, from (5.1),

$$x'(\eta(t))=Ax(\eta(t))+B_1x(\eta(t)-\tau_1)+\cdots+B_nx(\eta(t)-\tau_n)+f(\eta(t)),\quad t\geq t_0$$

which is precisely (5.5). However, the proof of Theorem 5.2 uses the properties of the newly defined general conformable Laplace transform.

Now consider the problem

$$\mathrm{D}_{\psi}^{p}x(t)=B_1x(t-\tau_1)+\cdots+B_nx(t-\tau_n)+f(t),\quad t\geq t_0 \tag{5.6}$$
$$x(t)=\varphi(t),\quad t\in[t_0-\tau,t_0]. \tag{5.7}$$

First we have to show that the Laplace transform can be applied.

Lemma 5.4. *If $f\colon[t_0,\infty)\to\mathbb{R}^N$ fulfills* (2.3) *with $c_1,c_2>0$, then there are $d_1,d_2>0$ such that the solution x of* (5.6)*,* (5.7) *satisfies*

$$|x(t)|\leq d_1\,\mathrm{e}^{d_2\eta(t)},\quad\forall t\geq t_0.$$

Proof. Applying the operator $\mathrm{I}_{t_0}^{p,\psi}$ to equation (5.6), we get

$$x(t)=\varphi(t_0)+\sum_{i=1}^{n}B_i\int_{t_0}^{t}x(\tilde{t}-\tau_i)\,\mathrm{d}_{p,\psi}\tilde{t}+\int_{t_0}^{t}f(\tilde{t})\,\mathrm{d}_{p,\psi}\tilde{t},\quad t\geq t_0.$$

Hence, for the norm we have

$$\begin{aligned}|x(t)| &\le |\varphi(t_0)| + \sum_{i=1}^{n} \|B_i\| \int_{t_0}^{t} |x(\tilde{t}-\tau_i)| \,\mathrm{d}_{p,\psi}\tilde{t} + c_1 \int_{t_0}^{t} \mathrm{e}^{c_2\eta(\tilde{t})} \,\mathrm{d}_{p,\psi}\tilde{t} \\ &\le C_\varphi + \sum_{i=1}^{n} \|B_i\| \int_{t_0}^{t} |x(\tilde{t}-\tau_i)| \,\mathrm{d}_{p,\psi}\tilde{t} + \frac{c_1}{c_2} \mathrm{e}^{c_2\eta(t)} =: z(t), \quad t \ge t_0,\end{aligned}$$

since

$$\int_{t_0}^{t} \mathrm{e}^{c_2\eta(\tilde{t})} \,\mathrm{d}_{p,\psi}\tilde{t} = \int_{0}^{\eta(t)} \mathrm{e}^{c_2\tilde{t}} \,\mathrm{d}\tilde{t} = \frac{\mathrm{e}^{c_2\eta(t)} - 1}{c_2}.$$

As in the proof of Lemma 5.1, function z is increasing and

$$|x(\tilde{t}-\tau_i)| \le 2z(\tilde{t}), \quad \forall \tilde{t} \in [t_0, t]$$

for each $i = 1, \dots, n$. Therefore,

$$z(t) \le C_\varphi + \frac{c_1}{c_2} \mathrm{e}^{c_2\eta(t)} + 2 \sum_{i=1}^{n} \|B_i\| \int_{t_0}^{t} z(\tilde{t}) \,\mathrm{d}_{p,\psi}\tilde{t}, \quad t \ge t_0.$$

Applying Corollary 4.2 we obtain

$$|x(t)| \le z(t) \le \left(C_\varphi + \frac{c_1}{c_2} \mathrm{e}^{c_2\eta(t)} \right) \mathrm{e}^{2\sum_{i=1}^{n} \|B_i\| \eta(t)}, \quad t \ge t_0.$$

So, the statement holds with $d_1 = C_\varphi + \frac{c_1}{c_2}$ and $d_2 = c_2 + 2\sum_{i=1}^{n} \|B_i\|$. □

Now we know that the general conformable Laplace transform of $x(t)$ and $\mathrm{D}_{\psi}^{p} x(t)$ exists if f fulfills (2.3). In the next result, we assume the empty sum property, i.e., $\sum_{i\in\emptyset} z(i) = 0$ for any function z.

Theorem 5.5. *Let $n \in \mathbb{N}$, $0 < T \in \mathbb{R}$, $0 < \tau_1, \dots, \tau_n \in \mathbb{R}$, $\tau_i = \lambda_i T$ for some $\lambda_i \in \mathbb{N}$, $i = 1, \dots, n$, $B_1, \dots, B_n$ be pairwise permutable $N \times N$ matrices, $\varphi \in C([t_0 - \tau, t_0], \mathbb{R}^N)$, and $f\colon [t_0, \infty) \to \mathbb{R}^N$ be a given function satisfying* (2.3) *for some $c_1, c_2 > 0$. If $p \in (0, 1]$ and ψ is a positive fractional conformable function T–periodic in t, then the solution of the Cauchy problem* (5.6), (5.7) *has the form*

$$x(t) = \begin{cases} \varphi(t), & t_0 - \tau \le t < t_0, \\ \mathcal{A}(\eta(t))\varphi(t_0) + \sum_{j=1}^{n} B_j \int_{t_0}^{t_0+\tau_j} \mathcal{A}(\eta(t) - \eta(s))\varphi(s - \tau_j) \,\mathrm{d}_{p,\psi} s & \\ \quad + \int_{t_0}^{t} \mathcal{A}(\eta(t) - \eta(s)) f(s) \,\mathrm{d}_{p,\psi} s, & t_0 \le t \end{cases}$$

where

$$\mathcal{A}(t) = \sum_{\substack{\sum_{m=1}^{n} k_m\lambda_m\eta(t_0+T) \le t \\ k_1, \dots, k_n \ge 0}} \frac{\left(t - \sum_{m=1}^{n} k_m\lambda_m\eta(t_0+T) \right)^{\sum_{m=1}^{n} k_m}}{k_1! \dots k_n!} \prod_{m=1}^{n} B_m^{k_m}$$

for any $t \in \mathbb{R}$.

Proof. Let us consider the T-periodic extension of $\psi(\cdot,p)$ on $\mathbb{R}$ and denote it again by ψ. Then the function η is defined on the whole of $\mathbb{R}$. As in Example 3.12.iv, it holds

$$\begin{aligned}\eta(t+\tau_i)-\eta(t) &= \eta(t_0+\tau_i)\\ =\int_{t_0}^{t_0+T}\mathrm{d}_{p,\psi}t+\int_{t_0+T}^{t_0+2T}\mathrm{d}_{p,\psi}t+\dots+\int_{t_0+(\lambda_i-1)T}^{t_0+\tau_i}\mathrm{d}_{p,\psi}t\\ =\int_{t_0}^{t_0+T}\mathrm{d}_{p,\psi}t+\int_{t_0}^{t_0+T}\mathrm{d}_{p,\psi}t+\dots+\int_{t_0}^{t_0+T}\mathrm{d}_{p,\psi}t=\lambda_i\eta(t_0+T)\end{aligned} \tag{5.8}$$

for each $i=1,\dots,n$ and any $t\in\mathbb{R}$. In particular,

$$\eta(t)-\eta(t-\tau_i)=\lambda_i\eta(t_0+T).$$

Consequently,

$$x(t-\tau_i)=x(\omega(\eta(t-\tau_i)))=x(\omega(\eta(t)-\lambda_i\eta(t_0+T))) \tag{5.9}$$

for each $i=1,\dots,n$ and any $t\in[t_0,\infty)$. Note that for each $i=1,\dots,n$,

$$\begin{aligned}\mathrm{e}^{-s\lambda_i\eta(t_0+T)}\int_{-\lambda_i\eta(t_0+T)}^{0}\mathrm{e}^{-st}\,x(\omega(t))\,\mathrm{d}t &= \int_{t_0-\tau_i}^{t_0}\frac{\mathrm{e}^{-s(\lambda_i\eta(t_0+T)+\eta(t))}\,x(t)}{\psi(t+\tau_i,p)}\,\mathrm{d}t\\ =\int_{t_0-\tau_i}^{t_0}\frac{\mathrm{e}^{-s\eta(t+\tau_i)}\,x(t)}{\psi(t+\tau_i,p)}\,\mathrm{d}t &= \int_{t_0}^{t_0+\tau_i}\mathrm{e}^{-s\eta(t)}\,\varphi(t-\tau_i)\,\mathrm{d}_{p,\psi}t\end{aligned}$$

where the first identity follows from (5.8). By Lemma 5.4, the general conformable Laplace transform can be applied to equation (5.6). Then by Lemma 3.2.iii, Lemma 3.9 (with $f=x\circ\omega$ and $\tau=\lambda_i\eta(t_0+T)$, see (5.9)) and Example 3.12.iv, we get

$$\begin{aligned}&sX_{p,\psi}^{t_0}(s)-x(t_0)\\ &=\sum_{i=1}^{n}B_i\left(\int_{t_0}^{t_0+\tau_i}\frac{\mathrm{e}^{-s\eta(t)}\,\varphi(t-\tau_i)}{\psi(t,p)}\,\mathrm{d}t+\mathrm{e}^{-s\lambda_i\eta(t_0+T)}\,X_{p,\psi}^{t_0}(s)\right)+F_{p,\psi}^{t_0}(s)\\ &=\sum_{i=1}^{n}B_i\left(\mathcal{L}_{p,\psi}^{t_0}\{\Phi(t-\tau_i)\}(s)+\mathrm{e}^{-s\lambda_i\eta(t_0+T)}\,X_{p,\psi}^{t_0}(s)\right)+F_{p,\psi}^{t_0}(s)\end{aligned}$$

for $s>c_2+2\sum_{i=1}^{n}\|B_i\|$ (cf. Lemma 5.4 and its proof), where

$$\Phi(t)=\begin{cases}\varphi(t), & t\in[t_0-\tau,t_0],\\ 0, & t\in(t_0,\infty).\end{cases}$$

Consequently,

$$\left(s\mathbb{I}-\sum_{i=1}^{n}B_i\,\mathrm{e}^{-s\lambda_i\eta(t_0+T)}\right)X_{p,\psi}^{t_0}(s)=\varphi(t_0)+\sum_{i=1}^{n}B_i\mathcal{L}_{p,\psi}^{t_0}\{\Phi(t-\tau_i)\}(s)+F_{p,\psi}^{t_0}(s)$$

with $\mathbb{I}$ being the $N\times N$ identity matrix. It is known (see e.g. [16, Proposition 7.5]) that if s is such that

$$\left\|\sum_{i=1}^{n}B_i\,\mathrm{e}^{-s\lambda_i\eta(t_0+T)}\right\|<s,$$

then the matrix $\mathbb{I} - \sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-s\lambda_i \eta(t_0+T)}}{s}$ is invertible and it holds

$$\left(\mathbb{I} - \sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-s\lambda_i \eta(t_0+T)}}{s}\right)^{-1} = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-s\lambda_i \eta(t_0+T)}}{s}\right)^{k}.$$

So for s sufficiently large, we can write

$$X_{p,\psi}^{t_0}(s) = A_0 + \sum_{j=1}^{n} A_j + A_f$$

where

$$A_0 = \frac{1}{s} \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-s\lambda_i \eta(t_0+T)}}{s}\right)^{k} \varphi(t_0),$$

$$A_j = \frac{1}{s} \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-s\lambda_i \eta(t_0+T)}}{s}\right)^{k} B_j \mathcal{L}_{p,\psi}^{t_0}\{\Phi(t-\tau_j)\}(s), \quad j = 1, \ldots, n,$$

$$A_f = \frac{1}{s} \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-s\lambda_i \eta(t_0+T)}}{s}\right)^{k} F_{p,\psi}^{t_0}(s).$$

By Example 3.4.i,

$$A_0 = \mathcal{L}_{p,\psi}^{t_0}\{1\}(s)\varphi(t_0) + \frac{1}{s} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-s\lambda_i \eta(t_0+T)}}{s}\right)^{k} \varphi(t_0).$$

Next, by multinomial theorem [2],

$$\begin{aligned} A_0 &= \mathcal{L}_{p,\psi}^{t_0}\{1\}(s)\varphi(t_0) + \sum_{k=1}^{\infty} \frac{1}{s} \sum_{\substack{k_1+\cdots+k_n=k \\ k_1,\ldots,k_n \geq 0}} \binom{k}{k_1, \ldots, k_n} \prod_{m=1}^{n} \left(\frac{B_m \mathrm{e}^{-s\lambda_m \eta(t_0+T)}}{s}\right)^{k_m} \varphi(t_0) \\ &= \mathcal{L}_{p,\psi}^{t_0}\{1\}(s)\varphi(t_0) + \sum_{k=1}^{\infty} \sum_{\substack{k_1+\cdots+k_n=k \\ k_1,\ldots,k_n \geq 0}} \binom{k}{k_1, \ldots, k_n} \left(\frac{1}{s} \prod_{m=1}^{n} \left(\frac{\mathrm{e}^{-s\lambda_m \eta(t_0+T)}}{s}\right)^{k_m}\right) \left(\prod_{m=1}^{n} B_m^{k_m} \varphi(t_0)\right) \end{aligned}$$

where

$$\binom{k}{k_1, \ldots, k_n} = \frac{k!}{k_1! \ldots k_n!}$$

is the multinomial coefficient. Using Example 3.10.iv and the linearity of the general conformable Laplace transform, we get

$$\begin{aligned} A_0 = {} & \mathcal{L}_{p,\psi}^{t_0}\{1\}(s)\varphi(t_0) + \sum_{k=1}^{\infty} \sum_{\substack{k_1+\cdots+k_n=k \\ k_1,\ldots,k_n \geq 0}} \binom{k}{k_1, \ldots, k_n} \\ & \times \mathcal{L}_{p,\psi}^{t_0} \left\{ \frac{\left(\eta(t) - \sum_{m=1}^{n} k_m \lambda_m \eta(t_0+T)\right)^{\sum_{m=1}^{n} k_m}}{\left(\sum_{m=1}^{n} k_m\right)!} \sigma\left(\eta(t) - \sum_{m=1}^{n} k_m \lambda_m \eta(t_0+T)\right) \right\}(s) \end{aligned}$$

$$\times \prod_{m=1}^{n} B_m^{k_m} \varphi(t_0) = \mathcal{L}_{p,\psi}^{t_0}\{\mathcal{A}(\eta(t))\varphi(t_0)\}(s).$$

Next, using the same arguments we obtain

$$A_j = B_j \mathcal{L}_{p,\psi}^{t_0}\{\mathcal{A}(\eta(t))\}(s)\mathcal{L}_{p,\psi}^{t_0}\{\Phi(t-\tau_j)\}(s)$$

for each $j = 1, \dots, n$. Then by Lemma 3.5,

$$\begin{aligned} A_j &= B_j \mathcal{L}_{p,\psi}^{t_0}\{(\mathcal{A} * \Phi(\omega(\cdot)-\tau_j))(\eta(t))\}(s) \\ &= B_j \mathcal{L}_{p,\psi}^{t_0}\left\{\int_0^{\eta(t)} \mathcal{A}(\eta(t)-q)\Phi(\omega(q)-\tau_j)\,\mathrm{d}q\right\}(s) \\ &= B_j \mathcal{L}_{p,\psi}^{t_0}\left\{\int_{t_0}^{t} \mathcal{A}(\eta(t)-\eta(u))\Phi(u-\tau_j)\,\mathrm{d}_{p,\psi}u\right\}(s). \end{aligned}$$

Analogously,

$$A_f = \mathcal{L}_{p,\psi}^{t_0}\{\mathcal{A}(\eta(t))\}(s)\mathcal{L}_{p,\psi}^{t_0}\{f(t)\}(s) = \mathcal{L}_{p,\psi}^{t_0}\left\{\int_{t_0}^{t} \mathcal{A}(\eta(t)-\eta(u))f(u)\,\mathrm{d}_{p,\psi}u\right\}(s).$$

Summarizing,

$$\begin{aligned} X_{p,\psi}^{t_0}(s) = \mathcal{L}_{p,\psi}\Bigg\{&\mathcal{A}(\eta(t))\varphi(t_0) + \sum_{j=1}^{n} B_j \int_{t_0}^{t} \mathcal{A}(\eta(t)-\eta(u))\Phi(u-\tau_j)\,\mathrm{d}_{p,\psi}u \\ &+ \int_{t_0}^{t} \mathcal{A}(\eta(t)-\eta(u))f(u)\,\mathrm{d}_{p,\psi}u\Bigg\}(s) \end{aligned}$$

for all s sufficiently large. Moreover, note that $\mathcal{A}(t) = 0$ whenever $t < 0$ due to the empty sum property. Therefore

$$\begin{aligned} &\int_{t_0}^{t} \mathcal{A}(\eta(t)-\eta(u))\Phi(u-\tau_j)\,\mathrm{d}_{p,\psi}u \\ &= \int_{t_0}^{\min\{t,t_0+\tau_j\}} \mathcal{A}(\eta(t)-\eta(u))\varphi(u-\tau_j)\,\mathrm{d}_{p,\psi}u \\ &= \int_{t_0}^{t_0+\tau_j} \mathcal{A}(\eta(t)-\eta(u))\varphi(u-\tau_j)\,\mathrm{d}_{p,\psi}u, \end{aligned}$$

and the statement is proved. □

Next, we present a result on the solution of initial-function problem consisting of the equation (5.6) with a linear nondelayed term, i.e.,

$$\mathrm{D}_{\psi}^{p} x(t) = Ax(t) + B_1 x(t-\tau_1) + \cdots + B_n x(t-\tau_n) + f(t), \quad t \geq t_0, \tag{5.10}$$

and initial condition (5.7).

Corollary 5.6. *Let $n \in \mathbb{N}$, $0 < T \in \mathbb{R}$, $0 < \tau_1, \ldots, \tau_n \in \mathbb{R}$, $\tau_i = \lambda_i T$ for some $\lambda_i \in \mathbb{N}$, $i = 1, \ldots, n$, A, B_1, $\ldots$, B_n be pairwise permutable $N \times N$ matrices, $\varphi \in C([t_0 - \tau, t_0], \mathbb{R}^N)$, and $f\colon [t_0, \infty) \to \mathbb{R}^N$ be a given function satisfying* (2.3) *for some $c_1, c_2 > 0$. If $p \in (0,1]$ and ψ is a positive fractional conformable function T–periodic in t, then the solution of the Cauchy problem* (5.10), (5.7) *has the form*

$$x(t) = \begin{cases} \varphi(t), & t_0 - \tau \le t < t_0, \\ \widetilde{\mathscr{A}}(\eta(t))\varphi(t_0) + \sum_{j=1}^n B_j \int_{t_0}^{t_0+\tau_j} \widetilde{\mathscr{A}}(\eta(t) - \eta(s))\varphi(s - \tau_j)\,\mathrm{d}_{p,\psi} s \\ + \int_{t_0}^t \widetilde{\mathscr{A}}(\eta(t) - \eta(s)) f(s)\,\mathrm{d}_{p,\psi} s, & t_0 \le t \end{cases}$$

with

$$\widetilde{\mathscr{A}}(t) = \mathrm{e}^{At} \sum_{\substack{\sum_{m=1}^n k_m \lambda_m \eta(t_0+T) \le t \\ k_1, \ldots, k_n \ge 0}} \frac{\left(t - \sum_{m=1}^n k_m \lambda_m \eta(t_0 + T)\right)^{\sum_{m=1}^n k_m}}{k_1! \ldots k_n!} \prod_{m=1}^n \widetilde{B}_m^{k_m}$$

for any $t \in \mathbb{R}$, where $\widetilde{B}_m = \mathrm{e}^{-A\lambda_m \eta(t_0+T)} B_m$, $m = 1, \ldots, n$.

Proof. As in the proof of Theorem 5.5, we have identity (5.8). Let us denote $y(t) = \mathrm{e}^{-A\eta(t)} x(t)$. Then by CFD of a product [17, Theorem 5] and (2.1), y satisfies

$$\begin{aligned} \mathrm{D}_\psi^p y(t) &= \mathrm{D}_\psi^p \left(\mathrm{e}^{-A\eta(t)}\right) x(t) + \mathrm{e}^{-A\eta(t)} \mathrm{D}_\psi^p x(t) = -A\,\mathrm{e}^{-A\eta(t)} x(t) + \mathrm{e}^{-A\eta(t)} \mathrm{D}_\psi^p x(t) \\ &= -Ay(t) + \mathrm{e}^{-A\eta(t)} \left(Ax(t) + \sum_{i=1}^n B_i \,\mathrm{e}^{A\eta(t-\tau_i)} y(t - \tau_i) + f(t) \right) \\ &= \sum_{i=1}^n B_i \,\mathrm{e}^{-A(\eta(t) - \eta(t-\tau_i))} y(t - \tau_i) + \mathrm{e}^{-A\eta(t)} f(t) \end{aligned}$$

for any $t \ge t_0$. So, we get the initial-function problem for y:

$$\begin{aligned} &\mathrm{D}_\psi^p y(t) = \widetilde{B}_1 y(t - \tau_1) + \cdots + \widetilde{B}_n y(t - \tau_n) + \tilde{f}(t), \quad t \ge t_0 \\ &y(t) = \widetilde{\varphi}(t), \quad t \in [t_0 - \tau, t_0] \end{aligned}$$

where $\tilde{f}(t) = \mathrm{e}^{-A\eta(t)} f(t)$ and $\widetilde{\varphi}(t) = \mathrm{e}^{-A\eta(t)} \varphi(t)$; which is of the form (5.6), (5.7). Note that $\widetilde{\varphi}(t_0) = \varphi(t_0)$ and

$$\widetilde{B}_i \widetilde{\varphi}(s - \tau_i) = B_i \,\mathrm{e}^{-A\lambda_i \eta(t_0+T) - A\eta(s-\tau_i)} \varphi(s - \tau_i) = B_i \,\mathrm{e}^{-A\eta(s)} \varphi(s - \tau_i)$$

for each $i = 1, \ldots, n$. Application of Theorem 5.5 and returning back to x proves the statement. □

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