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(ω,c)-Periodic Solutions for Impulsive Differential Systems

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Abstract

In this paper, we prove the existence of (ω,c)-periodic solutions for a nonhomogeneous linear impulsive system by constructing Green functions and adjoint systems, respectively. In addition, we study the existence and uniqueness of (ω,c)-periodic solutions for a semilinear impulsive system via fixed point approach. Two examples are provided to illustrate our results.

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1 Introduction

Periodic motion is a very important and special phenomenon not only in natural science, but also in social science. Impulsive periodic systems serve as the models to study the dynamics of periodic evolution processes that are subject to sudden changes in their states. There are many interesting existence results for periodic solutions for impulsive periodic system on finite and infinite dimensional spaces in the past decades (see for example [3, 4, 5, 6, 7]).

Alvarez et al. [2] introduced the concept of (ω,c)-periodic functions by observing the property $x(\cdot+\omega) = cx(\cdot)$, $c \in \mathbb{C}$ of any solution $x(t)$ of the well-known Mathieu's equations $x'' + ax = 2q\cos(2t)x$, which is a Hill's equation with only one harmonic mode. Obviously, (ω,c)-periodic functions reduce to the standard ω-periodic and ω-antiperiodic functions when $c = 1$ and $c = -1$, respectively.

Very recently, Agaoglou et al. [1] adopted the concept of (ω,c)-periodic functions to study the existence and uniqueness of (ω,c)-periodic solutions of semilinear evolution equations of the type $x' = Ax + f(t,x)$ in complex Banach spaces, where A is not necessary be a bounded linear operator and f is a (ω,c)-periodic function. Motivated by [1, 2], we study (ω,c)-periodic solutions of the following impulsive differential systems

$$\begin{cases} \dot{x} = Ax + f(t,x), \ t \neq \tau_i, \ i \in \mathbb{N} = \{1,2,\cdots\}, \\ \triangle x \mid_{t=\tau_i} = x(\tau_i^+) - x(\tau_i^-) = Bx(\tau_i^-) + c_i, \end{cases} \tag{1.1}$$

where A, B are $n \times n$ matrices, $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ and $\tau_i < \tau_{i+1}$, $i \in \mathbb{N}$. The symbols $x(\tau_i^+)$ and $x(\tau_i^-)$ represent the right and left limits of $x(t)$ at $t = \tau_i$. In addition, we set $x(\tau_i^-) = x(\tau_i)$ and $i(t,s)$ denotes the number of impulsive points $\tau_i \in (s,t)$. Concerning on (1.1), we impose the following assumptions:

(H_1) A, B are $n \times n$ commutative matrices, i.e., $AB = BA$.

(H_2) Constant vectors c_i and the time sequence $\{\tau_i\}_{i\in\mathbb{N}}$ are such that $c_{i+m} = c_i$, $\tau_{i+m} = \tau_i + \omega$ for some fixed m, $i \in \mathbb{N}$.

(H_3) $c \notin \sigma(e^{A\omega}(I+B)^m)$, where $\sigma(D)$ denotes the spectrum of a matrix D and I is the unit matrix.

(H_4) $c \in \sigma(e^{A\omega}(I+B)^m)$.

(H_5) For all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, it holds $f(t+\omega, cx) = cf(t,x)$.

(H_6) There is a constant $L > 0$ such that $\|f(t,x_1) - f(t,x_2)\| \le L\|x_1 - x_2\|$ for all $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$.

(H_7) There are constants $\mu, \nu \ge 0$ such that $\|f(t,x)\| \le \mu + \nu\|x\|$ for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

(H_8) There are constants $K \ge 1$ and $\eta \in \mathbb{R}$ such that $\|e^{At}\| \le Ke^{\eta t}$ for any $t \ge 0$.

We first find a sufficient and necessity condition to guarantee the homogeneous linear problem of (1.1) may have a (ω,c)-periodic solution. Secondly, we study the existence of (ω,c)-periodic solutions of nonhomogeneous linear problems of (1.1) by assuming a non-resonance condition via constructing a Green function of the corresponding boundary value problem. We also study the existence of (ω, c)-periodic solutions by assuming a resonance condition via constructing adjoint impulsive systems. Further, we transfer the existence of (ω, c)-periodic solutions into seeking a fixed point of a Fredholm integral equation with impulsive sources via Banach and Schauder fixed point theorems.

The rest of this paper is organized as follows. In Section 2, we present some necessary notations, definition and lemmas. In Section 3, we give two conditions for the existence of (ω, c)-periodic solutions of nonhomogeneous linear impulsive problem (3.1) and give two necessary estimations on $G(\cdot,\cdot)$ defined in (3.3). In Section 4, we prove the existence and uniqueness of (ω, c)-periodic solutions for semilinear impulsive problem via fixed point theorems. Two examples are given in Section 5 to demonstrate the application of our main results.

2 Preliminaries

Denote by $\|x\|$ a norm on $\mathbb{R}^n$. We introduce a Banach space $PC(\mathbb{R}, \mathbb{R}^n) = \{x : \mathbb{R} \to \mathbb{R}^n : x \in C((t_i, t_{i+1}], \mathbb{R}^n)$, and $x(t_i^-) = x(t_i)$, $x(t_i^+)$ exist for any $i \in \mathbb{N}\}$ endowed with the norm $\|x\| = \sup_{t\in\mathbb{R}} \|x(t)\|$.

The following definitions and lemmas will be used in this paper.

Definition 2.1. ([2]) Let $c \in \mathbb{R}\backslash\{0\}$ and $\omega > 0$. A function $f : \mathbb{R} \to \mathbb{R}^n$ is called (ω, c)-periodic if $f(t + \omega) = cf(t)$ for all $t \in \mathbb{R}$.

Set $\Phi_{\omega,c} := \{x : x \in PC(\mathbb{R}, \mathbb{R}^n)$ and $cx(\cdot) = x(\cdot + \omega)\}$, i.e. $\Phi_{\omega,c}$ denotes the set of all piecewise continuous and (ω, c)-periodic functions.

Lemma 2.2. *(see [1, Lemma 2.2]) $x \in \Phi_{\omega,c}$ if and only if it holds*

$$x(\omega) = cx(0). \tag{2.1}$$

Lemma 2.3. *Assuming that (H_1) and (H_2) hold. The following homogeneous linear impulsive differential equation*

$$\begin{cases} \dot{x} = Ax, \ t \neq \tau_i, \ i \in \mathbb{N}, \\ \triangle x \,|_{t=\tau_i} = Bx, \end{cases} \tag{2.2}$$

has a solution $x \in \Phi_{\omega,c}$ if and only if

$$(cI - e^{A\omega}(I + B)^m)x_0 = 0,$$

where $x(0) = x_0$.

Proof. For any $t_0 \in [0,\infty) \setminus \xi$, $\xi = \{\tau_i\}_{i\in\mathbb{N}}$, the solution $x \in PC(\mathbb{R}, \mathbb{R}^n)$ of (2.2) with $x(t_0) = x_0$ can be formulated by [7, (2.27)]

$$x(t) = e^{A(t-t_0)}(I + B)^{i(t_0,t)}x_0, \ t \geq t_0.$$

Then for $t_0 = 0$ and $t \in [0,\infty) \setminus \xi$, we have

$$\begin{aligned} x(t + \omega) = cx(t) \quad &\Longleftrightarrow \quad e^{A(t+\omega)}(I + B)^{i(0,t+\omega)}x_0 = ce^{At}(I + B)^{i(0,t)}x_0 \\ &\Longleftrightarrow \quad e^{A\omega}(I + B)^{i(t,t+\omega)}x_0 = cx_0 \\ &\Longleftrightarrow \quad (cI - e^{A\omega}(I + B)^m)x_0 = 0. \end{aligned}$$

Since $x(\tau_i) = x(\tau_i^-)$, we get $x(\tau_i + \omega) = cx(\tau_i)$. The proof is finished. □

3 Nonhomogeneous linear impulsive problems

In the section, we consider (ω, c)-periodic solutions of nonhomogeneous linear problems

$$\begin{cases} \dot{x} = Ax + g(t), \ t \neq \tau_i, \ i \in \mathbb{N}, \\ \triangle x \mid_{t=\tau_i} = Bx + c_i, \end{cases} \tag{3.1}$$

where $g \in C(\mathbb{R}, \mathbb{R}^n)$ and g is (ω, c)-periodic. In order to study the existence of (ω, c)-periodic solutions of (3.1), we consider the following two cases:

3.1 Case 1: $c \notin \sigma(e^{A\omega}(I+B)^m)$.

Lemma 3.1. *Assuming that $(H_1)-(H_3)$ hold, the solution $x \in \Omega := PC([0,\omega], \mathbb{R}^n)$ of (3.1) satisfying (2.1) is given by*

$$x(t) = \int_0^\omega G(t,\tau)g(\tau)d\tau + \sum_{i=1}^m G(t,\tau_i)c_i, \tag{3.2}$$

where $G(\cdot,\cdot)$ is a Green function defined by

$$G(t,\tau) = \begin{cases} ce^{A(t-\tau)}(I+B)^{i(\tau,t)}(cI - e^{A\omega}(I+B)^m)^{-1}, & 0 < \tau < t, \\ e^{A(t+\omega-\tau)}(I+B)^{i(0,t)+i(\tau,\omega)}(cI - e^{A\omega}(I+B)^m)^{-1}, & t \leq \tau < \omega. \end{cases} \tag{3.3}$$

Proof. The solution $x \in \Omega$ of (3.1) with $x(0) = x_0$ on the interval $[0,\omega]$ is formulated by (see [7, (2.21)])

$$x(t) = e^{At}(I+B)^{i(0,t)}x_0 + \int_0^t e^{A(t-\tau)}(I+B)^{i(\tau,t)}g(\tau)d\tau + \sum_{0<\tau_i<t} e^{A(t-\tau_i)}(I+B)^{i(\tau_i,t)}c_i. \tag{3.4}$$

Then

$$x(\omega) = e^{A\omega}(I+B)^m x_0 + \int_0^\omega e^{A(\omega-\tau)}(I+B)^{i(\tau,\omega)}g(\tau)d\tau + \sum_{i=1}^m e^{A(\omega-\tau_i)}(I+B)^{i(\tau_i,\omega)}c_i = cx_0,$$

which is equivalent to

$$\begin{aligned} x_0 &= (cI - e^{A\omega}(I+B)^m)^{-1}\Big(\int_0^\omega e^{A(\omega-\tau)}(I+B)^{i(\tau,\omega)}g(\tau)d\tau \\ &\quad + \sum_{i=1}^m e^{A(\omega-\tau_i)}(I+B)^{i(\tau_i,\omega)}c_i\Big), \end{aligned}$$

where (H_3) is used here. Therefore we arrive at a formula

$$\begin{aligned}
x(t) &= e^{At}(I+B)^{i(0,t)}(cI-e^{A\omega}(I+B)^m)^{-1}\Big(\int_0^\omega e^{A(\omega-\tau)}(I+B)^{i(\tau,\omega)}g(\tau)d\tau \\
&\quad +\sum_{i=1}^m e^{A(\omega-\tau_i)}(I+B)^{i(\tau_i,\omega)}c_i\Big)+\int_0^t e^{A(t-\tau)}(I+B)^{i(\tau,t)}g(\tau)d\tau \\
&\quad +\sum_{0<\tau_i<t} e^{A(t-\tau_i)}(I+B)^{i(\tau_i,t)}c_i \\
&= \int_0^t e^{A(t-\tau)}(I+B)^{i(\tau,t)}\Big((I+B)^m(cI-e^{A\omega}(I+B)^m)^{-1}e^{A\omega}+I\Big)g(\tau)d\tau \\
&\quad +\int_t^\omega e^{At}(I+B)^{i(0,t)}(cI-e^{A\omega}(I+B)^m)^{-1}e^{A(\omega-\tau)}(I+B)^{i(\tau,\omega)}g(\tau)d\tau \\
&\quad +\sum_{0<\tau_i<t} e^{A(t-\tau_i)}(I+B)^{i(\tau_i,t)}\Big((I+B)^m(cI-e^{A\omega}(I+B)^m)^{-1}e^{A\omega}+I\Big)c_i \\
&\quad +\sum_{t\le\tau_i<\omega} e^{At}(I+B)^{i(0,t)}(cI-e^{A\omega}(I+B)^m)^{-1}e^{A(\omega-\tau_i)}(I+B)^{i(\tau_i,\omega)}c_i \\
&= \int_0^t ce^{A(t-\tau)}(I+B)^{i(\tau,t)}(cI-e^{A\omega}(I+B)^m)^{-1}g(\tau)d\tau \\
&\quad +\int_t^\omega e^{A(t+\omega-\tau)}(I+B)^{i(0,t)+i(\tau,\omega)}(cI-e^{A\omega}(I+B)^m)^{-1}g(\tau)d\tau \\
&\quad +\sum_{0<\tau_i<t} ce^{A(t-\tau_i)}(I+B)^{i(\tau_i,t)}(cI-e^{A\omega}(I+B)^m)^{-1}c_i \\
&\quad +\sum_{t\le\tau_i<\omega} e^{A(t+\omega-\tau_i)}(I+B)^{i(0,t)+i(\tau_i,\omega)}(cI-e^{A\omega}(I+B)^m)^{-1}c_i \\
&= \int_0^\omega G(t,\tau)g(\tau)d\tau+\sum_{i=1}^m G(t,\tau_i)c_i.
\end{aligned}$$

The proof is finished. □

Remark 3.2. By Lemma 2.2, (3.2) is the unique (ω, c)-periodic solution of (3.1).

3.2 Case 2: $c \in \sigma(e^{A\omega}(I+B)^m)$.

If $c \in \sigma(e^{A\omega}(I+B)^m)$ then the homogeneous system (2.2) has k linearly independent solutions for $1 \le k \le n$. This implies that rank $(cI - e^{A\omega}(I+B)^m) = n-k$. Assuming that $I+B$ is invertible, we consider the adjoint system of (2.2) given by (see [7, (2.79)])

$$\begin{cases} \dot{y} = -A^\top y,\ t \ne \tau_i,\ i \in \mathbb{N}, \\ \triangle y \mid_{t=\tau_i} = -(I+B^\top)^{-1}B^\top y. \end{cases} \tag{3.5}$$

By Lemma 8 of [7], we know that the solution of (3.5) with $y(0) = y_0$ is given by

$$y(t) = ((I+B^\top)^{i(0,t)}e^{A^\top t})^{-1}y_0 \tag{3.6}$$

for some $y_0 \in \mathbb{R}^n$. This solution is $(\omega, \frac{1}{c})$-periodic if and only if it holds

$$y_0 \in \mathcal{N}\left(cI - e^{A\omega}(I+B)^m\right)^\top. \tag{3.7}$$

Since

$$\dim \mathcal{N}\left(cI - e^{A\omega}(I+B)^m\right)^\top = n - \text{rank}\,(cI - e^{A\omega}(I+B)^m)^\top = n - \text{rank}\,(cI - e^{A\omega}(I+B)^m) = k,$$

the adjoint system (3.5) has k linearly independent $(\omega, \frac{1}{c})$-periodic solutions.

Lemma 3.3. *Let $\langle\cdot,\cdot\rangle$ be the standard scalar product on $\mathbb{R}^n$. Assuming (H_1), (H_2) and (H_4) hold, a solution $x(t)$ of the system (3.1) is (ω, c)-periodic if and only if*

$$\int_0^\omega \langle y(\tau), g(\tau)\rangle d\tau + \sum_{i=1}^m \langle y(\tau_i^+), c_i\rangle = 0,\ j = 1, 2, \cdots k \tag{3.8}$$

for any $(\omega, \frac{1}{c})$-periodic solution $y(t)$ of the adjoint system (3.5).

Proof. We take a solution $x(t)$ of (3.1) given by (3.4). Set $Z = \mathcal{N}\left(cI - e^{A\omega}(I+B)^m\right)^\top$. Then using

$$\begin{aligned} i(0,\tau) + i(\tau,\omega) = i(0,\omega) = m \qquad & \tau \in [0,\omega] \setminus \xi, \\ i(0,\tau) + i(\tau,\omega) = i(0,\omega) = m - 1 \qquad & \tau \in [0,\omega] \cap \xi, \end{aligned}$$

we obtain

$$\begin{aligned} & x(t) \text{ is } (\omega, c)\text{-periodic} \\ & \Leftrightarrow \left(cI - e^{A\omega}(I+B)^m\right)x_0 = \int_0^\omega e^{A(\omega-\tau)}(I+B)^{i(\tau,\omega)}g(\tau)d\tau + \sum_{i=1}^m e^{A(\omega-\tau_i)}(I+B)^{i(\tau_i,\omega)}c_i \\ & \Leftrightarrow \int_0^\omega e^{A(\omega-\tau)}(I+B)^{i(\tau,\omega)}g(\tau)d\tau + \sum_{i=1}^m e^{A(\omega-\tau_i)}(I+B)^{i(\tau_i,\omega)}c_i \\ & \in \mathcal{R}\left(cI - e^{A\omega}(I+B)^m\right) = \left[\mathcal{N}\left(cI - e^{A^\omega}(I+B)^m\right)^\top\right]^\perp \\ & \Leftrightarrow 0 = \left\langle y_0, \int_0^\omega e^{A(\omega-\tau)}(I+B)^{i(\tau,\omega)}g(\tau)d\tau + \sum_{i=1}^m e^{A(\omega-\tau_i)}(I+B)^{i(\tau_i,\omega)}c_i\right\rangle \quad \forall y_0 \in Z \\ & \Leftrightarrow 0 = \left\langle (e^{A\omega}(I+B)^m)^\top y_0, \int_0^\omega (e^{A\tau}(I+B)^{i(0,\tau)})^{-1}g(\tau)d\tau \right. \\ & \left. + \sum_{i=1}^m (e^{A\tau_i}(I+B)^{i(0,\tau_i)+1})^{-1}c_i\right\rangle \quad \forall y_0 \in Z \\ & \Leftrightarrow 0 = c\int_0^\omega \langle ((I+B^\top)^{i(0,\tau)}e^{A^\top\tau})^{-1}y_0, g(\tau)\rangle d\tau \\ & + c\sum_{i=1}^m \langle ((I+B^\top)^{-1}(I+B^\top)^{i(0,\tau_i)}e^{A^\top\tau_i})^{-1}y_0, c_i\rangle \quad \forall y_0 \in Z \\ = \quad & c\int_0^\omega \langle y(\tau), g(\tau)\rangle d\tau + c\sum_{i=1}^m \langle y(\tau_i^+), c_i\rangle, \end{aligned}$$

for any $y(t)$ given by (3.6) with y_0 satisfying (3.7), which are all $(\omega, \frac{1}{c})$-periodic solutions of (3.5). The proof is finished. □

Next, we give two useful lemmas.

Lemma 3.4. *It holds*

$$\sum_{i=1}^{m} \|G(t,\tau_i)c_i\| \le K_\eta := \begin{cases} K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \\ \times \| \max\{|c|, e^{\eta\omega}\} \sum\limits_{i=1}^{m} e^{\eta(\omega-\tau_i)}\|c_i\|, & \eta > 0, \\ K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \\ \times \max\{|c|, 1\} \sum\limits_{i=1}^{m} \|c_i\|, & \eta \le 0 \end{cases}$$

for any $t \in [0,\omega]$.

Proof. By (H_8) and according to (3.3), we obtain

$$\begin{aligned} & \sum_{i=1}^{m} \|G(t,\tau_i)\|\|c_i\| = \sum_{0<\tau_i<t} \|G(t,\tau_i)\|\|c_i\| + \sum_{t\le\tau_i<\omega} \|G(t,\tau_i)\|\|c_i\| \\ \le\ & \sum_{0<\tau_i<t} |c|\|e^{A(t-\tau_i)}\|\|(I+B)^{i(\tau_i,t)}\|\|(cI - e^{A\omega}(I+B)^m)^{-1}\|\|c_i\| \\ & + \sum_{t\le\tau_i<\omega} \|e^{A(t+\omega-\tau_i)}\|\|(I+B)^{i(0,t)+i(\tau_i,\omega)}\|\|(cI - e^{A\omega}(I+B)^m)^{-1}\|\|c_i\| \\ \le\ & K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \\ & \times\Big(\sum_{0<\tau_i<t} |c| e^{\eta(t-\tau_i)}\|c_i\| + \sum_{t\le\tau_i<\omega} e^{\eta(t+\omega-\tau_i)}\|c_i\|\Big) \end{aligned}$$

for any $t \in [0,\omega]$. Therefore, we consider the following two cases:

Case I: $\eta > 0$. Then we have

$$\begin{aligned} & \sum_{i=1}^{m} \|G(t,\tau_i)\|\|c_i\| \le K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \\ & \times\Big(\sum_{0<\tau_i<t} |c| e^{\eta(\omega-\tau_i)}\|c_i\| + \sum_{t\le\tau_i<\omega} e^{\eta(2\omega-\tau_i)}\|c_i\|\Big) \\ \le\ & K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \max\{|c|, e^{\eta\omega}\} \sum_{i=1}^{m} e^{\eta(\omega-\tau_i)}\|c_i\|. \end{aligned}$$

Case II: $\eta \le 0$. Then we have

$$\begin{aligned} & \sum_{i=1}^{m} \|G(t,\tau_i)\|\|c_i\| \le K\|(cI - e^{A\omega}(I+B)^m)^{-1}\|\|\max\{\|(I+B)\|^m, 1\} \\ & \times\Big(\sum_{0<\tau_i<t} |c|\|c_i\| + \sum_{t\le\tau_i<\omega} \|c_i\|\Big) \\ \le\ & K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \max\{|c|, 1\} \sum_{i=1}^{m} \|c_i\|. \end{aligned}$$

The proof is finished. □

Lemma 3.5. *It holds*

$$\int_0^{\omega} \|G(t,\tau)\| d\tau \le M_\eta := \begin{cases} K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \\ \times \max\{|c|, 1\} \frac{e^{\eta\omega}-1}{\eta}, & \eta \neq 0, \\ K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \\ \times \max\{|c|, 1\}\omega, & \eta = 0 \end{cases}$$

for any $t \in [0,\omega]$.

Proof. According to (H_8) and the formula of (3.3), we obtain

$$\begin{aligned} \int_0^{\omega} \|G(t,\tau)\| d\tau &\le K|c|\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \int_0^t e^{\eta(t-\tau)} d\tau \\ &+ K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \int_t^{\omega} e^{\eta(t+\omega-\tau)}\| d\tau \\ &= K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\}\Big(|c|\frac{e^{\eta t}-1}{\eta} + \frac{e^{\eta\omega}-e^{\eta t}}{\eta}\Big). \end{aligned}$$

Therefore, we consider the following two cases:

Case I: $\eta \neq 0$. Then we have

$$\begin{aligned} \int_0^{\omega} \|G(t,\tau)\| d\tau &\le K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\}\Big(|c|\frac{e^{\eta t}-1}{\eta} + \frac{e^{\eta\omega}-e^{\eta t}}{\eta}\Big) \\ &\le K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \max\{|c|, 1\}\frac{e^{\eta\omega}-1}{\eta}. \end{aligned}$$

Case II: $\eta = 0$. Then we have

$$\int_0^{\omega} \|G(t,\tau)\| d\tau \le K\|(cI - e^{A\omega}(I+B)^m)^{-1}\| \max\{\|(I+B)\|^m, 1\} \max\{|c|, 1\}\omega.$$

The proof is finished. □

4 Semilinear impulsive problems

In this section, we will apply the Banach and Schauder fixed point theorems to prove existence and uniqueness results for (1.1).

Theorem 4.1. *Assume that* (H_1), (H_2), (H_3), (H_5), (H_6) *and* (H_8) *hold. If* $0 < LM_\eta < 1$, *then (1.1) has a unique* (ω, c)*-periodic solution* $x \in \Phi_{\omega,c}$ *satisfying*

$$\|x\| \le \frac{\|\widetilde{f}\| M_\eta + K_\eta}{1 - LM_\eta},$$

where $\widetilde{f}(\cdot) = f(\cdot, 0)$.

Proof. Note that the condition (H_5) implies that if $x \in \Phi_{\omega,c}$, then $f(\cdot, x(\cdot)) \in \Phi_{\omega,c}$. By Lemmas 2.2 and 3.1, our task is equivalent with solving the fixed point problem

$$x(t) = \int_0^\omega G(t,\tau) f(\tau, x(\tau)) d\tau + \sum_{i=1}^m G(t,\tau_i) c_i, \quad t \in [0,\omega].$$

Hence we define an operator Λ on Ω as follows

$$(\Lambda x)(t) = \int_0^\omega G(t,\tau) f(\tau, x(\tau)) d\tau + \sum_{i=1}^m G(t,\tau_i) c_i. \tag{4.1}$$

It is easy to show that $\Lambda : \Omega \to \Omega$. Next, for any $x_1, x_2 \in \Omega$ we only check that Λ is a contraction mapping. According to Lemma 3.5, we have

$$\begin{aligned} ||(\Lambda x_1)(t) - (\Lambda x_2)(t)|| &\leq \int_0^\omega ||G(t,\tau)|| ||f(\tau, x_1(\tau) - f(\tau, x_2(\tau)|| d\tau \\ &\leq L \int_0^\omega ||G(t,\tau)|| ||x_1(\tau) - x_2(\tau)|| d\tau \leq L||x_1 - x_2|| \int_0^\omega ||G(t,\tau)|| d\tau \\ &\leq LM_\eta ||x_1 - x_2||, \end{aligned}$$

which implies that

$$||\Lambda x_1 - \Lambda x_2|| \leq LM_\eta ||x_1 - x_2||.$$

Then using $0 < LM_\eta < 1$, the uniqueness result follows by the contraction mapping principle.

Furthermore, we obtain

$$\begin{aligned} ||x|| &= ||\Lambda x|| \leq \int_0^\omega ||G(t,\tau) f(\tau, x(\tau))|| d\tau + \sum_{i=1}^m ||G(t,\tau_i) c_i|| \\ &\leq L \int_0^\omega ||G(t,\tau)|| ||x(\tau)|| d\tau + ||\widetilde{f}|| \int_0^\omega ||G(t,\tau)|| d\tau + \sum_{i=1}^m ||G(t,\tau_i)|| ||c_i|| \\ &\leq LM_\eta ||x|| + ||\widetilde{f}|| M_\eta + \sum_{i=1}^m ||G(t,\tau_i)|| ||c_i||. \end{aligned}$$

By using Lemma 3.4, we have

$$||x|| \leq \frac{||\widetilde{f}|| M_\eta + K_\eta}{1 - LM_\eta}.$$

The proof is finished. □

Theorem 4.2. *Assume that* (H_1), (H_2), (H_3), (H_5), (H_7) *and* (H_8) *hold. If* $0 < \nu M_\eta < 1$, *then (1.1) has a* (ω, c)*-periodic solution* $x \in \Phi_{\omega,c}$.

Proof. Consider operator Λ defined in (4.1) on B_r, where $B_r := \{x \in \Omega \mid \|x\| \le r\}$ and $r = \frac{\mu M_\eta + K_\eta}{1-\nu M_\eta}$. First, for any $x \in B_r$ and $t \in [0,\omega]$, by Lemmas 3.4 and 3.5 we have

$$\begin{aligned}
\|(\Lambda x)(t)\| &\le \int_0^\omega \|G(t,\tau)f(\tau,x(\tau))\|d\tau + \sum_{i=1}^m \|G(t,\tau_i)c_i\| \\
&\le \nu \int_0^\omega \|G(t,\tau)\|\|x(\tau)\|d\tau + \mu \int_0^\omega \|G(t,\tau)\|d\tau + \sum_{i=1}^m \|G(t,\tau_i)\|\|c_i\| \\
&\le \nu M_\eta \|x\| + \mu M_\eta + K_\eta = r,
\end{aligned}$$

which implies that $\|\Lambda x\| \le r$. So $\Lambda(B_r) \subset B_r$ for any $x \in B_r$. Next, it is standard to prove that Λ is continuous and that $\Lambda(B_r)$ is a relatively compact set. Thus Schauder fixed point theorem gives the result. The proof is finished. □

5 Examples

Example 5.1. Consider the following semilinear impulsive system

$$\begin{cases}
\dot{x}_1 = 2x_2 + \kappa \sin t \cos(x_1 + x_2), \ t \ne \tau_i, \ i = 1,2,\cdots, \\
\dot{x}_2 = -3x_1 - 5x_2 + \kappa \sin(x_1 - x_2), \ t \ne \tau_i, \ i = 1,2,\cdots, \\
\triangle x_1 \mid_{t=\tau_i} = 2x_1 + \frac{\pi}{6}, \\
\triangle x_2 \mid_{t=\tau_i} = 2x_2 + \frac{\pi}{12},
\end{cases} \tag{5.1}$$

where κ is a real parameter, $\tau_i = \frac{(2i-1)\pi}{4}$ and $\omega = \pi$, $c = -1$. Since $\tau_{i+2} = \tau_i + \pi$ and $c_{i+2} = c_i$ for $i \in \mathbb{N}$, we get $m = 2$ and (H_2) holds. System (5.1) has a form of the following impulsive system

$$\begin{cases}
\dot{x} = Ax + f(t,x), \ t \ne \tau_i, \ i = 1,2,\cdots, \\
\triangle x \mid_{t=\tau_i} = Bx + c_i,
\end{cases} \tag{5.2}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \ A = \begin{pmatrix} 0 & 2 \\ -3 & -5 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \ c_i = \begin{pmatrix} \frac{\pi}{6} \\ \frac{\pi}{12} \end{pmatrix},$$

$$f(t,x) = \kappa \begin{pmatrix} \sin t \cos(x_1 + x_2) \\ \sin(x_1 - x_2) \end{pmatrix}.$$

Since $c = -1 \notin \sigma(e^{\pi \cdot A}(I+B)^2) = \left\{9e^{-2\pi}, 9e^{-3\pi}\right\}$, (H_3) is satisfied. Then by elementary calculation, we have

$$\begin{aligned}
e^{At} &= \begin{pmatrix} -2e^{-3t} + 3e^{-2t} & -2e^{-3t} + 2e^{-2t} \\ 3e^{-3t} - 3e^{-2t} & 3e^{-3t} - 2e^{-2t} \end{pmatrix}, \\
(-I - e^{A \cdot \pi}(I+B)^2)^{-1} &= \begin{pmatrix} -\frac{18}{9+e^{3\pi}} - 1 + \frac{27}{9+e^{2\pi}} & 18\left(\frac{1}{9+e^{2\pi}} - \frac{1}{9+e^{3\pi}}\right) \\ 27\left(\frac{1}{9+e^{3\pi}} - \frac{1}{9+e^{2\pi}}\right) & \frac{27}{9+e^{3\pi}} - 1 - \frac{18}{9+e^{2\pi}} \end{pmatrix}.
\end{aligned}$$

Clearly (H_1) and (H_5) hold. Since $\sigma(A) = \{-3, -2\}$, (H_8) is verified for $\eta = -2$. Now we consider on $\mathbb{R}^2$ the norm $\|x\| = |x_1| + |x_2|$. Then

$$K = \sup_{t\geq 0} e^{2t}\|e^{At}\| = \sup_{t\geq 0}\max\{|-2e^{-t}+3|+|3e^{-t}-3|, |-2e^{-t}+2|+|3e^{-t}-2|\} = 6,$$

which specifies (H_8). Similarly, we obtain

$$\begin{aligned} K_\eta &= 27\left(-\frac{45}{9+e^{3\pi}} + 1 + \frac{36}{9+e^{2\pi}}\right)\pi \doteq 90.1234, \\ M_\eta &= 27\left(-\frac{5\,(9+e^{\pi})}{9+e^{3\pi}} + \frac{40}{9+e^{2\pi}} + 1\right) \doteq 28.6336. \end{aligned}$$

By calculation, we have $\|f(t,x)\| \leq 2|\kappa| = \mu$, $\nu = 0$ and $L = 2|\kappa|$.

If $|\kappa| < 0.034924$, then $LM_\eta < 1$ and by Theorem 4.1, (5.2) has a unique π-antiperiodic solution $x \in PC([0,\infty), \mathbb{R}^2)$.

Since $\nu M_\eta = 0 < 1$ and by Theorem 4.2, (5.2) has a π-periodic solution $x \in PC([0,\infty), \mathbb{R}^2)$ for any $\kappa \in \mathbb{R}$.

Example 5.2. Consider the following more complicated semilinear impulsive system

$$\begin{cases} \dot{x_1} = x_1 + 2x_2 + \kappa x_2 \sin(e^{-t}x_1),\ t \neq \tau_i,\ i = 1, 2, \cdots, \\ \dot{x_2} = -3x_1 - 5x_2 + \kappa x_1 \sin(e^{-t}x_2),\ t \neq \tau_i,\ i = 1, 2, \cdots, \\ \triangle x_1 \mid_{t=\tau_i} = 4x_1 + 2x_2 + \frac{\pi}{6}, \\ \triangle x_2 \mid_{t=\tau_i} = -3x_1 - 2x_2 + \frac{\pi}{12}, \end{cases} \tag{5.3}$$

where κ is a real parameter, $\tau_i = \frac{2i-1}{4}$ and $\omega = 1$, $c = e$. Since $\tau_{i+2} = \tau_i + 1$ and $c_{i+2} = c_i$ for $i \in \mathbb{N}$, we get $m = 2$ and (H_2) holds. Now we have

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -5 \end{pmatrix},\ B = \begin{pmatrix} 4 & 2 \\ -3 & -2 \end{pmatrix},\ c_i = \begin{pmatrix} \frac{\pi}{6} \\ \frac{\pi}{12} \end{pmatrix},\ f(t,x) = \kappa\begin{pmatrix} x_2\sin(e^{-t}x_1) \\ x_1\sin(e^{-t}x_2) \end{pmatrix}.$$

Since $AB = BA$ and $c = e \notin \sigma(e^A(I+B)^2) \doteq \{10.6543, 0.00171908\}$, (H_1) and (H_3) are satisfied. Clearly (H_5) holds. Next, we have

$$e^{At} = \begin{pmatrix} e^{-2t}\left(\cosh\left(\sqrt{3}t\right) + \sinh\left(\sqrt{3}t\right)\sqrt{3}\right) & \frac{2e^{-2t}\sinh\left(\sqrt{3}t\right)}{\sqrt{3}} \\ -\sqrt{3}e^{-2t}\sinh\left(\sqrt{3}t\right) & e^{-2t}\left(\cosh\left(\sqrt{3}t\right) - \sqrt{3}\sinh\left(\sqrt{3}t\right)\right) \end{pmatrix},$$

and $\sigma(A) = \{-\sqrt{3}-2, \sqrt{3}-2\}$, so (H_8) is verified for $\eta = \sqrt{3}-2$. Now we consider on $\mathbb{R}^2$ the norm $\|x\| = |x_1| + |x_2|$. Then

$$\begin{aligned} K &= \sup_{t\geq 0} e^{(2-\sqrt{3})t}\|e^{At}\| = \sup_{t\geq 0}\max\left\{\left|e^{-\sqrt{3}t}\left(\cosh\left(\sqrt{3}t\right) + \sinh\left(\sqrt{3}t\right)\sqrt{3}\right)\right|\right. \\ &\quad \left. + \left|\sqrt{3}e^{-\sqrt{3}t}\sinh\left(\sqrt{3}t\right)\right|, \left|\frac{2e^{-\sqrt{3}t}\sinh\left(\sqrt{3}t\right)}{\sqrt{3}}\right| + \left|e^{-\sqrt{3}t}\left(\cosh\left(\sqrt{3}t\right) - \sqrt{3}\sinh\left(\sqrt{3}t\right)\right)\right|\right\} \\ &= \sqrt{3} + \frac{1}{2}, \end{aligned}$$

which specifies (H_8). Similarly, we obtain

$$M_\eta = \frac{31\left(5\sqrt{3}+8\right)e\left(e^{\sqrt{3}}-e^2\right)\left(-3e^3+47\sqrt{3}\sinh\left(\sqrt{3}\right)+81\cosh\left(\sqrt{3}\right)\right)}{6\left(e^6+1-2e^3\left(4\sqrt{3}\sinh\left(\sqrt{3}\right)+7\cosh\left(\sqrt{3}\right)\right)\right)} \doteq 137.649.$$

By calculation, we have $\|f(t,x)\| \le |\kappa| = \nu$ and $\mu = 0$. Now the function $f(\cdot,x)$ is not globally Lipschitz continuous. So we can just verify (H_7). If $\kappa < 0.00726488$ then by Theorem 4.2, (5.3) has a $(1,e)$-periodic solution $x \in PC([0,\infty),\mathbb{R}^2)$.

Remark 5.3. Motivated by the above arguments, we can consider in (H_5) with $f(t,x) = E(e^{at}x)x$ for $E \in C(\mathbb{R}^n, L(\mathbb{R}^n))$ with $\sup_{x\in\mathbb{R}^n} \|E(x)\| < \infty$ and $e^{a\omega}c = 1$, when $c > 0$ and $f(t,x) = E(\sin(\frac{\pi}{\omega}t)e^{at}x)x$ with $e^{a\omega}c = -1$, when $c < 0$.

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