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Jensen-Type Inequalities on Time Scales For n-Convex Functions

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Abstract

By utilizing some scalar inequalities obtained via Hermite's interpolating polynomial, we will obtain lower and upper bounds for the difference in Jensen's inequality and in the Edmundson-Lah-Ribarič inequality in time scale calculus that hold for the class of n-convex functions. Main results are then applied to generalized means, with a particular emphasis to power means, and in that way some new reverse relations for generalized and power means that correspond to n-convex functions are obtained.

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1 Introduction and preliminaries

1.1 On time scale calculus

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [14] in 1988 as a unification of the theory of difference equations with that of differential equations, unifying integral and differential calculus with the calculus of finite differences, extending to cases "in between" and offering a formalism for studying hybrid discrete-continuous dynamic systems. It has applications in any field that requires simultaneous modelling of

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discrete and continuous time. Now, we briefly introduce the time scales calculus and refer to [1, 15, 16] and the books [11, 12] for further details.

By a time scale $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$. The two most popular examples of time scales are the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$. Since the time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the *backward jump operator* by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, the convention is $\inf\,\emptyset = \sup\,\mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup\,\emptyset = \inf\,\mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t). If $\sigma(t) > t$, then we say that t is *right-scattered*, and if $\rho(t) < t$, then we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Also, if $\sigma(t) = t$, then t is said to be *right-dense*, and if $\rho(t) = t$, then t is said to be *left-dense*. Points that are simultaneously right-dense and left-dense are called *dense*. The mapping $\mu : \mathbb{T} \to [0,\infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

is called the *graininess function*. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

In the following considerations, $\mathbb{T}$ will denote a time scale, $I_\mathbb{T} = I \cap \mathbb{T}$ will denote a time scale interval (for any open or closed interval I in $\mathbb{R}$), and $[0,\infty)_\mathbb{T}$ will be used for the time scale interval $[0,\infty) \cap \mathbb{T}$.

Definition 1.1. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \le \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in U_\mathbb{T}.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t. We say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 1.2. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by $\mathrm{C_{rd}}$ the set of all rd-continuous functions. We say that f is rd-continuously delta differentiable (and write $f \in \mathrm{C}^1_{\mathrm{rd}}$) if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and $f^\Delta \in \mathrm{C_{rd}}$.

Definition 1.3. A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Then we define the *delta integral* by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

The importance of rd-continuous function is revealed by the following result.

Theorem 1.4. *Every rd-continuous function has a delta antiderivative.*

1.2 On positive linear functionals and time scale integrals

First we recall the following definition from [21].

Definition 1.5. Let E be a nonempty set and L be a linear class of real-valued functions $f : E \to \mathbb{R}$ having the following properties.

(L_1) If $f,g \in L$ and $\alpha,\beta \in \mathbb{R}$, then $(\alpha f + \beta g) \in L$.

(L_2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

A positive linear functional is a functional $A : L \to \mathbb{R}$ having the following properties.

(A_1) If $f,g \in L$ and $\alpha,\beta \in \mathbb{R}$, then $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$.

(A_2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

In [3, 8, 5, 6], the authors presented a series of inequalities for the time scale integral and showed that it is not necessary to prove such kind of inequalities "from scratch" in the time scale setting as they can be obtained easily from well-known inequalities for positive linear functionals since the time scale integral is in fact a positive linear functional. This method is extensively studied in the monograph [4], which is the main monograph in the area of Jensen-type inequalities on time scales. For some recent results concerning inequalities on time scale, the reader is reffer to [13, 19].

Now we quote three theorems from [3] that we need in our research.

Theorem 1.6. *Let $\mathbb{T}$ be a time scale. For $a,b \in \mathbb{T}$ with $a < b$, let*

$$E = [a,b) \cap \mathbb{T} \quad \text{and} \quad L = \mathrm{C_{rd}}(E,\mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, the delta integral $\int_a^b f(t)\Delta t$ is a positive linear functional which satisfies conditions (A_1) and (A_2).

Corresponding versions of Theorem 1.6 for nabla and α-diamond integrals are also given in [3].

Multiple Riemann integration and multiple Lebesgue integration on time scale was introduced in [9] and [10], respectively, and both integrals are also positive linear functionals.

Theorem 1.7. *Let $\mathbb{T}_1,\ldots,\mathbb{T}_n$ be time scales. For $a_i,b_i \in \mathbb{T}_i$ with $a_i < b_i$, $1 \leq i \leq n$, let*

$$\mathcal{E} \subset ([a_1,b_1) \cap \mathbb{T}_1) \times \cdots \times ([a_n,b_n) \cap \mathbb{T}_n)$$

be Lebesgue Δ-measurable and let L be the set of all Δ-measurable functions from $\mathcal{E}$ to $\mathbb{R}$. Then (L_1) and (L_2) are satisfied. Moreover, the multiple Lebesgue delta integral on time scales $\int_{\mathcal{E}} f(t)\Delta t$ is a positive linear functional and satisfies conditions (A_1) and (A_2).

Theorem 1.8. *Under the assumptions of Theorem 1.7, the delta integral $\frac{\int_{\mathcal{E}} h(t)f(t)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}$, where $h : \mathcal{E} \to \mathbb{R}$ is nonnegative, Δ-integrable and $\int_{\mathcal{E}} h(t)\Delta t > 0$, is also a positive linear functional satisfying (A_1), (A_2) and $A(1) = 1$.*

1.3 On the Jensen and Edmundson-Lah-Ribarič inequalities

Using the known Jensen inequality for positive linear functionals ([21]) and Theorem 1.8, M. Anwar, R. Bibi, M. Bohner and J. Pečarić proved in [3] the following generalization of Jensen's inequality on time scales.

Theorem 1.9. *Assume $\phi \in C(I,\mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval. Let $\mathcal{E} \subset \mathbb{R}^n$ be as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E}) = I$. Moreover, let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative, Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t > 0$. Then*

$$\phi\left(\frac{\int_{\mathcal{E}} h(t)f(t)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}\right) \le \frac{\int_{\mathcal{E}} h(t)\phi(f(t))\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}. \tag{1.1}$$

The authors in [17] proved the converse of Jensen's inequality for convex functions (see also [20]). Beesack and Pečarić gave in [7] the generalization of Edmundson-Lah-Ribarič's inequality for positive linear functionals. Applying the fact that the multiple Lebesgue delta time scale integral is a positive linear functional (Theorem 1.8) to Beesack-Pečarić's result from [7], the following theorem is proved in [3].

Theorem 1.10. *Assume $\phi \in C(I,\mathbb{R})$ is convex, where $I = [a,b] \subset \mathbb{R}$, with $a < b$. Let $\mathcal{E} \subset \mathbb{R}^n$ be as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E}) = I$. Moreover, let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative, Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t > 0$. Then*

$$\frac{\int_{\mathcal{E}} h(t)\phi(f(t))\Delta t}{\int_{\mathcal{E}} h(t)\Delta t} \le \frac{b - \frac{\int_{\mathcal{E}} h(t)f(t)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}}{b-a}\phi(a) + \frac{\frac{\int_{\mathcal{E}} h(t)f(t)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t} - a}{b-a}\phi(b). \tag{1.2}$$

1.4 On n-convex functions

Definition of n-convex functions is characterized by n-th order divided differences. The n-th order divided difference of a function $f\colon [a,b] \to \mathbb{R}$ at mutually distinct points $t_0, t_1, ..., t_n \in [a,b]$ is defined recursively by

$$f[t_i] = f(t_i), \quad i = 0, ..., n,$$
$$f[t_0, ..., t_n] = \frac{f[t_1, ..., t_n] - f[t_0, ..., t_{n-1}]}{t_n - t_0}.$$

The value $f[t_0, ..., t_n]$ is independent of the order of the points $t_0, ..., t_n$.

Definition of divided differences can be extended to include the cases in which some or all the points coincide (see e.g. [2], [21]):

$$f[\underbrace{a, ..., a}_{n\ times}] = \frac{1}{(n-1)!} f^{(n-1)}(a), \quad n \in \mathbb{N}.$$

A function $f\colon [a,b]\to\mathbb{R}$ is said to be n-convex ($n\geq 0$) if and only if for all choices of $(n+1)$ distinct points $t_0,t_1,...,t_n\in[a,b]$, we have $f[t_0,...,t_n]\geq 0$.

Recently, in the paper [18], R. Mikić Đ. Pečarić and J. Pečarić proved some representations of the left side in the scalar Edmundson-Lah-Ribarič inequality via Hermite's interpolating polynomials in terms of divided differences. Now, we quote their results.

Lemma 1.11. *Let a,b be real numbers such that $a<b$. For a function $\phi\in C^n([a,b])$, $n\geq 3$ the following identities hold:*

$$\phi(t)-\frac{b-t}{b-a}\phi(a)-\frac{t-a}{b-a}\phi(b)=\sum_{k=2}^{n-1}\phi[a;\underbrace{b,...,b}_{k\ times}](t-a)(t-b)^{k-1}+R_1(t) \tag{1.3}$$

$$\begin{aligned}\phi(t)-\frac{b-t}{b-a}\phi(a)-\frac{t-a}{b-a}\phi(b)&=\phi[a,a;b](t-a)(t-b)\\&+\sum_{k=2}^{n-2}\phi[a,a;\underbrace{b,...,b}_{k\ times}](t-a)^2(t-b)^{k-1}+R_2(t),\end{aligned} \tag{1.4}$$

where

$$R_m(t)=(t-a)^m(t-b)^{n-m}\phi[t;\underbrace{a,...,a}_{m\ times};\underbrace{b,b,...,b}_{(n-m)\ times}]. \tag{1.5}$$

In addition, if $n>m\geq 3$, then we have

$$\begin{aligned}\phi(t)-\frac{b-t}{b-a}\phi(a)-\frac{t-a}{b-a}\phi(b)&=(t-a)\left(\phi[a,a]-\phi[a,b]\right)+\sum_{k=2}^{m-1}\frac{\phi^{(k)}(a)}{k!}(t-a)^k\\&+\sum_{k=1}^{n-m}\phi[\underbrace{a,...,a}_{m\ times};\underbrace{b,...,b}_{k\ times}](t-a)^m(t-b)^{k-1}+R_m(t).\end{aligned} \tag{1.6}$$

Lemma 1.12. *Let a,b be real numbers such that $a<b$. For a function $\phi\in C^n([a,b])$, $n\geq 3$, the following identities hold:*

$$\phi(t)-\frac{b-t}{b-a}\phi(a)-\frac{t-a}{b-a}\phi(b)=\sum_{k=2}^{n-1}\phi[b;\underbrace{a,...,a}_{k\ times}](t-b)(t-a)^{k-1}+R_1^*(t) \tag{1.7}$$

$$\begin{aligned}\phi(t)-\frac{b-t}{b-a}\phi(a)-\frac{t-a}{b-a}\phi(b)&=\phi[b,b;a](t-b)(t-a)\\&+\sum_{k=2}^{n-2}\phi[b,b;\underbrace{a,...,a}_{k\ times}](t-b)^2(t-a)^{k-1}+R_2^*(t),\end{aligned} \tag{1.8}$$

where

$$R_m^*(t)=\phi[t;\underbrace{b,...,b}_{m\ times};\underbrace{a,a,...,a}_{(n-m)\ times}](t-b)^m(t-a)^{n-m}. \tag{1.9}$$

If $n > m \geq 3$, then additionally we have

$$\phi(t) - \frac{b-t}{b-a}\phi(a) - \frac{t-a}{b-a}\phi(b) = (b-t)\left(\phi[a,b] - \phi[b,b]\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!}(t-b)^k$$
$$+ \sum_{k=1}^{n-m} \phi[\underbrace{b,...,b}_{m\ times};\underbrace{a,...,a}_{k\ times}](t-b)^m (t-a)^{k-1} + R_m^*(t). \tag{1.10}$$

The rest of this paper is organized as follows: in Section 2 we will obtain various lower and upper bounds for the difference in the Edmundson-Lah-Ribarič inequality in time scale calculus that hold for the class of n-convex functions; in Section 3 we will utilize our results from Section 2 in order to get different lower and upper bounds for the difference in the Jensen inequality in the same settings, and finally in Section 4 we will apply all of the results to generalized means, with a particular emphasis to power means, and in that way we will get some new reverse relations for generalized and power means that correspond to the class of n-convex functions.

2 Inequalities of the Edmundson-Lah-Ribarič type on time scales

For simplicity, we introduce the notations

$$L_\Delta(f) = \int_{\mathcal{E}} f(t)\Delta t \quad \text{and} \quad \overline{L}_\Delta(f,h) = \frac{\int_{\mathcal{E}} f(t)h(t)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t},$$

where $f : \mathcal{E} \to \mathbb{R}$ is Δ-integrable and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t > 0$.

Throughout this paper, whenever mentioning the interval $[a,b]$, we assume that a, b are finite real numbers such that $a < b$. We can write the Edmundson-Lah-Ribarič inequality (1.2) in the form

$$\alpha_\phi \overline{L}_\Delta(f,h) + \beta_\phi - \overline{L}_\Delta(\phi(f),h) \geq 0 \tag{2.1}$$

with standard notation

$$\alpha_\phi = \frac{\phi(b)-\phi(a)}{b-a} \quad \text{and} \quad \beta_\phi = \frac{b\phi(a) - a\phi(b)}{b-a}.$$

A generalization of the Edmundson-Lah-Ribarič inequality (1.2) obtained from Lemma 1.11 is given in the following theorem.

Theorem 2.1. *Let $\phi \in C^n([a,b])$ be an n-convex function. Assume $\mathcal{E}$ is as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E}) = [a,b]$. Moreover, let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t > 0$. If $n > m \geq 3$ are of different parity, then*

$$\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \tag{2.2}$$
$$\leq (\overline{L}_\Delta(f,h) - a)\left(\phi'(a) - \phi[a,b]\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(a)}{k!} \overline{L}_\Delta\left((f - a\mathbf{1})^k, h\right)$$

$$+\sum_{k=1}^{n-m}\phi[\underbrace{a,...,a}_{m\ times};\underbrace{b,...,b}_{k\ times}]\overline{L}_\Delta\left((f-a\mathbf{1})^m(f-b\mathbf{1})^{k-1},h\right).$$

Inequality (2.2) also holds when the function ϕ is n-concave and n and m are of equal parity. In case when the function ϕ is n-convex and n and m are of equal parity, or when the function ϕ is n-concave and n and m are of different parity, the inequality sign in (2.2) is reversed.

Proof. Since $f(\mathcal{E})=[a,b]$ by the assumptions, we have $a\leq f(t)\leq b$, so we can replace t with $f(t)$ in (1.6) and obtain:

$$\phi(f(t))-\alpha_\phi f(t)-\beta_\phi=(f(t)-a)\left(\phi'(a)-\phi[a,b]\right)+\sum_{k=2}^{m-1}\frac{\phi^{(k)}(a)}{k!}(f(t)-a)^k$$
$$+\sum_{k=1}^{n-m}\phi[\underbrace{a,...,a}_{m\ times};\underbrace{b,...,b}_{k\ times}](f(t)-a)^m(f(t)-b)^{k-1}+R_m(f(t)).$$

Since the multiple Lebesgue delta time scale integral is a positive linear functional, multiplying the previous inequality by $\frac{h(t)}{\int_{\mathcal{E}} h(t)\Delta t}$ and then integrating the resulting inequality yields

$$\overline{L}_\Delta(\phi(f),h)-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi \tag{2.3}$$
$$=(\overline{L}_\Delta(f,h)-a)\left(\phi'(a)-\phi[a,b]\right)+\sum_{k=2}^{m-1}\frac{\phi^{(k)}(a)}{k!}\cdot\frac{\int_{\mathcal{E}}(f(t)-a)^k h(t)\Delta t}{\int_{\mathcal{E}}h(t)\Delta t}$$
$$+\sum_{k=1}^{n-m}\phi[\underbrace{a,...,a}_{m\ times};\underbrace{b,...,b}_{k\ times}]\frac{\int_{\mathcal{E}}(f(t)-a)^m(f(t)-b)^{k-1}h(t)\Delta t}{\int_{\mathcal{E}}h(t)\Delta t}+\frac{\int_{\mathcal{E}}R_m(f(t))h(t)\Delta t}{\int_{\mathcal{E}}h(t)\Delta t}.$$

Now we set our focus on positivity and negativity of the term

$$\frac{\int_{\mathcal{E}}R_m(f(t))h(t)\Delta t}{\int_{\mathcal{E}}h(t)\Delta t}.$$

Because multiple Lebesgue delta time scale integral takes nonnegative values for positive functions, it is enough to study positivity and negativity of:

$$R_m(f(t))=(f(t)-a)^m\,(f(t)-b)^{n-m}\,\phi[f(t);\underbrace{a,...,a}_{m\ times};\underbrace{b,b,...,b}_{(n-m)\ times}].$$

Since by assumptions we have $a\leq f(t)\leq b$, we have $(f(t)-a)^m\geq 0$ for any choice of m. For the same reason we have $(f(t)-b)\leq 0$. Trivially it follows that $(f(t)-b)^{n-m}\leq 0$ when n and m are of different parity, and $(f(t)-b)^{n-m}\geq 0$ when n and m are of equal parity.

If the function ϕ is n-convex, then $\phi[f(t);\underbrace{a,...,a}_{m\ times};\underbrace{b,b,...,b}_{(n-m)\ times}]\geq 0$, and if the function ϕ is n-concave, then $\phi[f(t);\underbrace{a,...,a}_{m\ times};\underbrace{b,b,...,b}_{(n-m)\ times}]\leq 0$ for any $t\in[a,b]$. Inequality (2.2) now easily follows from (2.3). □

Next result is another generalization of the Edmundson-Lah-Ribarič inequality in terms of divided differences, obtained from Lemma 1.12 that also holds for the class of n-convex functions.

Theorem 2.2. *Let $\phi \in C^n([a,b])$ be an n-convex function. Assume $\mathcal{E}$ is as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E}) = [a,b]$. Moreover, let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t > 0$. For an odd number $m \geq 3$ such that $m < n$ we have*

$$\begin{aligned}
&\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (2.4)\\
\leq &(b - \overline{L}_\Delta(f,h))\,(\phi[a,b] - \phi'(b)) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!} \overline{L}_\Delta\left((f - b\mathbf{1})^k, h\right)\\
&+ \sum_{k=1}^{n-m} \phi[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}]\overline{L}_\Delta\left((f - b\mathbf{1})^m (f - a\mathbf{1})^{k-1}, h\right).
\end{aligned}$$

Inequality (2.4) also holds when the function ϕ is n-concave and m is even. In case when the function ϕ is n-convex and m is even, or when the function ϕ is n-concave and m is odd, the inequality sign in (2.4) is reversed.

Proof. In a similar manner as in the proof of the previous theorem, we can replace t with $f(t)$ in (1.10), multiply the obtained inequality by $\frac{h(t)}{\int_{\mathcal{E}} h(t)\Delta t}$ and then integrate the resulting inequality. In that way we get

$$\begin{aligned}
&\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (2.5)\\
= &(b - \overline{L}_\Delta(f,h))\,(\phi[a,b] - \phi'(b)) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!} \cdot \frac{\int_{\mathcal{E}} (f(t) - b)^k h(t)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}\\
&+ \sum_{k=1}^{n-m} \phi[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}] \frac{\int_{\mathcal{E}} (f(t) - b)^m (f(t) - a)^{k-1} h(t)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t} + \frac{\int_{\mathcal{E}} R_m^*(f(t)) h(t)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}.
\end{aligned}$$

Next, we study positivity and negativity of the term

$$\frac{\int_{\mathcal{E}} R_m^*(f(t)) h(t)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}.$$

Again, it is enough to study positivity and negativity of the function:

$$R_m^*(f(t)) = (f(t) - b)^m (f(t) - a)^{n-m} \phi[f(t); \underbrace{b,\dots,b}_{m\ times};\ \underbrace{a,a,\dots,a}_{(n-m)\ times}].$$

Since $f(t) \in [a,b]$, we have $(f(t) - a)^{n-m} \geq 0$ for every t and any choice of m. For the same reason we have $(f(t) - b) \leq 0$. Trivially it follows that $(f(t) - b)^m \leq 0$ when m is odd, and $(f(t) - b)^m \geq 0$ when m is even. If the function ϕ is n-convex, then its n-th order divided differences are greater of equal to zero, and if the function ϕ is n-concave, then its n-th order divided differences are less or equal to zero. Now (2.4) easily follows from (2.5). □

The following corollary is a direct consequence of the previous two theorems, and it provides us with a lower and an upper bound for the difference in the Edmundson-Lah-Ribarič inequality for time scales.

Corollary 2.3. *Let $\phi \in C^n([a,b])$ be an n-convex function. Assume $\mathcal{E}$ is as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E}) = [a,b]$. Moreover, let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t > 0$. If $m \geq 3$ is odd and $m < n$, then*

$$\begin{aligned}
&(\overline{L}_\Delta(f,h) - a)(\phi'(a) - \phi[a,b]) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(a)}{k!} \overline{L}_\Delta((f - a\mathbf{1})^k, h) \\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}] \overline{L}_\Delta((f-a\mathbf{1})^m (f-b\mathbf{1})^{k-1}, h) \\
&\leq \overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (2.6) \\
&\leq (b - \overline{L}_\Delta(f,h))(\phi[a,b] - \phi'(b)) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!} \overline{L}_\Delta((f - b\mathbf{1})^k, h) \\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}] \overline{L}_\Delta((f-b\mathbf{1})^m (f-a\mathbf{1})^{k-1}, h).
\end{aligned}$$

Inequality (2.6) also holds when the function ϕ is n-concave and m is even. In case when the function ϕ is n-convex and m is even, or when the function ϕ is n-concave and m is odd, the inequality signs in (2.6) are reversed.

In our next result we establish another set of bounds for the difference in the Edmundson-Lah-Ribarič inequality. It is obtained from Lemma 1.11.

Theorem 2.4. *Let $\phi \in C^n([a,b])$ be an n-convex function. Assume $\mathcal{E}$ is as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E}) = [a,b]$. Moreover, let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t > 0$. If $n \geq 3$ is odd, then*

$$\begin{aligned}
&\sum_{k=2}^{n-1} \phi[a;\underbrace{b,\dots,b}_{k\ times}] \overline{L}_\Delta((f-a\mathbf{1})(f-b\mathbf{1})^{k-1}, h) \\
&\leq \overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \leq \phi[a,a;b] \overline{L}_\Delta((f-a\mathbf{1})(f-b\mathbf{1}), h) \qquad (2.7) \\
&\quad + \sum_{k=2}^{n-2} \phi[a,a;\underbrace{b,\dots,b}_{k\ times}] \overline{L}_\Delta((f-a\mathbf{1})^2 (f-b\mathbf{1})^{k-1}, h).
\end{aligned}$$

Inequalities (2.7) also hold when the function ϕ is n-concave and n is even. In case when the function ϕ is n-convex and n is even, or when the function ϕ is n-concave and n is odd, the inequality signs in (2.7) are reversed.

Proof. Again, we can replace t with $f(t)$ in (1.3) and (1.4), multiply the obtained inequality by $\frac{h(t)}{\int_{\mathcal{E}} h(t)\Delta t}$ and then integrate the resulting inequality. In that way we get

$$\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (2.8)$$

$$= \sum_{k=2}^{n-1} \phi[a;\underbrace{b,...,b}_{k\ times}] \frac{\int_{\mathcal{E}} h(t)(f(t)-a)(f(t)-b)^{k-1}\Delta t}{\int_{\mathcal{E}} h(t)\Delta t} + \frac{\int_{\mathcal{E}} h(t)R_1(f(t))\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}$$

and

$$\overline{L}_{\Delta}(\phi(f),h) - \alpha_{\phi}\overline{L}_{\Delta}(f,h) - \beta_{\phi} = \phi[a,a;b] \frac{\int_{\mathcal{E}} h(t)(f(t)-a)(f(t)-b)\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}$$

$$+ \sum_{k=2}^{n-2} \phi[a,a;\underbrace{b,...,b}_{k\ times}] \frac{\int_{\mathcal{E}} h(t)(f(t)-a)^2(f(t)-b)^{k-1}\Delta t}{\int_{\mathcal{E}} h(t)\Delta t} + \frac{\int_{\mathcal{E}} h(t)R_2(f(t))\Delta t}{\int_{\mathcal{E}} h(t)\Delta t}. \tag{2.9}$$

From the discussion about positivity and negativity of the term

$$\overline{L}_{\Delta}(R_m(f),h) = \frac{\int_{\mathcal{E}} h(t)R_m(f(t))\Delta t}{\int_{\mathcal{E}} h(t)\Delta t},$$

that is, about positivity and negativity of the function $R_m(f(t))$ in the proof of Theorem 2.1, for $m = 1$ it follows that

* $\overline{L}_{\Delta}(R_1(f),h) \geq 0$ when the function ϕ is n-convex and n is odd, or when ϕ is n-concave and n even;
* $\overline{L}_{\Delta}(R_1(f),h) \leq 0$ when the function ϕ is n-concave and n is odd, or when ϕ is n-convex and n even.

Now the relation (2.8) becomes inequality

$$\overline{L}_{\Delta}(\phi(f),h) - \alpha_{\phi}\overline{L}_{\Delta}(f,h) - \beta_{\phi} \geq \sum_{k=2}^{n-1} \phi[a;\underbrace{b,...,b}_{k\ times}]\overline{L}_{\Delta}((f-a\mathbf{1})(f-b\mathbf{1})^{k-1},h)$$

that holds for $\overline{L}_{\Delta}(R_1(f),h) \geq 0$, and in case $\overline{L}_{\Delta}(R_1(f),h) \leq 0$ the inequality sign is reversed.

In the same manner, for $m = 2$ it follows that

* $\overline{L}_{\Delta}(R_2(f),h) \leq 0$ when the function ϕ is n-convex and n is odd, or when ϕ is n-concave and n even;
* $\overline{L}_{\Delta}(R_2(f),h) \geq 0$ when the function ϕ is n-concave and n is odd, or when ϕ is n-convex and n even.

In this case the relation (2.9) for $\overline{L}_{\Delta}(R_2(f),h) \leq 0$ gives us

$$\overline{L}_{\Delta}(\phi(f),h) - \alpha_{\phi}\overline{L}_{\Delta}(f,h) - \beta_{\phi} \leq \phi[a,a;b]\overline{L}_{\Delta}((f-a\mathbf{1})(f-b\mathbf{1}),h)$$

$$+\sum_{k=2}^{n-2}\phi[a,a;\underbrace{b,...,b}_{k\ times}]\overline{L}_\Delta((f-a\mathbf{1})^2(f-b\mathbf{1})^{k-1},h),$$

and when $\overline{L}_\Delta(R_2(f),h)\ge 0$ the inequality sign is reversed.

When we combine the two inequalities obtained above, we get exactly (2.7). □

By utilizing Lemma 1.12 we can get a similar lower and upper bound for the difference in the Edmundson-Lah-Ribarič inequality that holds for all $n\in\mathbb{N}$, not only the odd ones.

Theorem 2.5. *Let $\phi\in C^n([a,b])$ be an n-convex function, $n\ge 3$. Assume $\mathcal{E}$ is as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$. Moreover, let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. Then we have*

$$\phi[b,b;a]\overline{L}_\Delta((f-b\mathbf{1})(f-a\mathbf{1}),h)+\sum_{k=2}^{n-2}\phi[b,b;\underbrace{a,...,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})^2(f-a\mathbf{1})^{k-1},h)$$
$$\le\overline{L}_\Delta(\phi(f),h)-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi\le\sum_{k=2}^{n-1}\phi[b;\underbrace{a,...,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})(f-a\mathbf{1})^{k-1},h) \tag{2.10}$$

If the function ϕ is n-concave, the inequality signs in (2.10) are reversed.

Proof. We follow the lines from the proof of Theorem 2.4, with the difference that we start with equalities (1.7) and (1.8) from Lemma 1.12, and then we return to the discussion about positivity and negativity of the term $\overline{L}_\Delta(R_m^*(f),h)$ from the proof of Theorem 2.2 for $m=1$ and $m=2$. □

3 Inequalities of the Jensen type on time scales

In this section we will utilize the results from the previous section, as well Lemma 1.11 and Lemma 1.12, in order to obtain some Jensen-type inequalities that hold for n-convex functions.

Our first result is a consequence of Corollary 2.3, and it provides us with a lower and an upper bound for the difference in the Jensen inequality for time scales (1.1).

Theorem 3.1. *Let $\phi\in C^n([a,b])$ be an n-convex function. Assume $\mathcal{E}$ is as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$. Moreover, let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. If $m\ge 3$ is odd and $m<n$, then*

$$\phi(a)-\phi(b)+b\phi'(b)-a\phi'(a)+(\phi'(a)-\phi'(b))\overline{L}_\Delta(f,h)$$
$$+\sum_{k=2}^{m-1}\left(\frac{\phi^{(k)}(a)}{k!}\overline{L}_\Delta((f-a\mathbf{1})^k,h)-\frac{\phi^{(k)}(b)}{k!}(\overline{L}_\Delta(f,h)-b)^k\right)$$
$$+\sum_{k=1}^{n-m}\phi[\underbrace{a,...,a}_{m\ times};\underbrace{b,...,b}_{k\ times}]\overline{L}_\Delta((f-a\mathbf{1})^m(f-b\mathbf{1})^{k-1},h)$$

$$
\begin{aligned}
&-\sum_{k=1}^{n-m}\phi[\underbrace{b,...,b}_{m\ times};\underbrace{a,...,a}_{k\ times}](\overline{L}_\Delta(f,h)-b)^m(\overline{L}_\Delta(f,h)-a)^{k-1}\\
\leq&\overline{L}_\Delta(\phi(f),h)-\phi(\overline{L}_\Delta(f,h)) \qquad (3.1)\\
\leq&\phi(b)-\phi(a)+a\phi'(a)-b\phi'(b)+(\phi'(b)-\phi'(a))\overline{L}_\Delta(f,h)\\
&+\sum_{k=2}^{m-1}\left(\frac{\phi^{(k)}(b)}{k!}\overline{L}_\Delta((f-b\mathbf{1})^k,h)-\frac{\phi^{(k)}(a)}{k!}(\overline{L}_\Delta(f,h)-a)^k\right)\\
&+\sum_{k=1}^{n-m}\phi[\underbrace{b,...,b}_{m\ times};\underbrace{a,...,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})^m(f-a\mathbf{1})^{k-1},h)\\
&-\sum_{k=1}^{n-m}\phi[\underbrace{a,...,a}_{m\ times};\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-a)^m(\overline{L}_\Delta(f,h)-b)^{k-1}
\end{aligned}
$$

Inequalities (3.1) also hold when the function ϕ is n-concave and m is even. In case when the function ϕ is n-convex and m is even, or when the function ϕ is n-concave and m is odd, the inequality signs in (3.1) are reversed.

Proof. Because $f(\mathcal{E})=[a,b]$, we have $ah(t)\leq f(t)h(t)\leq bh(t)$, and consequently $\overline{L}_\Delta(f,h)\in[a,b]$, so we can substitute t with $\overline{L}_\Delta(f,h)$ in (1.6) and obtain

$$
\begin{aligned}
&\phi(\overline{L}_\Delta(f,h))-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi \qquad (3.2)\\
=&(\overline{L}_\Delta(f,h)-a)(\phi'(a)-\phi[a,b])+\sum_{k=2}^{m-1}\frac{\phi^{(k)}(a)}{k!}(\overline{L}_\Delta(f,h)-a)^k\\
&+\sum_{k=1}^{n-m}\phi[\underbrace{a,...,a}_{m\ times};\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-a)^m(\overline{L}_\Delta(f,h)-b)^{k-1}+R_m(\overline{L}_\Delta(f,h)).
\end{aligned}
$$

We need to study positivity and negativity of the term:

$$
R_m(\overline{L}_\Delta(f,h))=\left(\overline{L}_\Delta(f,h)-a\right)^m\left(\overline{L}_\Delta(f,h)-b\right)^{n-m}\phi[\overline{L}_\Delta(f,h);\underbrace{a,...,a}_{m\ times};\underbrace{b,b,...,b}_{(n-m)\ times}].
$$

Since $\overline{L}_\Delta(f,h)\in[a,b]$, we have $\left(\overline{L}_\Delta(f,h)-a\right)^m\geq 0$ for any choice of m, and $(\overline{L}_\Delta(f,h)-b)^{n-m}\leq 0$ when n and m are of different parity, and $(\overline{L}_\Delta(f,h)-b)^{n-m}\geq 0$ when n and m are of equal parity.

If the function ϕ is n-convex, then $\phi[\overline{L}_\Delta(f,h);\underbrace{a,...,a}_{m\ times};\underbrace{b,b,...,b}_{(n-m)\ times}]\geq 0$, and if the function ϕ is n-concave, then the inequality sign is reversed.

Now the relation (3.2) for n-convex function ϕ and n and $m\geq 3$ of different parity, or n-concave function ϕ and n and $m\geq 3$ of the same parity, becomes

$$
\begin{aligned}
&\phi(\overline{L}_\Delta(f,h))-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi \qquad (3.3)\\
\leq&(\overline{L}_\Delta(f,h)-a)(\phi'(a)-\phi[a,b])+\sum_{k=2}^{m-1}\frac{\phi^{(k)}(a)}{k!}(\overline{L}_\Delta(f,h)-a)^k
\end{aligned}
$$

$$+\sum_{k=1}^{n-m}\phi[\underbrace{a,...,a}_{m\ times};\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-a)^m(\overline{L}_\Delta(f,h)-b)^{k-1},$$

and for n-convex function ϕ and n and $m \geq 3$ of the same parity, or n-concave function ϕ and n and $m \geq 3$ of different parity, the inequality sign is reversed.

In the same way we can replace t with $\overline{L}_\Delta(f,h)$ in (1.10) and get

$$\begin{aligned}&\phi(\overline{L}_\Delta(f,h))-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi \\ =&(b-\overline{L}_\Delta(f,h))(\phi[a,b]-\phi'(b))+\sum_{k=2}^{m-1}\frac{\phi^{(k)}(b)}{k!}(\overline{L}_\Delta(f,h)-b)^k \\ &+\sum_{k=1}^{n-m}\phi[\underbrace{b,...,b}_{m\ times};\underbrace{a,...,a}_{k\ times}](\overline{L}_\Delta(f,h)-b)^m(\overline{L}_\Delta(f,h)-a)^{k-1}+R_m^*(\overline{L}_\Delta(f,h)).\end{aligned}\tag{3.4}$$

As before, we study positivity and negativity of the term $R_m^*(\overline{L}_\Delta(f,h))$:

$$R_m^*(\overline{L}_\Delta(f,h))=(\overline{L}_\Delta(f,h)-b)^m(\overline{L}_\Delta(f,h)-a)^{n-m}\phi[\overline{L}_\Delta(f,h);\underbrace{b,...,b}_{m\ times};\underbrace{a,a,...,a}_{(n-m)\ times}].$$

Again, since $\overline{L}_\Delta(f,h)\in[a,b]$, we have $\left(\overline{L}_\Delta(f,h)-a\right)^{n-m}\geq 0$ for any choice of m, and $(\overline{L}_\Delta(f,h)-b)^m\leq 0$ when m is odd, and $(\overline{L}_\Delta(f,h)-b)^m\geq 0$ when m is even. If the function ϕ is n-convex, then its n-th order divided differences are greater of equal to zero, and if the function ϕ is n-concave, then its n-th order divided differences are less or equal to zero.

Equality (3.4) now turns into

$$\begin{aligned}&\phi(\overline{L}_\Delta(f,h))-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi \\ \leq&(b-\overline{L}_\Delta(f,h))(\phi[a,b]-\phi'(b))+\sum_{k=2}^{m-1}\frac{\phi^{(k)}(b)}{k!}(\overline{L}_\Delta(f,h)-b)^k \\ &+\sum_{k=1}^{n-m}\phi[\underbrace{b,...,b}_{m\ times};\underbrace{a,...,a}_{k\ times}](\overline{L}_\Delta(f,h)-b)^m(\overline{L}_\Delta(f,h)-a)^{k-1}\end{aligned}\tag{3.5}$$

for n-convex function ϕ and an odd number $m \geq 3$ or n-concave function ϕ and an even number $m \geq 3$. If ϕ is n-convex and m is even, or if ϕ is n-concave and m is odd, the inequality is reversed.

By combining inequalities (3.3) and (3.5) we get that

$$\begin{aligned}&(\overline{L}_\Delta(f,h)-a)(\phi'(a)-\phi[a,b])+\sum_{k=2}^{m-1}\frac{\phi^{(k)}(a)}{k!}(\overline{L}_\Delta(f,h)-a)^k \\ &+\sum_{k=1}^{n-m}\phi[\underbrace{a,...,a}_{m\ times};\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-a)^m(\overline{L}_\Delta(f,h)-b)^{k-1} \\ \leq&\phi(\overline{L}_\Delta(f,h))-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi\end{aligned}\tag{3.6}$$

$$\leq (b-\overline{L}_\Delta(f,h))\,(\phi[a,b]-\phi'(b))+\sum_{k=2}^{m-1}\frac{\phi^{(k)}(b)}{k!}(\overline{L}_\Delta(f,h)-b)^k$$
$$+\sum_{k=1}^{n-m}\phi[\underbrace{b,...,b}_{m\ times};\underbrace{a,...,a}_{k\ times}](\overline{L}_\Delta(f,h)-b)^m(\overline{L}_\Delta(f,h)-a)^{k-1}$$

holds if n is odd and ϕ is n-convex and m is odd, or ϕ is n-concave and m is even. If ϕ is n-convex and m is even, or ϕ is n-concave and m is odd, then the inequality signs are reversed.

When we multiply series of inequalities (3.6) by -1 and add to (2.6), we get exactly (3.1), and the proof is complete. □

Next result also provides us with a lower and upper bound for the difference in the Jensen inequality for time scales, and it is obtained from Theorem 2.4 and Lemma 1.11.

Theorem 3.2. *Let $\phi \in C^n([a,b])$ be an n-convex function. Assume $\mathcal{E}$ is as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$. Moreover, let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. If $n\geq 3$ is odd, then*

$$\phi[a,a;b](b-\overline{L}_\Delta(f,h))(\overline{L}_\Delta(f,h)-a)+\sum_{k=2}^{n-1}\phi[a;\underbrace{b,...,b}_{k\ times}]\overline{L}_\Delta((f-a\mathbf{1})(f-b\mathbf{1})^{k-1},h)$$
$$-(\overline{L}_\Delta(f,h)-a)^2\sum_{k=2}^{n-2}\phi[a,a;\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-b)^{k-1}$$
$$\leq \overline{L}_\Delta(\phi(f),h)-\phi(\overline{L}_\Delta(f,h)) \tag{3.7}$$
$$\leq \phi[a,a;b]\overline{L}_\Delta((f-a\mathbf{1})(f-b\mathbf{1}),h)-(\overline{L}_\Delta(f,h)-a)\sum_{k=2}^{n-1}\phi[a;\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-b)^{k-1}$$
$$+\sum_{k=2}^{n-2}\phi[a,a;\underbrace{b,...,b}_{k\ times}]\overline{L}_\Delta((f-a\mathbf{1})^2(f-b\mathbf{1})^{k-1},h).$$

Inequalities (3.7) also hold when the function ϕ is n-concave and n is even. In case when the function ϕ is n-convex and n is even, or when the function ϕ is n-concave and n is odd, the inequality signs in (3.7) are reversed.

Proof. By following a similar procedure as in the proof of the previous theorem, we start by replacing t with $\overline{L}_\Delta(f,h)$ in with relations (1.3) and (1.4) from Lemma 1.11. We get

$$\phi(\overline{L}_\Delta(f,h))-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi=\sum_{k=2}^{n-1}\phi[a;\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-a)(\overline{L}_\Delta(f,h)-b)^{k-1}+R_1(\overline{L}_\Delta(f,h)) \tag{3.8}$$

and

$$\phi(\overline{L}_\Delta(f,h))-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi \tag{3.9}$$

$$=\phi[a,a;b](\overline{L}_\Delta(f,h)-a)(\overline{L}_\Delta(f,h)-b)$$
$$+\sum_{k=2}^{n-2}\phi[a,a;\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-a)^2(\overline{L}_\Delta(f,h)-b)^{k-1}+R_2(\overline{L}_\Delta(f,h))$$

respectively. After discussing the positivity an negativity of terms $R_1(\overline{L}_\Delta(f,h))$ and $R_2(\overline{L}_\Delta(f,h))$ in the same way as in the proof Theorem 3.1, from relations (3.8) and (3.9) we get a series of inequalities

$$(\overline{L}_\Delta(f,h)-a)\sum_{k=2}^{n-1}\phi[a;\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-b)^{k-1}\leq\phi(\overline{L}_\Delta(f,h))-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi \tag{3.10}$$
$$\leq\phi[a,a;b](\overline{L}_\Delta(f,h)-a)(\overline{L}_\Delta(f,h)-b)+(\overline{L}_\Delta(f,h)-a)^2\sum_{k=2}^{n-2}\phi[a,a;\underbrace{b,...,b}_{k\ times}](\overline{L}_\Delta(f,h)-b)^{k-1}$$

that holds when n is odd and ϕ is n-convex, or when n is even and ϕ is n-concave. If n is odd and ϕ is n-concave, or if n is even and ϕ is n-convex, then the inequality signs in (3.10) are reversed.

Inequalities (3.7) are obtained after multiplying (3.10) by -1 and adding it to (2.7). □

In the analogue way as described in the proof of the previous theorem, but with utilizing Lemma 1.12 and Theorem 2.5, we can get a similar lower and upper bound for the difference in the Jensen inequality (1.1) that holds for all $n\in\mathbb{N}$, not only the odd ones.

Theorem 3.3. *Let $\phi\in C^n([a,b])$ be an n-convex function, $n\geq 3$. Assume $\mathcal{E}$ is as in Theorem 1.7 and suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$. Moreover, let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}}h(t)\Delta t>0$. We have*

$$\phi[b,b;a]\overline{L}_\Delta((f-b\mathbf{1})(f-a\mathbf{1}),h)-(\overline{L}_\Delta(f,h)-b)\sum_{k=2}^{n-1}\phi[b;\underbrace{a,...,a}_{k\ times}](\overline{L}_\Delta(f,h)-a)^{k-1}$$
$$+\sum_{k=2}^{n-2}\phi[b,b;\underbrace{a,...,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})^2(f-a\mathbf{1})^{k-1},h)$$
$$\leq\overline{L}_\Delta(\phi(f),h)-\phi(\overline{L}_\Delta(f,h)) \tag{3.11}$$
$$\leq f[b,b;a](b-\overline{L}_\Delta(f,h))(\overline{L}_\Delta(f,h)-a)+\sum_{k=2}^{n-1}\phi[b;\underbrace{a,...,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})(f-a\mathbf{1})^{k-1},h)$$
$$-(\overline{L}_\Delta(f,h)-b)^2\sum_{k=2}^{n-2}\phi[b,b;\underbrace{a,...,a}_{k\ times}](\overline{L}_\Delta(f,h)-a)^{k-1}.$$

If the function ϕ is n-concave, the inequality signs in (3.11) are reversed.

4 Applications

In this section, we use the results from the previous sections to get new converse inequalities for generalized means and power means in the time scale setting.

4.1 Generalized means

Suppose $\Psi\colon I \to \mathbb{R}$ is continuous and strictly monotone and f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E}) = I$, where $\mathcal{E} \subset \mathbb{R}^n$ is as in Theorem 1.7. Let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t > 0$. The generalized mean with respect to the multiple Lebesgue delta time scale integral is defined by

$$M_\Psi\left(f, \overline{L}_\Delta(f,h)\right) = \Psi^{-1}\left(\overline{L}_\Delta(\Psi(f),h)\right). \tag{4.1}$$

Now, our intention is to derive mutual bounds for generalized means in the time scale setting. In such a way, we will obtain some new reverse relations for generalized means that correspond to n-convex functions.

Before we state such results, we have to introduce some notations arising from this particular setting. Throughout this section we denote

$$\Phi = \chi \circ \psi^{-1}, \quad \alpha_\Phi = \frac{\chi(b) - \chi(a)}{\psi(b) - \psi(a)}, \quad \beta_\Phi = \frac{\psi(b)\chi(a) - \psi(a)\chi(b)}{\psi(b) - \psi(a)}$$

and

$$\psi_a = \min\{\psi(a), \psi(b)\}, \quad \psi_b = \max\{\psi(a), \psi(b)\},$$

where χ and ψ are strictly monotone functions. It is obvious that if the function ψ is increasing, then $\psi_a = \psi(a)$, $\psi_b = \psi(b)$, and if ψ is decreasing, then $\psi_a = \psi(b)$, $\psi_b = \psi(a)$.

Since for a Δ-integrable function f on $\mathcal{E}$ such that $f(\mathcal{E}) = [a,b]$ we have $\psi(f(\mathcal{E})) = [\psi_a, \psi_b]$, all of the results from previous sections can be exploited in establishing some new reverses of Jensen's inequality and the Edmundson-Lah-Ribarič inequality for selfadjoint operators related to quasi-arithmetic means by substituting ϕ with $\Phi = \chi \circ \psi^{-1}$ and f with $\psi(f)$.

We start with some Edmundson-Lah-Ribarič type inequalities for qgeneralized means which arise from the results from Section 2. The first result in this section is carried out by virtue of our Theorem 2.1.

Corollary 4.1. *Suppose $\psi, \chi : [a,b] \to \mathbb{R}$ are continuous and strictly monotone and $\Phi = \chi \circ \psi^{-1} \in C^n([a,b])$ is n-convex. Assume f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E}) = [a,b]$, where $\mathcal{E} \subset \mathbb{R}^n$ is as in Theorem 1.7. Let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t > 0$. If $n > m \geq 3$ are of different parity, then*

$$\begin{aligned}
&\chi\left(M_\chi(f, \overline{L}_\Delta(f,h))\right) - \alpha_\Phi \psi\left(M_\psi(f, \overline{L}_\Delta(f,h))\right) - \beta_\Phi \\
\leq &(\overline{L}_\Delta(\psi(f),h) - \psi_a)\left(\Phi'(\psi_a) - \Phi[\psi_a,\psi_b]\right) + \sum_{k=2}^{m-1} \frac{\Phi^{(k)}(\psi_a)}{k!} \overline{L}_\Delta\left((\psi(f) - \psi_a \mathbf{1})^k, h\right)
\end{aligned} \tag{4.2}$$

$$+\sum_{k=1}^{n-m}\Phi[\underbrace{\psi_a,\dots,\psi_a}_{m\ times};\underbrace{\psi_b,\dots,\psi_b}_{k\ times}]\overline{L}_\Delta\left((\psi(f)-\psi_a\mathbf{1})^m(\psi(f)-\psi_b\mathbf{1})^{k-1},h\right).$$

Inequality (4.2) also holds when the function Φ is n-concave and n and m are of equal parity. In case when the function Φ is n-convex and n and m are of equal parity, or when the function Φ is n-concave and n and m are of different parity, the inequality sign in (4.2) is reversed.

The following result is a direct consequence of Theorem 2.2.

Corollary 4.2. *Suppose $\psi,\chi:[a,b]\to\mathbb{R}$ are continuous and strictly monotone and $\Phi=\chi\circ\psi^{-1}\in C^n([a,b])$ is n-convex. Assume f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$, where $\mathcal{E}\subset\mathbb{R}^n$ is as in Theorem 1.7. Let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. For an odd number $m\geq 3$ such that $m<n$, we have*

$$\begin{aligned}&\chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right)-\alpha_\Phi\psi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right)-\beta_\Phi \\ \leq&(\psi_b-\overline{L}_\Delta(\psi(f),h))\left(\Phi[\psi_a,\psi_b]-\Phi'(\psi_b)\right)+\sum_{k=2}^{m-1}\frac{\Phi^{(k)}(\psi_b)}{k!}\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^k,h) \\ &+\sum_{k=1}^{n-m}\Phi[\underbrace{\psi_b,\dots,\psi_b}_{m\ times};\underbrace{\psi_a,\dots,\psi_a}_{k\ times}]\overline{L}_\Delta(f,h)(\psi(f)-\psi_b\mathbf{1})^m(\psi(f)-\psi_a\mathbf{1})^{k-1}.\end{aligned}\tag{4.3}$$

Inequality (4.3) also holds when the function Φ is n-concave and m is even. In case when the function Φ is n-convex and m is even, or when the function Φ is n-concave and m is odd, the inequality sign in (4.3) is reversed.

Our next result arises from Theorem 2.4.

Corollary 4.3. *Suppose $\psi,\chi:[a,b]\to\mathbb{R}$ are continuous and strictly monotone and $\Phi=\chi\circ\psi^{-1}\in C^n([a,b])$ is n-convex. Assume f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$, where $\mathcal{E}\subset\mathbb{R}^n$ is as in Theorem 1.7. Let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. If $n\geq 3$ is odd, then*

$$\begin{aligned}&\sum_{k=2}^{n-1}\Phi[\psi_a;\underbrace{\psi_b,\dots,\psi_b}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})(\psi(f)-\psi_b\mathbf{1})^{k-1},h) \\ \leq&\chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right)-\alpha_\Phi\psi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right)-\beta_\Phi \\ \leq&\Phi[\psi_a,\psi_a;\psi_b]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})(\psi(f)-\psi_b\mathbf{1}),h) \\ &+\sum_{k=2}^{n-2}\Phi[\psi_a,\psi_a;\underbrace{\psi_b,\dots,\psi_b}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})^2(\psi(f)-\psi_b\mathbf{1})^{k-1},h).\end{aligned}\tag{4.4}$$

Inequalities (4.4) also hold when the function Φ is n-concave and n is even. In case when the function Φ is n-convex and n is even, or when the function Φ is n-concave and n is odd, the inequality signs in (4.4) are reversed.

As a consequence of Theorem 2.5, we have the following result.

Corollary 4.4. *Suppose $\psi,\chi:[a,b]\to\mathbb{R}$ are continuous and strictly monotone and $\Phi=\chi\circ\psi^{-1}\in C^n([a,b])$ is n-convex, $n\geq 3$. Assume f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$, where $\mathcal{E}\subset\mathbb{R}^n$ is as in Theorem 1.7. Let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. Then we have*

$$\begin{aligned}
&\Phi[\psi_b,\psi_b;\psi_a]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})(\psi(f)-\psi_a\mathbf{1}),h)\\
&\quad+\sum_{k=2}^{n-2}\Phi[\psi_b,\psi_b;\underbrace{\psi_a,\dots,\psi_a}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^2(\psi(f)-\psi_a\mathbf{1})^{k-1},h)\\
&\leq\chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right)-\alpha_\Phi\psi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right)-\beta_\Phi \qquad (4.5)\\
&\leq\sum_{k=2}^{n-1}\Phi[\psi_b;\underbrace{\psi_a,\dots,\psi_a}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})(\psi(f)-\psi_a\mathbf{1})^{k-1},h).
\end{aligned}$$

If the function Φ is n-concave, the inequality signs in (4.5) are reversed.

The corollaries below arise from the results from Section 3 and give us Jensen type inequalities for quasi-arithmetic means. They are obtained from Theorem 3.1, 3.2 and 3.3 respectively.

Corollary 4.5. *Suppose $\psi,\chi:[a,b]\to\mathbb{R}$ are continuous and strictly monotone and $\Phi=\chi\circ\psi^{-1}\in C^n([a,b])$ is n-convex. Assume f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$, where $\mathcal{E}\subset\mathbb{R}^n$ is as in Theorem 1.7. Let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. If $m\geq 3$ is odd and $m<n$, then*

$$\begin{aligned}
&\Phi(\psi_a)-\Phi(\psi_b)+\psi_b\Phi'(\psi_b)-\psi_a\Phi'(\psi_a)+(\Phi'(\psi_a)-\Phi'(\psi_b))\overline{L}_\Delta(\psi(f),h)\\
&\quad+\sum_{k=2}^{m-1}\left(\frac{\Phi^{(k)}(\psi_a)}{k!}\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})^k,h)-\frac{\Phi^{(k)}(g_b)}{k!}(\overline{L}_\Delta(\psi(f),h)-\psi_b)^k\right)\\
&\quad+\sum_{k=1}^{n-m}\Phi[\underbrace{\psi_a,\dots,\psi_a}_{m\ times};\underbrace{\psi_b,\dots,\psi_b}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})^m(\psi(f)-\psi_b\mathbf{1})^{k-1},h)\\
&\quad-\sum_{k=1}^{n-m}\Phi[\underbrace{\psi_b,\dots,\psi_b}_{m\ times};\underbrace{\psi_a,\dots,\psi_a}_{k\ times}](\overline{L}_\Delta(\psi(f),h)-\psi_b)^m(\overline{L}_\Delta(\psi(f),h)-\psi_a)^{k-1}\\
&\leq\chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right)-\chi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right) \qquad (4.6)\\
&\leq\Phi(\psi_b)-\Phi(\psi_a)+\psi_a\Phi'(\psi_a)-\psi_b\Phi'(\psi_b)+(\Phi'(\psi_b)-\Phi'(\psi_a))\overline{L}_\Delta(\psi(f),h)\\
&\quad+\sum_{k=2}^{m-1}\left(\frac{\Phi^{(k)}(\psi_b)}{k!}\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^k,h)-\frac{\Phi^{(k)}(\psi_a)}{k!}(\overline{L}_\Delta(\psi(f),h)-\psi_a)^k\right)\\
&\quad+\sum_{k=1}^{n-m}\Phi[\underbrace{\psi_b,\dots,\psi_b}_{m\ times};\underbrace{\psi_a,\dots,\psi_a}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^m(\psi(f)-\psi_a\mathbf{1})^{k-1},h)
\end{aligned}$$

$$-\sum_{k=1}^{n-m}\Phi[\underbrace{\psi_a,...,\psi_a}_{m\ times};\underbrace{\psi_b,...,\psi_b}_{k\ times}](\overline{L}_\Delta(\psi(f),h)-\psi_a)^m(\overline{L}_\Delta(\psi(f),h)-\psi_b)^{k-1}$$

Inequalities (4.6) also hold when the function Φ is n-concave and m is even. In case when the function Φ is n-convex and m is even, or when the function Φ is n-concave and m is odd, the inequality signs in (4.6) are reversed.

Corollary 4.6. *Suppose $\psi,\chi:[a,b]\to\mathbb{R}$ are continuous and strictly monotone and $\Phi=\chi\circ\psi^{-1}\in C^n([a,b])$ is n-convex. Assume f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$, where $\mathcal{E}\subset\mathbb{R}^n$ is as in Theorem 1.7. Let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. If $n\geq 3$ is odd, then*

$$\begin{aligned}
&\Phi[\psi_a,\psi_a;\psi_b](\psi_b-\overline{L}_\Delta(\psi(f),h))(\overline{L}_\Delta(\psi(f),h)-\psi_a)\\
&\quad+\sum_{k=2}^{n-1}\Phi[\psi_a;\underbrace{\psi_b,...,\psi_b}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})(\psi(f)-\psi_b\mathbf{1})^{k-1},h)\\
&\quad-(\overline{L}_\Delta(\psi(f),h)-\psi_a)^2\sum_{k=2}^{n-2}\Phi[\psi_a,\psi_a;\underbrace{\psi_b,...,\psi_b}_{k\ times}](\overline{L}_\Delta(\psi(f),h)-\psi_b)^{k-1}\\
&\leq\chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right)-\chi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right) \qquad (4.7)\\
&\leq\Phi[\psi_a,\psi_a;\psi_b]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})(\psi(f)-\psi_b\mathbf{1}),h)\\
&\quad-(\overline{L}_\Delta(\psi(f),h)-\psi_a)\sum_{k=2}^{n-1}\Phi[\psi_a;\underbrace{\psi_b,...,\psi_b}_{k\ times}](\overline{L}_\Delta(\psi(f),h)-\psi_b)^{k-1}\\
&\quad+\sum_{k=2}^{n-2}\Phi[\psi_a,\psi_a;\underbrace{\psi_b,...,\psi_b}_{k\ times}](\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})^2(\psi(f)-\psi_b\mathbf{1})^{k-1},h).
\end{aligned}$$

Inequalities (4.7) also hold when the function Φ is n-concave and n is even. In case when the function Φ is n-convex and n is even, or when the function Φ is n-concave and n is odd, the inequality signs in (4.7) are reversed.

Corollary 4.7. *Suppose $\psi,\chi:[a,b]\to\mathbb{R}$ are continuous and strictly monotone and $\Phi=\chi\circ\psi^{-1}\in C^n([a,b])$ is n-convex, $n\geq 3$. Assume f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=[a,b]$, where $\mathcal{E}\subset\mathbb{R}^n$ is as in Theorem 1.7. Let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. Then we have*

$$\begin{aligned}
&\Phi[\psi_b,\psi_b;\psi_a]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})(\psi(f)-\psi_a\mathbf{1}),h)\\
&\quad-(\overline{L}_\Delta(\psi(f),h)-\psi_b)\sum_{k=2}^{n-1}\Phi[\psi_b;\underbrace{\psi_a,...,\psi_a}_{k\ times}](\overline{L}_\Delta(\psi(f),h)-\psi_a)^{k-1}\\
&\quad+\sum_{k=2}^{n-2}\Phi[\psi_b,\psi_b;\underbrace{\psi_a,...,\psi_a}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^2(\psi(f)-\psi_a\mathbf{1})^{k-1},h)
\end{aligned}$$

$$
\begin{aligned}
&\leq \chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right)-\chi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right) \qquad (4.8)\\
&\leq \Phi[\psi_b,\psi_b;\psi_a](\psi_b-\overline{L}_\Delta(\psi(f),h))(\overline{L}_\Delta(\psi(f),h)-\psi_a)\\
&\quad+\sum_{k=2}^{n-1}\Phi[\psi_b;\underbrace{\psi_a,...,\psi_a}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})(\psi(f)-\psi_a\mathbf{1})^{k-1},h)\\
&\quad-(\overline{L}_\Delta(\psi(f),h)-\psi_b)^2\sum_{k=2}^{n-2}\Phi[\psi_b,\psi_b;\underbrace{\psi_a,...,\psi_a}_{k\ times}](\overline{L}_\Delta(\psi(f),h)-\psi_a)^{k-1}.
\end{aligned}
$$

If the function Φ is n-concave, the inequality signs in (4.8) are reversed.

4.2 Examples with power means

Assume $\mathcal{E}\subset\mathbb{R}^n$ is as in Theorem 1.7 and f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E})=I$ and $f(t)>0$, $t\in\mathcal{E}$. Let $h:\mathcal{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\Delta t>0$. For $r\in\mathbb{R}$, suppose f^r and $(\log f)$ are Δ-integrable on $\mathcal{E}$. The power mean with respect to the multiple Riemann delta time scale integral is defined by

$$
M^{[r]}\left(f,\overline{L}_\Delta(f,h)\right)=\begin{cases}\left(\overline{L}_\Delta(f^r,h)\right)^{\frac{1}{r}}, & \text{if } r\neq 0\\ \exp\left(\overline{L}_\Delta(\log f,h)\right), & \text{if } r=0.\end{cases} \qquad (4.9)
$$

Since power means are a special case of generalized means for particular choices of functions χ and ψ, first let us set $\chi(t)=t^s$ and $\psi(t)=t^r$, where s and r are real parameters such that $r\neq 0$ and $t>0$.

Now, the function $\Phi(t)=(\chi\circ\psi^{-1})(t)=t^{s/r}$ belongs to the class $C^n(\mathbb{R})$ for any $n\in\mathbb{N}$, and we have

$$
\Phi^{(n)}(t)=\frac{s}{r}\left(\frac{s}{r}-1\right)\left(\frac{s}{r}-2\right)\cdots\left(\frac{s}{r}-n+1\right)t^{\frac{s}{r}-n}.
$$

It is straightforward to check that:

- if $r<0<s$ or $s<0<r$, then the function Φ is n-convex for any even $n\in\mathbb{N}$, and n-concave for any odd number n;
- if $0<s<r$ or $r<s<0$, then the function Φ is n-convex for any odd $n\in\mathbb{N}$, and n-concave for any even number n;
- if $0<r<s$ or $s<r<0$, then the function Φ is n-convex when $\lfloor\frac{s}{r}\rfloor$ is even and n is odd, or when $\lfloor\frac{s}{r}\rfloor$ is odd and n is even, and Φ is n-concave when $\lfloor\frac{s}{r}\rfloor$ and n are both either even or odd.

It remains to consider the cases when one of the parameters r and s is equal to zero. If $s=0$, then setting $\chi(t)=\log t$ and $\psi(t)=t^r$, it follows that $\Phi(t)=(\chi\circ\psi^{-1})(t)=\frac{1}{r}\log t$ belongs to the class $C^n(\mathbb{R})$ for any $n\in\mathbb{N}$, and we have

$$
\Phi^{(n)}(t)=\frac{1}{r}(-1)^{n-1}(n-1)!\,t^{-n}.
$$

It is easy to see that:

- the function Φ is n-convex if $r > 0$ and $n \in \mathbb{N}$ is odd, or if $r < 0$ and $n \in \mathbb{N}$ is even;
- the function Φ is n-concave if $r > 0$ and $n \in \mathbb{N}$ is even, or if $r < 0$ and $n \in \mathbb{N}$ is odd.

In cases when $r < 0$ the function $\psi(t) = t^r$ is strictly decreasing, so we have $\psi_a = b^r$ and $\psi_b = a^r$, and in cases when $0 < r$ the function ψ is strictly increasing, so we have $\psi_a = a^r$ and $\psi_b = b^r$.

Finally, if $r = 0$, then setting $\chi(t) = t^s$ and $\psi(t) = \log t$, it follows that the function $\Phi(t) = (\phi \circ \psi^{-1})(t) = e^{st}$ belongs to the class $C^n(\mathbb{R})$ for any $n \in \mathbb{N}$, and we have

$$\Phi^{(n)}(t) = s^n \, e^{st}.$$

Trivially,

- if $s > 0$, then the function Φ is n-convex for any $n \in \mathbb{N}$;
- if $s < 0$, then Φ is n-convex for any even number n, and n-concave for any odd number n.

The function $\psi(t) = \log t$ is strictly increasing, so in this case we have $\psi_a = \log a$ and $\psi_b = \log b$.

We see that all of the results regarding quasi-arithmetic means from the previous subsection can be applied to power means, considering our discussion from above.

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