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A Uniform Ergodic Theorem for Some Nörlund Means

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Abstract

We obtain a uniform ergodic theorem for the sequence $\frac{1}{s(n)}\sum_{k=0}^{n}(\varDelta s)(n-k)T^k$, where $\varDelta$ is the inverse of the endomorphism on the vector space of scalar sequences which maps each sequence into the sequence of its partial sums, T is a bounded linear operator on a Banach space and s is a divergent nondecreasing sequence of strictly positive real numbers, such that $\lim_{n\to+\infty} s(n+1)/s(n) = 1$ and $\varDelta^q s \in \ell_1$ for some positive integer q. Indeed, we prove that if $T^n/s(n)$ converges to zero in the uniform operator topology, then the sequence of averages above converges in the same topology if and only if 1 is either in the resolvent set of T, or a simple pole of the resolvent function of T.

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1 Introduction

Throughout this paper, we will write $\mathbb{N}$ and $\mathbb{Z}_+$ for the sets of nonnegative integers and of strictly positive integers, respectively. Also, for each $\nu \in \mathbb{N}$, we will write $\mathbb{N}_\nu$ for the set of all nonnegative integers n satisfying $n \geq \nu$.

$\mathbb{K}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$, and we will denote by $\mathbb{K}^{\mathbb{N}}$ the vector space (over $\mathbb{K}$) of all sequences in $\mathbb{K}$. For each vector space V over $\mathbb{K}$, let 0_V and I_V denote respectively the zero element of V and the identity operator on V. If V and W are vector spaces over $\mathbb{K}$ and $\varLambda : V \longrightarrow W$ is a linear map, let $\mathcal{N}(\varLambda)$ and $\mathcal{R}(\varLambda)$ stand respectively for the kernel and the range of $\varLambda$.

For each normed space X, we will write $\| \ \|_X$ for the norm of X, and $L(X)$ for the normed algebra of all bounded linear operators on X. Henceforth, by *convergence in* $L(X)$ of a sequence of bounded linear operators on X, we will mean convergence with respect to the topology induced by $\| \ \|_{L(X)}$, that is, the uniform operator topology.

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If X is a complex nonzero Banach space, then $L(X)$ is a complex Banach algebra—with identity I_X.
For each $T \in L(X)$, let $r(T)$ and $\sigma(T)$ stand respectively for the spectral radius and for the spectrum of T. Also, let $\rho(T)$ and $\Re_T$ stand respectively for the resolvent set and for the resolvent function of T. Namely, $\rho(T) = \mathbb{C} \setminus \sigma(T)$ and $\Re_T : \rho(T) \ni \lambda \longmapsto (\lambda I_X - T)^{-1} \in L(X)$. It is well known that $\Re_T$ is analytic on the open set $\rho(T)$.

In [3], N. Dunford obtained several results about convergence of the sequence $f_n(T)$ in different topologies (where $T \in L(X)$ for a complex Banach space X, and, for each $n \in \mathbb{N}$, f_n is a complex-valued function, holomorphic in some open neighborhood of $\sigma(T)$). The uniform ergodic theorem, establishing equivalence between convergence of the sequence $\frac{1}{n}\sum_{k=0}^{n-1} T^k$ in $L(X)$ and 1 being either in $\rho(T)$ or a simple pole of $\Re_T$, under the hypothesis $\lim_{n\to+\infty} \frac{1}{n}\|T^n\|_{L(X)} = 0$, is a special case of one of these results (see [3], 3.16; see also [4], comments following Theorem 8). Notice that if the sequence $\frac{1}{n}\sum_{k=0}^{n-1} T^k$ converges in $L(X)$, then $\frac{1}{n}\|T^n\|_{L(X)}$ necessarily converges to zero, as $\frac{1}{n}T^n = \frac{n+1}{n}\Big(\frac{1}{n+1}\sum_{k=0}^{n} T^k\Big) - \frac{1}{n}\sum_{k=0}^{n-1} T^k$ for each $n \in \mathbb{Z}_+$.

More general means of the sequence of the iterates of the bounded linear operator T than the arithmetical ones involved in the uniform ergodic theorem, that is, the (C,α) *means* $\frac{1}{A_\alpha(n)}\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k$, $n \in \mathbb{N}$ (where $\alpha \in (0,+\infty)$, and A_α and $A_{\alpha-1}$ denote respectively the sequences of Cesàro numbers—whose definition is recalled here in Section 2—of order α and $\alpha-1$; notice that for $\alpha = 1$ we have $\frac{1}{A_\alpha(n)}\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k = \frac{1}{n+1}\sum_{k=0}^{n} T^k$ for each $n \in \mathbb{N}$), were considered by E. Hille in [8]. Indeed, in [8], Theorem 6 he proved that if the sequence $\frac{1}{A_\alpha(n)}\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k$ converges to some $E \in L(X)$ in $L(X)$, then $\frac{\|T^n\|_{L(X)}}{n^\alpha} \longrightarrow 0$ as $n \to +\infty$ and $\lim_{\lambda\to1^+} \|(\lambda-1)\Re_T(\lambda) - E\|_{L(X)} = 0$. Notice that the former of these two conditions yields $r(T) \le 1$, and then the latter is equivalent to 1 being either in $\rho(T)$, or a simple pole of $\Re_T$, and moreover E being the residue of $\Re_T$ at 1 (see the result recorded here as Theorem 2.4). Theorem 6 of [8] also provides a partial converse of this, establishing that if T is power-bounded and $\lim_{\lambda\to1^+} \|(\lambda-1)\Re_T(\lambda) - E\|_{L(X)} = 0$, then $\lim_{n\to+\infty} \left\| \frac{1}{A_\alpha(n)}\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k - E \right\|_{L(X)} = 0$ for each $\alpha \in (0,+\infty)$.

More recently, an improvement of [8], Theorem 6 was obtained by T. Yoshimoto, who in [12], Theorem 1 replaced power-boundedness of T by $\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{n^\omega} = 0$ (where $\omega = \min\{1,\alpha\}$). Finally, in [5], E. Ed-dari was able to complete the (C,α) uniform ergodic theorem, by proving that the sequence $\frac{1}{A_\alpha(n)}\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k$ converges to E in $L(X)$ if and only if $\frac{\|T^n\|_{L(X)}}{n^\alpha} \longrightarrow 0$ as $n \to +\infty$ and $\lim_{\lambda\to1^+} \|(\lambda-1)\Re_T(\lambda) - E\|_{L(X)} = 0$. E. Ed-dari's result is recorded here as Theorem 2.6.

We are interested here in obtaining a uniform ergodic theorem for the *Nörlund means* of the sequence T^n, that is, for the means $\frac{1}{s(n)}\sum_{k=0}^{n} (\varDelta s)(n-k)T^k$, $n \in \mathbb{N}$, where s is a divergent nondecreasing sequence of strictly positive real numbers (and $\Delta : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is as in the abstract; see Definition 4.1 here). Notice that for $s = A_\alpha$, $\alpha \in (0,+\infty)$, one obtains the (C,α) means.

In Section 2 we collect some preliminaries, in order to make this paper as self-contained as possible.

In Sections 3, 4 and 5 we derive some properties of real sequences, that we use in the final section dealing with bounded linear operators.

In Section 3 we are concerned with the least concave majorant of a real sequence.

In particular, in Theorem 3.9 we prove that if b is a real sequence such that the sequence $(\frac{b(n)}{n})_{n\in\mathbb{Z}_+}$ is bounded from above and b is not, then the least concave majorant of b, besides being strictly increasing and divergent, has a subsequence that is asymptotic to the corresponding subsequence of b.

In Section 4 we mainly deal with the real sequences s for which $\Delta^p s$ is concave for some $p \in \mathbb{N}$.

The main result of this section is Theorem 4.7, in which we derive several properties of a sequence s of nonnegative real numbers such that $\Delta^p s$ is concave and unbounded from above for some $p \in \mathbb{N}$. In particular, we prove that s is strictly increasing and divergent, $\lim\limits_{n\to+\infty} \frac{s(n+1)}{s(n)} = 1$, and $\Delta^{p+2} s \in \ell_1$. Also, in Example 4.9 we show that if $\alpha \in (0,+\infty)$ the sequence A_α satisfies the hypotheses of Theorem 4.7 (for $p = [\alpha]$ if $\alpha \notin \mathbb{Z}_+$; for $p = \alpha - 1$ if $\alpha \in \mathbb{Z}_+$).

In Section 5 we introduce an index $\mathcal{H}(b)$ $(\in \mathbb{N} \cup \{+\infty\})$ for a real sequence b, such that $\mathcal{H}(b) < +\infty$ if and only if the sequence $(\frac{b(n)}{n^m})_{n\in\mathbb{Z}_+}$ is bounded from above for some $m \in \mathbb{N}$, in which case $\mathcal{H}(b)$ is the minimum of such nonnegative integers m.

In Theorem 5.3 we use Theorem 3.9 to prove that if b is unbounded from above and such that $\mathcal{H}(b) < +\infty$, then b has a majorant s which satisfies the hypotheses of Theorem 4.7 for $p = \mathcal{H}(b) - 1$, and moreover is such that $\limsup\limits_{n\to+\infty} \frac{b(n)}{s(n)} \in [\frac{1}{\mathcal{H}(b)}, 1]$. We also prove (in Proposition 5.4) that if a is a real sequence such that $\Delta^q a \in \ell_1$ for some $q \in \mathbb{Z}_+$, then $\mathcal{H}(a) \le q - 1$.

Section 6 contains our main result, that is Theorem 6.7: we prove that if T is a bounded linear operator on a complex Banach space, and b is a divergent sequence of strictly positive real numbers, such that $\mathcal{H}(b) < +\infty$ and $\lim\limits_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{b(n)} = 0$ (which gives $r(T) \le 1$), then, for each divergent nondecreasing sequence s of strictly positive real numbers, such that $\lim\limits_{n\to+\infty} \frac{s(n+1)}{s(n)} = 1$, $\Delta^q s \in \ell_1$ for some $q \in \mathbb{N}_2$, and the sequence $(\frac{b(n)}{s(n)})_{n\in\mathbb{N}}$ is bounded (which gives $\lim\limits_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{s(n)} = 0$), the sequence $\frac{1}{s(n)} \sum\limits_{k=0}^{n} (\Delta s)(n-k) T^k$ converges in $L(X)$ if and only if 1 is either in $\rho(T)$, or a simple pole of $\Re_T$. The sequence s can be chosen so that it is not infinite of higher order than b, and $\Delta^p s$ is concave and unbounded from above for some $p \in \mathbb{N}$.

We conclude this section—and the paper—with an example (Example 6.10), showing that, contrary to the case of the sequence A_α considered in Theorem 6 of [8], convergence in $L(X)$ of the sequence $\frac{1}{s(n)} \sum\limits_{k=0}^{n} (\Delta s)(n-k) T^k$ does not imply $\lim\limits_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{s(n)} = 0$, even if s satisfies the hypotheses of Theorem 4.7.

2 Preliminaries

If X is a Banach space, and Y, Z are closed subspaces of X, satisfying $X = Y \oplus Z$, by the *projection of X onto Y along Z* we mean the bounded linear map $P : X \longrightarrow X$ such that $Px \in Y$ and $x - Px \in Z$ for every $x \in X$. Notice that $I_X - P$ is the projection of X onto Z along Y, and that $P^2 = P$. On the other hand, if $E \in L(X)$ satisfies $E^2 = E$, it is easily seen that $\mathcal{R}(E)$ is closed in X, $X = \mathcal{R}(E) \oplus \mathcal{N}(E)$, and E is the projection of X onto $\mathcal{R}(E)$ along $\mathcal{N}(E)$.

We begin by recalling a classical characterization of simple poles of $\mathfrak{R}_T$, that will be useful to us in this paper.

Theorem 2.1 (see [11], V, 10.1, 10.2, 6.2, 6.3 and 6.4, and IV, 5.10)**.** *Let X be a complex nonzero Banach space, $T \in L(X)$ and $\lambda_0 \in \mathbb{C}$. If λ_0 is a simple pole of $\mathfrak{R}_T$, then λ_0 is an eigenvalue of T, $\mathcal{N}((\lambda_0 I_X - T)^n) = \mathcal{N}(\lambda_0 I_X - T)$ and $\mathcal{R}((\lambda_0 I_X - T)^n) = \mathcal{R}(\lambda_0 I_X - T)$ for every $n \in \mathbb{Z}_+$, $\mathcal{R}(\lambda_0 I_X - T)$ is closed in X, $X = \mathcal{N}(\lambda_0 I_X - T) \oplus \mathcal{R}(\lambda_0 I_X - T)$, and the projection of X onto $\mathcal{N}(\lambda_0 I_X - T)$ along $\mathcal{R}(\lambda_0 I_X - T)$ coincides with the residue of $\mathfrak{R}_T$ at λ_0. Conversely, if $X = \mathcal{N}(\lambda_0 I_X - T) \oplus \mathcal{R}(\lambda_0 I_X - T)$, then λ_0 is either in $\rho(T)$, or else a simple pole of $\mathfrak{R}_T$.*

If X is a complex nonzero Banach space and $T \in L(X)$, following [11], Definition on page 310, we denote by $\mathfrak{A}(T)$ the set of all complex-valued holomorphic functions f whose domain $\mathrm{Dom}(f)$ is an open neighbourhood of $\sigma(T)$. For each $f \in \mathfrak{A}(T)$, the operator $f(T) \in L(X)$ is defined as follows:

$$f(T) = \frac{1}{2\pi i} \int_{+\partial D} f(\lambda)\, \mathfrak{R}_T(\lambda)\, d\lambda,$$

where $+\partial D$ denotes the positively oriented boundary of D, and D is any open bounded subset of $\mathbb{C}$, such that $D \supseteq \sigma(T)$, $\overline{D} \subseteq \mathrm{Dom}(f)$, D has a finite number of components, with pairwise disjoint closures, and ∂D consists of a finite number of simple closed rectifiable curves, no two of which intersect; the integral above does not depend on the particular choice of D (see [11], comment 2 on pages 310–311; see also [3], 2.2, 2.3 and 2.6). We recall that for each polynomial $\mathfrak{p} : \mathbb{C} \ni \lambda \longmapsto \sum_{k=0}^{n} a_k \lambda^k \in \mathbb{C}$ (where $n \in \mathbb{N}$, and $a_0, \ldots, a_n \in \mathbb{C}$), we have $\mathfrak{p}(T) = \sum_{k=0}^{n} a_k T^k$ (see [11], V, 8.1).

We will use the following convergence result for the elements of $\mathfrak{A}(T)$, due to N. Dunford, a special case of which is the classical uniform ergodic theorem.

Theorem 2.2 (see [3], 3.16)**.** *Let X be a complex nonzero Banach space, $T \in L(X)$, and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}(T)$, satisfying $1 \in \mathrm{Dom}(f_n)$ for each $n \in \mathbb{N}$, such that $\lim_{n\to+\infty} f_n(1) = 1$ and $(I_X - T) f_n(T) \longrightarrow 0_{L(X)}$ in $L(X)$ as $n \to +\infty$. Then the following three conditions are equivalent:*

(2.2.1) *there exists $E \in L(X)$ such that $E^2 = E$, $\mathcal{R}(E) = \mathcal{N}(I_X - T)$, and $f_n(T) \longrightarrow E$ in $L(X)$ as $n \to +\infty$;*

(2.2.2) *1 is either in $\rho(T)$, or a simple pole of $\mathfrak{R}_T$;*

(2.2.3) *$\mathcal{R}(I_X - T)$ is closed and $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$.*

Remark 2.3. We remark that, under the hypotheses of Theorem 2.2, each of conditions (2.2.1)–(2.2.3) is actually equivalent to each of the following two conditions (which at first glance might respectively appear to be weaker and stronger than them):

(2.3.1) *the sequence* $(f_n(T))_{n\in\mathbb{N}}$ *converges in* $L(X)$*;*

(2.3.2) $\mathcal{R}(I_X - T)$ *is closed,* $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$*, and the sequence* $(f_n(T))_{n\in\mathbb{N}}$ *converges in* $L(X)$ *to the projection of* X *onto* $\mathcal{N}(I_X - T)$ *along* $\mathcal{R}(I_X - T)$*.*

Equivalence between (2.2.1) and (2.3.1) is observed in [4], comments following Theorem 8. For the convenience of the reader, we give here a proof of equivalence of these five conditions. Indeed, it suffices to prove that (2.3.1) implies (2.3.2). Suppose that (2.3.1) is satisfied, and let $E \in L(X)$ be such that $f_n(T) \longrightarrow E$ in $L(X)$ as $n \to +\infty$. We prove that then $E^2 = E$ and $\mathcal{R}(E) = \mathcal{N}(I_X - T)$.
We begin by proving that for each $x \in \mathcal{N}(I_X - T)$ we have $Ex = x$. This is clear if $\mathcal{N}(I_X - T) = \{0_X\}$. If instead $\mathcal{N}(I_X - T) \neq \{0_X\}$, then $1 \in \sigma(T)$, and $\mathfrak{R}_T(\lambda)x = \frac{1}{\lambda-1}x$ for every $\lambda \in \rho(T)$. Hence (see [11], V, 1.3) $f_n(T)x = f_n(1)x$ for every $n \in \mathbb{N}$. Since $\lim_{n\to+\infty} f_n(1) = 1$, we conclude that $Ex = x$. This gives the desired result, which in turn yields $\mathcal{N}(I_X - T) \subseteq \mathcal{R}(E)$. On the other hand, since $(I_X - T)E = \lim_{n\to+\infty}(I_X - T)f_n(T) = 0_{L(X)}$, we have $\mathcal{R}(E) \subseteq \mathcal{N}(I_X - T)$. Hence $\mathcal{R}(E) = \mathcal{N}(I_X - T)$, and $E^2 = E$.
We have thus proved that the equivalent conditions (2.2.1)–(2.2.3) are satisfied. Now we observe that, since $f_n(T)$ commutes with $I_X - T$ for each $n \in \mathbb{N}$ by [11], V, 8.1, and consequently E also does, we have $E(I_X - T) = 0_{L(X)}$. Hence $\mathcal{R}(I_X - T) \subseteq \mathcal{N}(E)$. Since $E^2 = E$ and $\mathcal{R}(E) = \mathcal{N}(I_X - T)$ give $X = \mathcal{N}(I_X - T) \oplus \mathcal{N}(E)$, and condition (2.2.3) in turn gives $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$, we conclude that $\mathcal{N}(E) = \mathcal{R}(I_X - T)$. Then condition (2.3.2) is satisfied.

We also recall the following consequence of [3], 3.16.

Theorem 2.4 ([5], 1.3; [9], 18.8.1)**.** *Let* X *be a complex nonzero Banach space and* $T, E \in L(X)$*. If there exists a sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *in* $\rho(T)$ *such that* $\lim_{n\to+\infty} \lambda_n = 1$ *and* $(\lambda_n - 1)\mathfrak{R}_T(\lambda_n) \longrightarrow E$ *in* $L(X)$ *as* $n \to +\infty$*, then* 1 *is either in* $\rho(T)$*, or a simple pole of* $\mathfrak{R}_T$*. Furthermore,* $\mathcal{R}(I_X - T)$ *is closed in* X*,* $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$ *and* E *is the projection of* X *onto* $\mathcal{N}(I_X - T)$ *along* $\mathcal{R}(I_X - T)$*.*

For each $\alpha \in \mathbb{R}$, let $A_\alpha : \mathbb{N} \to \mathbb{R}$ denote the sequence of the *Cesàro numbers of order* α. That is,

$$A_\alpha(n) = \binom{n+\alpha}{n} = \begin{cases} 1 & \text{if } n = 0 \\ \dfrac{\prod_{j=1}^{n}(\alpha+j)}{n!} & \text{if } n \in \mathbb{Z}_+. \end{cases}$$

Hence $A_\alpha(n) > 0$ for each $n \in \mathbb{N}$ if $\alpha > -1$. Notice also that $A_0(n) = 1$ for all $n \in \mathbb{N}$. We recall that

(2.1) $\sum_{k=0}^{n} A_\alpha(k) = A_{\alpha+1}(n)$ *for each* $n \in \mathbb{N}$ *and each* $\alpha \in \mathbb{R}$

and

(2.2) $\lim\limits_{n\to+\infty} \frac{A_\alpha(n)}{n^\alpha} = \frac{1}{\Gamma(\alpha+1)}$ *for each* $\alpha \in \mathbb{R}\setminus\{-k : k \in \mathbb{Z}_+\}$,

where Γ denotes Euler's gamma function (see for instance [13], III, (1-11) and (1-15)).
The following well known identity—which we will need in the sequel—can be obtained from (2.1) as a straightforward consequence, or else is not difficult to check directly, by induction on n.

(2.3) $\sum\limits_{k=j}^{n} \binom{k}{j} = \binom{n+1}{j+1}$ *for every* $j \in \mathbb{N}$ *and every* $n \in \mathbb{N}_j$.

Remark 2.5. Let X be a complex nonzero Banach space, and let $T \in L(X)$. We recall that if the sequence $\left(\frac{\|T^n\|_{L(X)}}{n^\alpha}\right)_{n\in\mathbb{Z}_+}$ is bounded for some $\alpha \in (0,+\infty)$, then $r(T) \le 1$. Indeed, if $M \in (0,+\infty)$ is such that $\frac{\|T^n\|_{L(X)}}{n^\alpha} \le M$ for each $n \in \mathbb{Z}_+$, then

$$r(T) = \lim_{n\to+\infty} \|T^n\|_{L(X)}^{\frac{1}{n}} = \lim_{n\to+\infty} \left(\frac{\|T^n\|_{L(X)}}{n^\alpha}\right)^{\frac{1}{n}} \le \lim_{n\to+\infty} M^{\frac{1}{n}} = 1.$$

Finally, by also taking Theorem 2.4 into account, the improvement of E. Hille's (C,α) ergodic theorem obtained by E. Ed-dari can be formulated as follows.

Theorem 2.6 (see [5], Theorem 1)**.** *Let X be a complex nonzero Banach space, $T \in L(X)$, and $\alpha \in (0,+\infty)$. Then, given any $E \in L(X)$, we have*

$$\lim_{n\to+\infty} \left\| \frac{\sum\limits_{k=0}^{n} A_{\alpha-1}(n-k)T^k}{A_\alpha(n)} - E \right\|_{L(X)} = 0$$

if and only if

$$\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{n^\alpha} = 0 \qquad \text{and} \qquad \lim_{\lambda\to1^+} \|(\lambda-1)\mathfrak{R}_T(\lambda) - E\|_{L(X)} = 0.$$ [1]

Hence the following two conditions are equivalent:

(2.6.1) *the sequence* $\left(\frac{\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k}{A_\alpha(n)}\right)_{n\in\mathbb{N}}$ *converges in* $L(X)$*;*

(2.6.2) $\lim\limits_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{n^\alpha} = 0$ *and* 1 *is either in* $\rho(T)$*, or a simple pole of* $\mathfrak{R}_T$.

3 The least concave majorant of a real sequence

We begin with some results concerning the least concave majorant of a real sequence.

We recall that a real sequence $a : \mathbb{N} \to \mathbb{R}$ is called *concave* (*convex*) if the real sequence $(a(n+1)-a(n))_{n\in\mathbb{N}}$ is nonincreasing (nondecreasing). Notice that a is concave (convex) if and only if $a(n+1) \ge \frac{a(n)+a(n+2)}{2}$ ($a(n+1) \le \frac{a(n)+a(n+2)}{2}$) for every $n \in \mathbb{N}$.

[1] We point out that, by virtue of Remark 2.5, $\lim\limits_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{n^\alpha} = 0$ gives $r(T) \le 1$. Then $\rho(T)$ contains all real numbers λ satisfying $\lambda > 1$, which allows the limit $\lim\limits_{\lambda\to1^+} \|(\lambda-1)\mathfrak{R}_T(\lambda) - E\|_{L(X)}$ to be considered.

Definition 3.1. For each real sequence $a : \mathbb{N} \to \mathbb{R}$, let $\phi_a : [0,+\infty) \to \mathbb{R}$ be the function defined by

$$\phi_a(x) = a(n) + (x-n)(a(n+1) - a(n)) \qquad \text{for every } x \in [n,n+1] \text{ and every } n \in \mathbb{N}.$$

Notice that $\phi_a(x) = a(n)(n+1-x) + a(n+1)(x-n)$ for every $x \in [n,n+1]$ and every $n \in \mathbb{N}$. Hence $\phi_a(n) = a(n)$ for every $n \in \mathbb{N}$.

Proposition 3.2. *Let $a : \mathbb{N} \to \mathbb{R}$ be a real sequence. Then a is concave if and only if the function ϕ_a is concave.*

Proof. It is easily seen that a is concave if ϕ_a is. Conversely, suppose a to be concave. Notice that ϕ_a is continuous. Also, the right derivative $(\phi_a)'_+$ of ϕ_a exists at every point of $[0,+\infty)$, and $(\phi_a)'_+(x) = a(n+1) - a(n)$ for every $x \in [n,n+1)$ and every $n \in \mathbb{N}$. Since a is concave, it follows that $(\phi_a)'_+$ is nonincreasing, and consequently (see [10], 5, Proposition 18) ϕ_a is concave. □

We recall that a *majorant* of a real sequence $b : \mathbb{N} \to \mathbb{R}$ is a real sequence $c : \mathbb{N} \to \mathbb{R}$ satisfying $c(n) \geq b(n)$ for every $n \in \mathbb{N}$.
The following result is probably known. Indeed, for instance, the authors of [1] seem to be aware of it when (in the proof of Proposition 2.1) they derive that the sequence $(\rho_n)_{n\in\mathbb{N}}$ has a least concave majorant from being $\lim_{n\to+\infty} \frac{\rho_n}{n} = 0$. Anyway, we give a (short) proof here, for the convenience of the reader.

Proposition 3.3. *A real sequence $b : \mathbb{N} \to \mathbb{R}$ has a concave majorant if and only if the sequence $(\frac{b(n)}{n})_{n\in\mathbb{Z}_+}$ is bounded from above.*

Proof. By virtue of Proposition 3.2, it is easily seen that b has a concave majorant if and only if there exists a concave function $f : [0,+\infty) \to \mathbb{R}$ such that $f(x) \geq \phi_b(x)$ for every $x \in [0,+\infty)$. The latter condition is satisfied if and only if there exist α, $\beta \in \mathbb{R}$ such that $\phi_b(x) \leq \alpha + \beta x$ for every $x \in [0,+\infty)$ (see [6], Theorem 1.2) or, equivalently, $b(n) \leq \alpha + \beta n$ for every $n \in \mathbb{N}$. Now it is straightforward to observe that such α and β exist if and only if the sequence $(\frac{b(n)}{n})_{n\in\mathbb{Z}_+}$ is bounded from above. We have thus obtained the desired result. □

Remark 3.4. If $a : \mathbb{N} \to \mathbb{R}$ is a concave sequence, then the sequence $(\frac{a(n)}{n})_{n\in\mathbb{Z}_+}$ is bounded from above.

Remark 3.5. If a real sequence $b : \mathbb{N} \to \mathbb{R}$ has a concave majorant, then b has a *least* concave majorant c. Furthermore, we have

$$c(n) = \inf\{a(n) : a \in \mathbb{R}^{\mathbb{N}},\ a \text{ concave majorant of } b\} \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, once one observes that the real sequence c defined as above is concave and is a majorant of b, from the definition of c it follows that each concave majorant of b is also a majorant of c, that is, c is the least concave majorant of b.

Theorem 3.6. *Let $b : \mathbb{N} \to \mathbb{R}$ be a real sequence such that the sequence $(\frac{b(n)}{n})_{n\in\mathbb{Z}_+}$ is bounded from above, and let $c : \mathbb{N} \to \mathbb{R}$ be the least concave majorant of b. Then c satisfies the following properties.*

(3.6.1) $c(0) = b(0)$.

(3.6.2) $c(n+1) = c(n) + \sup\left\{\frac{b(k)-c(n)}{k-n} : k \in \mathbb{N}_{n+1}\right\}$ *for every*[2] $n \in \mathbb{N}$.

(3.6.3) *For each* $n \in \mathbb{N}$ *and each* $k \in \mathbb{N}_{n+2}$, *we have*

$$\frac{b(k)-c(n+1)}{k-n-1} \leq \frac{b(k)-c(n)}{k-n}.$$

If in addition

$$\frac{b(k)-c(n)}{k-n} = \max\left\{\frac{b(j)-c(n)}{j-n} : j \in \mathbb{N}_{n+1}\right\},$$

then

$$\begin{aligned}\frac{b(k)-c(h)}{k-h} &= \max\left\{\frac{b(j)-c(h)}{j-h} : j \in \mathbb{N}_{h+1}\right\}\\ &= \frac{b(k)-c(n)}{k-n} \qquad \textit{for all } h = n,\ldots,k-1,\end{aligned}$$

and consequently $c(h+1)-c(h) = c(n+1)-c(n)$ *for all* $h = n,\ldots,k-1$.

Proof. Let $a : \mathbb{N} \to \mathbb{R}$ be defined by

$$a(0) = b(0),$$

$$a(n+1) = a(n) + \sup\left\{\frac{b(k)-a(n)}{k-n} : k \in \mathbb{N}_{n+1}\right\} \qquad \text{for every } n \in \mathbb{N}.$$

We prove that $a = c$. First of all, we prove that $a(n) \geq b(n)$ for every $n \in \mathbb{N}$.
We proceed by induction. The desired result clearly holds for $n = 0$. Besides, if for some $n \in \mathbb{N}$ we have $a(n) \geq b(n)$, then

$$a(n+1) = a(n) + \sup\left\{\frac{b(k)-a(n)}{k-n} : k \in \mathbb{N}_{n+1}\right\} \geq a(n) + b(n+1) - a(n) = b(n+1).$$

We have thus proved that a is a majorant of b. Now we prove that a is concave.
For each $n \in \mathbb{N}$, we have

$$a(n+1) - a(n) = \sup\left\{\frac{b(k)-a(n)}{k-n} : k \in \mathbb{N}_{n+1}\right\} \tag{3.1}$$

and

$$a(n+2) - a(n+1) = \sup\left\{\frac{b(k)-a(n+1)}{k-n-1} : k \in \mathbb{N}_{n+2}\right\}. \tag{3.2}$$

Now, for each $k \in \mathbb{N}_{n+2}$, (3.1) yields

$$\begin{aligned}\frac{b(k)-a(n+1)}{k-n-1} &= \frac{b(k)-a(n)}{k-n-1} - \frac{a(n+1)-a(n)}{k-n-1}\\ &= \left(\frac{k-n}{k-n-1}\right)\left(\frac{b(k)-a(n)}{k-n}\right) - \left(\frac{1}{k-n-1}\right)\sup\left\{\frac{b(j)-a(n)}{j-n} : j \in \mathbb{N}_{n+1}\right\}\\ &\leq \left(\frac{k-n}{k-n-1}\right)\left(\frac{b(k)-a(n)}{k-n}\right) - \frac{1}{k-n-1}\left(\frac{b(k)-a(n)}{k-n}\right)\\ &= \left(\frac{b(k)-a(n)}{k-n}\right)\left(\frac{k-n}{k-n-1} - \frac{1}{k-n-1}\right) = \frac{b(k)-a(n)}{k-n}.\end{aligned} \tag{3.3}$$

[2] Notice that, since the sequence $(\frac{b(n)}{n})_{n\in\mathbb{Z}_+}$ is bounded from above, this supremum is finite for every $n \in \mathbb{N}$.

Now from (3.3), together with (3.1) and (3.2), we derive that a is concave. Hence a is a concave majorant of b. In order to conclude that $a = c$, it suffices to prove that $c(n) \geq a(n)$ for every $n \in \mathbb{N}$. We proceed by induction.

Since $c(0) \geq b(0) = a(0)$, the desired inequality holds for $n = 0$. Now let $n \in \mathbb{N}$ be such that $c(n) \geq a(n)$. Since c is concave and is a majorant of b, from Proposition 3.2 and from the three chord lemma we conclude that for each $k \in \mathbb{N}_{n+1}$ we have

$$c(n+1) - c(n) \geq \frac{c(k) - c(n)}{k-n} \geq \frac{b(k) - c(n)}{k-n}$$

and consequently, since $c(n) \geq a(n)$,

$$\begin{aligned} c(n+1) \geq c(n) + \frac{b(k) - c(n)}{k-n} &= c(n) + \frac{b(k) - a(n)}{k-n} + \frac{a(n) - c(n)}{k-n} \\ &= \frac{(k-n-1)}{k-n} c(n) + \frac{1}{k-n} a(n) + \frac{b(k) - a(n)}{k-n} \\ \geq \left(\frac{k-n-1}{k-n} + \frac{1}{k-n} \right) a(n) + \frac{b(k) - a(n)}{k-n} &= a(n) + \frac{b(k) - a(n)}{k-n}. \end{aligned}$$

Then

$$c(n+1) \geq a(n) + \sup \left\{ \frac{b(k) - a(n)}{k-n} : k \in \mathbb{N}_{n+1} \right\} = a(n+1).$$

We have thus proved that $c(n) \geq a(n)$ for every $n \in \mathbb{N}$. Then $a = c$, from which we obtain (3.6.1) and (3.6.2). Also, the inequality in (3.6.3) now follows from (3.3).

In order to complete the proof, it remains to prove that if $n \in \mathbb{N}$ and $k \in \mathbb{N}_{n+2}$ satisfy $\frac{b(k)-c(n)}{k-n} = \max\left\{\frac{b(j)-c(n)}{j-n} : j \in \mathbb{N}_{n+1}\right\}$, then $\frac{b(k)-c(h)}{k-h} = \max\left\{\frac{b(j)-c(h)}{j-h} : j \in \mathbb{N}_{h+1}\right\} = \frac{b(k)-c(n)}{k-n}$ for all $h = n, \ldots, k-1$.

Let $n \in \mathbb{N}$, $k \in \mathbb{N}_{n+2}$ be as above. As a straightforward consequence of (3.3), we obtain

$$\frac{b(k) - c(n+1)}{k-n-1} = \frac{b(k) - c(n)}{k-n}$$

and consequently

$$c(n+2) - c(n+1) = \sup \left\{ \frac{b(j) - c(n+1)}{j-n-1} : j \in \mathbb{N}_{n+2} \right\} \geq \frac{b(k) - c(n)}{k-n} = c(n+1) - c(n).$$

Since c is concave, the opposite inequality also holds. Hence

$$\frac{b(k) - c(n+1)}{k-n-1} = \max \left\{ \frac{b(j) - c(n+1)}{j-n-1} : j \in \mathbb{N}_{n+2} \right\} = \frac{b(k) - c(n)}{k-n}.$$

If $k = n+2$, the proof is complete. Otherwise, we finish the proof by applying the same argument again $k-n-2$ times, with $\frac{b(k)-c(n)}{k-n}$ replaced by $\frac{b(k)-c(h)}{k-h}$, $h = n+1, \ldots, k-2$. □

Lemma 3.7. *Let $a : \mathbb{N} \to \mathbb{R}$ be a real sequence. If there exists $\nu \in \mathbb{N}$ such that $a(\nu) \geq \limsup_{n \to +\infty} a(n)$, then the set $\{a(n) : n \in \mathbb{N}\}$ has a maximum.*

Proof. We set $\ell = \limsup_{n\to+\infty} a(n)$ and observe that $\ell \in [-\infty,+\infty)$. Since assuming $a(n) \le \ell$ for every $n \in \mathbb{N}$ yields $\ell \in \mathbb{R}$ and $a(\nu) = \ell \ge a(n)$ for every $n \in \mathbb{N}$—which means that $a(\nu)$ is the maximum of $\{a(n) : n \in \mathbb{N}\}$, we may assume that $a(\nu) > \ell$. Then there exists $n_0 \in \mathbb{N}$ such that $a(n) < a(\nu)$ for every $n \in \mathbb{N}_{n_0}$, from which we conclude that $\nu < n_0$. Now let $n_1 \in \{0,\dots,n_0-1\}$ be such that $a(n_1) \ge a(k)$ for all $k = 0,\dots,n_0-1$. It suffices to remark that for each $n \in \mathbb{N}_{n_0}$ we have $a(n) < a(\nu) \le a(n_1)$. Hence $a(n_1) \ge a(n)$ for every $n \in \mathbb{N}$. □

Theorem 3.8. *Let $b : \mathbb{N} \to \mathbb{R}$ be a real sequence such that the sequence $\left(\frac{b(n)}{n}\right)_{n\in\mathbb{Z}_+}$ is bounded from above, and let $c : \mathbb{N} \to \mathbb{R}$ be the least concave majorant of b.*

(3.8.1) *If we set $\ell = \limsup_{n\to+\infty} \frac{b(n)}{n}$, we have $\ell \in [-\infty,+\infty)$, $c(n+1) - c(n) \ge \ell$ for every $n \in \mathbb{N}$ and $\limsup_{k\to+\infty}\left(\frac{b(k)-c(n)}{k-n}\right) = \ell$ for every $n \in \mathbb{N}$.*

(3.8.2) *If we set*

$$\mathfrak{N} = \{0\} \cup \left\{ n \in \mathbb{Z}_+ : c(n) - c(n-1) = \max\left\{ \frac{b(k) - c(n-1)}{k - n + 1} : k \in \mathbb{N}_n \right\}\right\},$$

it follows that $n \in \mathfrak{N} \Longrightarrow \{0,\dots,n\} \subseteq \mathfrak{N}$.

(3.8.3) *If we set $N = \sup(\mathfrak{N})$ and $(\nu_k)_{k\in\mathbb{N}}$ is the nondecreasing sequence of nonnegative integers defined by $\nu_0 = 0$,*

$$\nu_{k+1} = \begin{cases} \min\left\{ n \in \mathbb{N}_{\nu_k+1} : \frac{b(n)-c(\nu_k)}{n-\nu_k} = c(\nu_k+1) - c(\nu_k)\right\} & \text{if}^3\ \nu_k < N \\ \nu_k & \text{if } \nu_k \ge N \end{cases}$$

for every $k \in \mathbb{N}$, it follows that $\nu_k \in \mathfrak{N}$ for every $k \in \mathbb{N}$ and $\{\nu_k : k \in \mathbb{N}\} = \{n \in \mathbb{N} : c(n) = b(n)\}$.

(3.8.4) *If $\mathfrak{N}$ is finite (that is, $N \in \mathbb{N}$, $N = \max(\mathfrak{N})$), then the sequence $(\nu_k)_{k\in\mathbb{N}}$ is eventually constant (and consequently $\nu_k \ge N$ for sufficiently large n). Furthermore, if we set $k_0 = min\{k \in \mathbb{N} : \nu_k \ge N\}$ we have: $\nu_k = N$ for every $k \in \mathbb{N}_{k_0}$, $\nu_k < \nu_{k+1}$ for each $k \in \mathbb{N}$ satisfying $k < k_0$, $\ell \in \mathbb{R}$, and, for each $n \in \mathbb{N}$,*

$$c(n) = \begin{cases} b(\nu_k) + \left(\frac{n-\nu_k}{\nu_{k+1}-\nu_k}\right)(b(\nu_{k+1}) - b(\nu_k)) & \text{if } \nu_k \le n \le \nu_{k+1} \text{ for some } k \in \mathbb{N} \\ & \text{satisfying } k < k_0 \\ b(N) + \ell(n-N) & \text{if } n \ge N \end{cases}$$

$$= \begin{cases} \left(\frac{\nu_{k+1}-n}{\nu_{k+1}-\nu_k}\right)b(\nu_k) + \left(\frac{n-\nu_k}{\nu_{k+1}-\nu_k}\right)b(\nu_{k+1}) & \text{if } \nu_k \le n \le \nu_{k+1} \text{ for some } k \in \mathbb{N} \\ & \text{satisfying } k < k_0 \\ b(N) + \ell(n-N) & \text{if } n \ge N. \end{cases}$$

Finally, $c(n) > b(n)$ for every $n \in \mathbb{N}_{N+1}$.

[3]Notice that if $\nu_k < N$, from (3.8.2) it follows that $\nu_k + 1 \in \mathfrak{N}$ and consequently $c(\nu_k+1) - c(\nu_k) = \frac{b(n)-c(\nu_k)}{n-\nu_k}$ for some $n \in \mathbb{N}_{\nu_k+1}$.

(3.8.5) *If $\mathfrak{N}$ is infinite (that is, $\mathfrak{N} = \mathbb{N}$), then the sequence $(\nu_k)_{k\in\mathbb{N}}$ is strictly increasing. Furthermore, for each $k \in \mathbb{N}$ we have*

$$c(n) = b(\nu_k) + \left(\frac{n-\nu_k}{\nu_{k+1}-\nu_k}\right)(b(\nu_{k+1}) - b(\nu_k)) = \left(\frac{\nu_{k+1}-n}{\nu_{k+1}-\nu_k}\right) b(\nu_k) + \left(\frac{n-\nu_k}{\nu_{k+1}-\nu_k}\right) b(\nu_{k+1})$$

for every $n \in \mathbb{N}$ satisfying $\nu_k \le n \le \nu_{k+1}$.

Proof. We begin by proving (3.8.1). As a straightforward consequence of $\left(\frac{b(n)}{n}\right)_{n\in\mathbb{Z}_+}$ being bounded from above, we have $\ell \in [-\infty, +\infty)$.
Now let $(k_j)_{j\in\mathbb{N}}$ be a strictly increasing sequence of strictly positive integers such that $\lim\limits_{j\to+\infty} \frac{b(k_j)}{k_j} = \ell$. Then for each $n \in \mathbb{N}$ we have

$$\lim_{j\to+\infty} \left(\frac{b(k_j) - c(n)}{k_j - n}\right) = \lim_{j\to+\infty} \left(\frac{k_j}{k_j - n}\right)\left(\frac{b(k_j)}{k_j} - \frac{c(n)}{k_j}\right) = \ell. \tag{3.4}$$

Hence, by virtue of (3.6.2),

$$c(n+1) - c(n) = \sup\left\{\frac{b(k) - c(n)}{k-n} : k \in \mathbb{N}_{n+1}\right\} \ge \ell.$$

Now if we set $s = \limsup\limits_{k\to+\infty}\left(\frac{b(k)-c(n)}{k-n}\right)$, from (3.4) we also derive that $s \ge \ell$. On the other hand, if $(m_j)_{j\in\mathbb{N}}$ is a strictly increasing sequence of nonnegative integers such that $\lim\limits_{j\to+\infty}\left(\frac{b(m_j)-c(n)}{m_j-n}\right) = s$, we obtain

$$\frac{b(m_j)}{mj} = \left(\frac{b(m_j) - c(n)}{m_j - n}\right)\left(\frac{m_j - n}{m_j}\right) + \frac{c(n)}{m_j} \xrightarrow[j\to+\infty]{} s,$$

which gives $s \le \ell$. Hence $s = \ell$.
(3.8.1) is thus proved. Now we prove (3.8.2).
Fix $n \in \mathfrak{N}$ and let $k \in \{0, \dots, n\}$. We prove that $k \in \mathfrak{N}$. This is clearly true if $k = 0$. If $k \in \mathbb{Z}_+$, then $n \in \mathbb{Z}_+$ and $c(n) - c(n-1) = \frac{b(p_n)-c(n-1)}{p_n-n+1}$ for some $p_n \in \mathbb{N}_n$. It is not restrictive to assume $k \le n-1$ (which gives $n-k \in \mathbb{Z}_+$, $n \in \mathbb{N}_2$, $k-1 \le n-2$). Since for each $j \in \{k-1, \dots, n-2\}$ we have $j+2 \le p_n$, and consequently $\frac{b(p_n)-c(j+1)}{p_n-j-1} \le \frac{b(p_n)-c(j)}{p_n-j}$ by (3.6.3), by taking (3.8.1) into account we obtain

$$\frac{b(p_n) - c(k-1)}{p_n - k + 1} \ge \frac{b(p_n) - c(n-1)}{p_n - n + 1} = c(n) - c(n-1) \ge \ell = \limsup_{m\to+\infty}\left(\frac{b(m) - c(k-1)}{m - k + 1}\right).$$

Now from Lemma 3.7 we conclude that the set $\left\{\frac{b(m)-c(k-1)}{m-k+1} : m \in \mathbb{N}_k\right\}$ has a maximum. This, together with (3.6.2), yields $k \in \mathfrak{N}$.
We prove (3.8.3).
We begin by proving that for each $k \in \mathbb{N}$ we have $\nu_k \in \mathfrak{N}$ and $\{n \in \{0, \dots \nu_k\} : c(n) = b(n)\} = \{\nu_j : j = 0, \dots, k\}$. We proceed by induction. We set

$$\mathcal{S} = \Big\{k \in \mathbb{N} : \nu_k \in \mathfrak{N} \text{ and } \{n \in \{0, \dots, \nu_k\} : c(n) = b(n)\} = \{\nu_j : j = 0, \dots, k\}\Big\}.$$

Since $\nu_0 = 0$, by the definition of $\mathfrak{N}$ and by (3.6.1) we have $0 \in \mathcal{S}$. Now suppose $k \in \mathcal{S}$. Then $\nu_k \in \mathfrak{N}$ and $\{n \in \{0, \dots \nu_k\} : c(n) = b(n)\} = \{\nu_j : j = 0, \dots, k\}$. If $\nu_k \ge N$, we have $\nu_{k+1} =$

$\nu_k \in \mathfrak{N}$ and $\{n \in \{0,\dots,\nu_{k+1}\} : c(n) = b(n)\} = \{n \in \{0,\dots \nu_k\} : c(n) = b(n)\} = \{\nu_j : j = 0,\dots,k\} = \{\nu_j : j = 0,\dots,k+1\}$, which gives $k+1 \in \mathcal{S}$. Thus, let us assume $\nu_k < N$. Then $\nu_{k+1} \geq \nu_k + 1$ and

$$\nu_{k+1} = \min\left\{n \in \mathbb{N}_{\nu_k+1} : \frac{b(n) - c(\nu_k)}{n - \nu_k} = c(\nu_k + 1) - c(\nu_k)\right\}. \tag{3.5}$$

From (3.6.3) we derive that for each $j \in \mathbb{N}$ satisfying $\nu_k \leq j \leq \nu_{k+1} - 1$ we have

$$\frac{b(\nu_{k+1}) - c(j)}{\nu_{k+1} - j} = \max\left\{\frac{b(m) - c(j)}{m - j} : m \in \mathbb{N}_{j+1}\right\} = \frac{b(\nu_{k+1}) - c(\nu_k)}{\nu_{k+1} - \nu_k}. \tag{3.6}$$

By letting $j = \nu_{k+1} - 1$, from (3.6)—together with (3.6.2)—we conclude that $\nu_{k+1} \in \mathfrak{N}$ and besides

$$b(\nu_{k+1}) - c(\nu_{k+1} - 1) = \max\left\{\frac{b(m) - c(\nu_{k+1} - 1)}{m - \nu_{k+1} + 1} : m \in \mathbb{N}_{\nu_{k+1}}\right\} = c(\nu_{k+1}) - c(\nu_{k+1} - 1),$$

which gives $c(\nu_{k+1}) = b(\nu_{k+1})$. Finally, for each $n \in \mathbb{N}_{\nu_k+1}$ satisfying $n < \nu_{k+1}$, (3.6.2), (3.6.3), (3.5) and (3.6) give

$$\begin{aligned} c(n) - c(n-1) &= \max\left\{\frac{b(m) - c(n-1)}{m - n + 1} : m \in \mathbb{N}_n\right\} \\ &= \frac{b(\nu_{k+1}) - c(\nu_k)}{\nu_{k+1} - \nu_k} > \frac{b(n) - c(\nu_k)}{n - \nu_k} \geq b(n) - c(n-1), \end{aligned}$$

the latter inequality being trivially an equality if $n = \nu_k + 1$, and being a consequence of (3.6.3) if $n \geq \nu_k + 2$ (as $\frac{b(n)-c(j)}{n-j} \geq \frac{b(n)-c(j+1)}{n-j-1}$ for all $j = \nu_k,\dots,n-2$). Consequently, $c(n) > b(n)$. Hence

$$\begin{aligned} &\{n \in \{0,\dots,\nu_{k+1}\} : c(n) = b(n)\} \\ &= \{n \in \{0,\dots \nu_k\} : c(n) = b(n)\} \cup \{n \in \{\nu_k + 1,\dots,\nu_{k+1}\} : c(n) = b(n)\} \\ &= \{\nu_j : j = 0,\dots,k\} \cup \{\nu_{k+1}\} = \{\nu_j : j = 0,\dots,k+1\}, \end{aligned}$$

which gives $k+1 \in \mathcal{S}$.

We have thus proved that $\nu_k \in \mathfrak{N}$ for every $k \in \mathbb{N}$. Also,

$$\{n \in \{0,\dots,\nu_k\} : c(n) = b(n)\} = \{\nu_j : j = 0,\dots,k\} \qquad \text{for every } k \in \mathbb{N}. \tag{3.7}$$

Now we prove that $\{n \in \mathbb{N} : c(n) = b(n)\} = \{\nu_k : k \in \mathbb{N}\}$.

If $N = +\infty$, then $\nu_{k+1} \geq \nu_k + 1$ for every $k \in \mathbb{N}$, which gives $\lim_{k\to+\infty} \nu_k = +\infty$, and consequently $\bigcup_{k\in\mathbb{N}} \{0,\dots,\nu_k\} = \mathbb{N}$. Hence

$$\begin{aligned} \{n \in \mathbb{N} : c(n) = b(n)\} &= \bigcup_{k\in\mathbb{N}} \{n \in \{0,\dots,\nu_k\} : c(n) = b(n)\} \\ &= \bigcup_{k\in\mathbb{N}} \{\nu_j : j = 0,\dots,k\} = \{\nu_k : k \in \mathbb{N}\}, \end{aligned}$$

which is the desired result.

If $N < +\infty$, then there exists $\overline{k} \in \mathbb{N}$ such that $\nu_{\overline{k}} \geq N$: otherwise, if $\nu_k < N$ for all $k \in \mathbb{N}$, the sequence $(\nu_k)_{k\in\mathbb{N}}$ would be strictly increasing and consequently we would have $+\infty = \lim_{k\to+\infty} \nu_k \le N$, a contradiction. Hence $\nu_k = \nu_{\overline{k}}$ for every $k \in \mathbb{N}_{\overline{k}}$. Furthermore, for each $n \in \mathbb{N}_{\nu_{\overline{k}}+1}$, we have $n > N$ and consequently $n \notin \mathfrak{N}$. Then $n \in \mathbb{Z}_+$ and, by virtue of (3.6.2), $c(n) - c(n-1) > b(n) - c(n-1)$, which gives $c(n) > b(n)$. From this, together with (3.7), we obtain

$$\begin{aligned}\{n \in \mathbb{N} : c(n) = b(n)\} &= \{n \in \{0, \dots, \nu_{\overline{k}}\} : c(n) = b(n)\}\\ &= \{\nu_j : j = 0, \dots, \overline{k}\} = \{\nu_k : k \in \mathbb{N}\}.\end{aligned}$$

We have thus finished the proof of (3.8.3).
We prove (3.8.4). Suppose $\mathfrak{N}$ to be finite. Then $N \in \mathbb{N}$ and $N = \max(\mathfrak{N})$. Also, we have already observed—in the proof of (3.8.3)—that the sequence $(\nu_k)_{k\in\mathbb{N}}$ is eventually constant and not less than N. We set $k_0 = \min\{k \in \mathbb{N} : \nu_k \ge N\}$. Then $\nu_{k_0} \ge N$. On the other hand, since $\nu_{k_0} \in \mathfrak{N}$ by (3.8.3), we have $\nu_{k_0} \le N$. Then $\nu_{k_0} = N$, and consequently $\nu_k = N$ for each $k \in \mathbb{N}_{k_0}$. Furthermore, for each $k \in \mathbb{N}$ satisfying $k < k_0$ we have $\nu_k < N$, and consequently $\nu_k < \nu_{k+1}$. From (3.6) and (3.6.2) we conclude that for each $n \in \mathbb{N}$ satisfying $\nu_k + 1 \le n \le \nu_{k+1}$ we have

$$c(j) - c(j-1) = \frac{b(\nu_{k+1}) - c(\nu_k)}{\nu_{k+1} - \nu_k} \qquad \text{for all } j = \nu_k + 1, \dots, n$$

and consequently

$$c(n) - c(\nu_k) = \sum_{j=\nu_k+1}^{n} (c(j) - c(j-1)) = (n - \nu_k)\left(\frac{b(\nu_{k+1}) - c(\nu_k)}{\nu_{k+1} - \nu_k}\right).$$

Hence

$$c(n) - c(\nu_k) = (n - \nu_k)\left(\frac{b(\nu_{k+1}) - c(\nu_k)}{\nu_{k+1} - \nu_k}\right) \qquad \text{for all } n = \nu_k, \dots, \nu_{k+1}. \tag{3.8}$$

Since $c(\nu_k) = b(\nu_k)$ by (3.8.3), from (3.8) we derive that for each $n \in \{\nu_k, \dots, \nu_{k+1}\}$ we have

$$c(n) = c(\nu_k) + (n - \nu_k)\left(\frac{b(\nu_{k+1}) - c(\nu_k)}{\nu_{k+1} - \nu_k}\right) = b(\nu_k) + \left(\frac{n - \nu_k}{\nu_{k+1} - \nu_k}\right)(b(\nu_{k+1}) - b(\nu_k)).$$

Now we prove that $\ell \in \mathbb{R}$ and $c(n) = b(N) + \ell(n - N)$ for every $n \in \mathbb{N}_N$.
For each $n \in \mathbb{N}_N$, we have $n + 1 > N$ and consequently $n + 1 \notin \mathfrak{N}$. From (3.6.2) we conclude that the set $\left\{\frac{b(k)-c(n)}{k-n} : k \in \mathbb{N}_{n+1}\right\}$ has no maximum. From Lemma 3.7 and from (3.8.1) we derive that $\frac{b(k)-c(n)}{k-n} < \ell$ for every $k \in \mathbb{N}_{n+1}$, and consequently $\ell \in \mathbb{R}$. Besides, from (3.6.2) we obtain

$$c(n+1) - c(n) = \sup\left\{\frac{b(k) - c(n)}{k - n} : k \in \mathbb{N}_{n+1}\right\} \le \ell,$$

which, together with (3.8.1), gives $c(n+1) - c(n) = \ell$. Notice also that by virtue of (3.8.3), $\nu_{k_0} = N$ yields $c(N) = c(\nu_{k_0}) = b(\nu_{k_0}) = b(N)$. Now, proceeding by induction, we conclude that $c(n) = b(N) + \ell(n - N)$ for every $n \in \mathbb{N}_N$. Finally, from (3.8.3) we derive that $c(n) > b(n)$ for every $n \in \mathbb{N}_{N+1}$ and the proof of (3.8.4) is complete.
We prove (3.8.5). If we assume $\mathfrak{N}$ to be infinite (or equivalently, by virtue of (3.8.2), $\mathfrak{N} = \mathbb{N}$), then $N = +\infty$ and consequently the sequence $(\nu_k)_{k\in\mathbb{N}}$ is strictly increasing. The remaining assertion can be derived from (3.6), (3.6.2) and (3.8.3), proceeding as in the proof of (3.8.4).
The proof is now finished. □

Theorem 3.9. *Let $b : \mathbb{N} \to \mathbb{R}$ be a real sequence such that the sequence $\left(\frac{b(n)}{n}\right)_{n\in\mathbb{Z}_+}$ is bounded from above and b is not, and let c be the least concave majorant of b. Then c is strictly increasing, $\lim_{n\to+\infty} c(n) = +\infty$ and $\limsup_{n\to+\infty} \frac{b(n)}{c(n)} = 1$.*

Proof. We set $\ell = \limsup_{n\to+\infty} \frac{b(n)}{n}$. We observe that $\limsup_{n\to+\infty} b(n) = +\infty$, and consequently $\ell \in [0, +\infty)$. From (3.8.1) it follows that c is nondecreasing, and consequently there exists $\lim_{n\to+\infty} c(n)$. Since $c(n) \geq b(n)$ for every $n \in \mathbb{N}$, we conclude that

$$\lim_{n\to+\infty} c(n) \geq \limsup_{n\to+\infty} b(n) = +\infty.$$

Hence $c(n) \longrightarrow +\infty$ as $n \to +\infty$. Now we prove that c is strictly increasing.
If c were not strictly increasing, then—being c nondecreasing—there would be $n_0 \in \mathbb{N}$ such that $c(n_0 + 1) - c(n_0) = 0$. Since c is concave as well as nondecreasing, we would conclude that c is eventually constant, in contradiction with $\lim_{n\to+\infty} c(n) = +\infty$.
Finally, we prove that $\limsup_{n\to+\infty} \frac{b(n)}{c(n)} = 1$. By virtue of Theorem 3.8, one of the following two conditions is satisfied:

(3.9.1) there exists a strictly increasing sequence $(\nu_k)_{k\in\mathbb{N}}$ of nonnegative integers such that $c(\nu_k) = b(\nu_k)$ for every $k \in \mathbb{N}$;

(3.9.2) there exists $N \in \mathbb{N}$ such that $c(n) > b(n)$ for every $n \in \mathbb{N}_{N+1}$ and $c(n) = b(N) + \ell(n - N)$ for every $n \in \mathbb{N}_N$.

If (3.9.1) holds, it suffices to observe that $\limsup_{n\to+\infty} \frac{b(n)}{c(n)} \geq \lim_{k\to+\infty} \frac{b(\nu_k)}{c(\nu_k)} = 1$. The opposite inequality follows from c being a majorant of b.
If (3.9.2) holds, then $\lim_{n\to+\infty} (b(N) + \ell(n - N)) = \lim_{n\to+\infty} c(n) = +\infty$ gives $\ell \in (0, +\infty)$. Hence

$$\limsup_{n\to+\infty} \left(\frac{b(n)}{c(n)}\right) = \limsup_{n\to+\infty} \left(\frac{b(n)}{n}\right) \cdot \frac{1}{\ell + \left(\frac{b(N)-\ell N}{n}\right)} = 1.$$

The desired result is thus proved. □

The following is a consequence of Remark 3.4 and Theorem 3.9. Alternatively, it can be derived from Proposition 3.2 and the properties of concave functions.

Corollary 3.10. *If a concave sequence $a : \mathbb{N} \to \mathbb{R}$ is not bounded from above, then a is strictly increasing and $\lim_{n\to+\infty} a(n) = +\infty$.*

4 Real sequences with concave p^{th}-difference

Definition 4.1. Let $\varSigma, \varDelta : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ be the linear operators defined by

$$(\varSigma a)(n) = \sum_{k=0}^{n} a(k) \qquad \text{and} \qquad (\varDelta a)(n) = \begin{cases} a(0) & \text{if } n = 0 \\ a(n) - a(n-1) & \text{if } n \in \mathbb{Z}_+ \end{cases}$$

for every $n \in \mathbb{N}$ and every $a \in \mathbb{K}^{\mathbb{N}}$.

Notice that both linear operators $\varSigma$ and $\varDelta$ are bijective. Besides, $\varDelta = \varSigma^{-1}$ (or, equivalently, $\varSigma = \varDelta^{-1}$). We also remark that $\varDelta(\ell_1) \subseteq \ell_1$. Finally, we observe that the operator $\varSigma$ preserves inequalities: indeed, if $a, b \in \mathbb{R}^{\mathbb{N}}$ satisfy $a(n) \le b(n)$ for each $n \in \mathbb{N}$, then $(\varSigma a)(n) \le (\varSigma b)(n)$ for each $n \in \mathbb{N}$.

The following is a consequence of Proposition 3.2 and of the the three chord lemma.

Lemma 4.2. *Let $a : \mathbb{N} \to \mathbb{R}$ be a concave sequence. Then*

$$n(a(k) - a(0)) \ge k(a(n) - a(0)) \qquad \textit{for every } n \in \mathbb{N} \textit{ and every } k = 0, \dots, n.$$

Theorem 4.3. *Let $a : \mathbb{N} \to \mathbb{R}$ be a nondecreasing concave sequence. Then for each $p \in \mathbb{N}$ we have*

$$\frac{1}{p+1}\binom{n+p}{p} a(n) + \frac{p}{p+1}\binom{n+p}{p} a(0) \le (\varSigma^p a)(n) \le \binom{n+p}{p} a(n) \qquad \textit{for every } n \in \mathbb{N}.$$

Proof. We begin by proving that $(\varSigma^p a)(n) \le \binom{n+p}{p} a(n)$ for all n, $p \in \mathbb{N}$. We proceed by induction on p.

For $p = 0$, the desired inequality trivially holds for every $n \in \mathbb{N}$. Now let $p \in \mathbb{N}$ be such that $(\varSigma^p a)(n) \le \binom{n+p}{p} a(n)$ for every $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, since a is nondecreasing we have

$$(\varSigma^{p+1} a)(n) = \sum_{k=0}^{n} (\varSigma^p a)(k) \le \sum_{k=0}^{n} \binom{k+p}{p} a(k) \le a(n) \sum_{k=0}^{n} \binom{k+p}{p} = \binom{n+p+1}{p+1} a(n)$$

by (2.3).

We have thus proved the desired inequality. Now, proceeding again by induction on p, we prove that $(\varSigma^p a)(n) \ge \frac{1}{p+1}\binom{n+p}{p} a(n) + \frac{p}{p+1}\binom{n+p}{p} a(0)$ for all n, $p \in \mathbb{N}$.

For $p = 0$, the desired inequality is trivially satisfied for every $n \in \mathbb{N}$. Now let $p \in \mathbb{N}$ be such that $(\varSigma^p a)(n) \ge \frac{1}{p+1}\binom{n+p}{p} a(n) + \frac{p}{p+1}\binom{n+p}{p} a(0)$ for every $n \in \mathbb{N}$. We prove that $(\varSigma^{p+1} a)(n) \ge \frac{1}{p+2}\binom{n+p+1}{p+1} a(n) + \frac{p+1}{p+2}\binom{n+p+1}{p+1} a(0)$ for every $n \in \mathbb{N}$.

For $n = 0$, since $(\varSigma^{p+1} a)(0) = a(0)$ the desired inequality is straightforward. Now fix $n \in \mathbb{Z}_+$. Then from Lemma 4.2 and (2.3) we obtain

$$\begin{aligned}
(\varSigma^{p+1} a)(n) &= \sum_{k=0}^{n} (\varSigma^p a)(k) \ge \frac{1}{p+1} \sum_{k=0}^{n} \binom{k+p}{p} a(k) + \frac{p}{p+1}\, a(0) \sum_{k=0}^{n} \binom{k+p}{p} \\
&\ge \frac{1}{n(p+1)} \sum_{k=0}^{n} \binom{k+p}{p} (k a(n) + (n-k)\, a(0)) + \frac{p}{p+1}\, a(0) \binom{n+p+1}{p+1} \\
&= \frac{(a(n) - a(0))}{n} \sum_{k=1}^{n} \frac{k}{p+1} \binom{k+p}{p} + \frac{a(0)}{p+1} \sum_{k=0}^{n} \binom{k+p}{p} + \frac{p}{p+1}\, a(0) \binom{n+p+1}{p+1} \\
&= \frac{(a(n) - a(0))}{n} \sum_{k=1}^{n} \binom{k+p}{p+1} + a(0) \binom{n+p+1}{p+1} \\
&= \frac{(a(n) - a(0))}{n} \sum_{k=0}^{n-1} \binom{k+p+1}{p+1} + a(0) \binom{n+p+1}{p+1}
\end{aligned}$$

$$= \frac{(a(n)-a(0))}{n}\binom{n+p+1}{p+2}+a(0)\binom{n+p+1}{p+1}$$
$$= \frac{(a(n)-a(0))}{p+2}\cdot\frac{(n+p+1)!}{n\,(p+1)!\,(n-1)!}+a(0)\binom{n+p+1}{p+1}$$
$$= \frac{(a(n)-a(0))}{p+2}\binom{n+p+1}{p+1}+a(0)\binom{n+p+1}{p+1}$$
$$= \frac{1}{p+2}\binom{n+p+1}{p+1}a(n)+\frac{p+1}{p+2}\binom{n+p+1}{p+1}a(0),$$

which is the desired result. The proof is now complete. □

Lemma 4.4. *Let $b : \mathbb{N} \to \mathbb{R}$ be a nondecreasing sequence, satisfying $b(0) \geq 0$ and $b(1) > 0$. Then for each $k \in \mathbb{Z}_+$ the sequence $\Sigma^k b$ is convex and strictly increasing.*

Proof. It suffices to prove that Σb is convex and strictly increasing. Indeed, once this is proved, the desired result follows by induction on k (as $(\Sigma^k b)(0) = b(0) \geq 0$ for every $k \in \mathbb{Z}_+$, and then $\Sigma^k b$ strictly increasing gives $(\Sigma^k b)(1) > 0$).
Sice b is nondecreasing, $b(1) > 0$ yields $b(n) > 0$ for every $n \in \mathbb{Z}_+$, and consequently $(\Sigma b)(n+1) = (\Sigma b)(n) + b(n+1) > (\Sigma b)(n)$ for every $n \in \mathbb{N}$. Hence Σb is strictly increasing. Furthermore, being the sequence $((\Sigma b)(n+1) - (\Sigma b)(n))_{n\in\mathbb{N}} = (b(n+1))_{n\in\mathbb{N}}$ nondecreasing, Σb is convex. We have thus obtained the desired result. □

Lemma 4.5. *Let $c : \mathbb{N} \to \mathbb{R}$ be a concave nondecreasing sequence. Then Δc is convergent, and $\Delta^2 c \in \ell_1$.*

Proof. Since c is concave and nondecreasing, it follows that the sequence $((\Delta c)(n+1))_{n\in\mathbb{N}}$ is nonincreasing and $(\Delta c)(n) \geq 0$ for each $n \in \mathbb{Z}_+$. Then $\lim_{n\to+\infty} (\Delta c)(n) = \lambda$ for some $\lambda \in [0, +\infty)$ (and so Δc converges). Besides, $(\Delta^2 c)(n) \leq 0$ for each $n \in \mathbb{N}_2$. Since

$$\sum_{k=0}^{n} (\Delta^2 c)(n) = (\Sigma\Delta^2 c)(n) = (\Delta c)(n) \xrightarrow[n\to+\infty]{} \lambda$$

(that is, the series $\sum_{n=0}^{+\infty} (\Delta^2 c)(n)$ converges), being $\Delta^2 c$ eventually nonpositive it follows that the series $\sum_{n=0}^{+\infty} |(\Delta^2 c)(n)|$ also converges. Hence $\Delta^2 c \in \ell_1$. □

Remark 4.6. Let $s \in \mathbb{K}^{\mathbb{N}}$ be an eventually nonzero sequence. If we fix $\nu \in \mathbb{Z}_+$ such that $s(n) \neq 0$ for all $n \in \mathbb{N}_\nu$, then for each $n \in \mathbb{N}_\nu$ we have $\frac{(\Delta s)(n)}{s(n)} = 1 - \frac{s(n-1)}{s(n)}$. Hence

$$\lim_{n\to+\infty} \frac{s(n+1)}{s(n)} = 1 \qquad \Longleftrightarrow \qquad \lim_{n\to+\infty} \frac{(\Delta s)(n)}{s(n)} = 0.$$

Theorem 4.7. *Let $s : \mathbb{N} \to \mathbb{R}$ be a real sequence satisfying $s(0) \geq 0$, and let $p \in \mathbb{N}$ be such that the sequence $\Delta^p s$ is concave and is not bounded from above. Then:*

(4.7.1) *$s(n) > 0$ for every $n \in \mathbb{Z}_+$;*

(4.7.2) *s is strictly increasing;*

(4.7.3) $\lim_{n\to+\infty} \frac{s(n)}{n^p} = +\infty$ *(and consequently* $\lim_{n\to+\infty} s(n) = +\infty$*);*

(4.7.4) *the sequence* $\left(\frac{s(n)}{n^{p+1}}\right)_{n\in\mathbb{Z}_+}$ *is bounded;*

(4.7.5) $\lim_{n\to+\infty} \frac{s(n+1)}{s(n)} = 1$*;*

(4.7.6) $\Delta^{p+2} s \in \ell_1$.

If in addition $p \in \mathbb{Z}_+$*, then the sequence* s *is also convex.*

Proof. From Corollary 3.10 it follows that $\Delta^p s$ is strictly increasing and $\lim_{n\to+\infty} (\Delta^p s)(n) = +\infty$. Also, $(\Delta^p s)(0) = s(0) \geq 0$, and consequently $(\Delta^p s)(n) > 0$ for every $n \in \mathbb{Z}_+$. By applying Lemma 4.4 in case $p \in \mathbb{Z}_+$, since $s = \Sigma^p(\Delta^p s)$ we conclude that s is strictly increasing. Hence $s(n) > 0$ for every $n \in \mathbb{Z}_+$. From Lemma 4.4 we also derive that if $p \in \mathbb{Z}_+$, then s is convex. We have thus proved (4.7.1) and (4.7.2), plus the final assertion.
Now we prove (4.7.3). If $p = 0$, the desired result holds as $+\infty = \lim_{n\to+\infty} (\Delta^0 s)(n) = \lim_{n\to+\infty} s(n)$. Thus, let us assume $p \in \mathbb{Z}_+$. Since $\Delta^p s$ is nondecreasing, from Theorem 4.3 it follows that, for each $n \in \mathbb{Z}_+$, we have

$$s(n) = (\Sigma^p(\Delta^p s))(n) \geq \frac{1}{p+1}\binom{n+p}{p}(\Delta^p s)(n) + \frac{p}{p+1}\binom{n+p}{p} s(0) \geq \frac{1}{p+1}\binom{n+p}{p}(\Delta^p s)(n)$$

and consequently

$$\frac{s(n)}{n^p} \geq \frac{\prod_{k=1}^{p}(n+k)}{(p+1)!\,n^p} \cdot (\Delta^p s)(n) \geq \frac{(\Delta^p s)(n)}{(p+1)!}.$$

Since $\lim_{n\to+\infty} (\Delta^p s)(n) = +\infty$, we obtain the desired result.
We prove (4.7.4). Since $s(n) \geq 0$ for every $n \in \mathbb{N}$, it suffices to prove that the sequence $\left(\frac{s(n)}{n^{p+1}}\right)_{n\in\mathbb{Z}_+}$ is bounded from above. By Remark 3.4, the sequence $\left(\frac{(\Delta^p s)(n)}{n}\right)_{n\in\mathbb{Z}_+}$ is bounded from above, which is the desired result if $p = 0$. Now suppose $p \in \mathbb{Z}_+$. From Theorem 4.3 it follows that

$$\frac{s(n)}{n^{p+1}} = \frac{(\Sigma^p(\Delta^p s))(n)}{n^{p+1}} \leq \frac{1}{p!}\left(\frac{\prod_{k=1}^{p}(n+k)}{n^p}\right)\left(\frac{(\Delta^p s)(n)}{n}\right) \qquad \text{for every } n \in \mathbb{Z}_+.$$

Since $\lim_{n\to+\infty}\left(\frac{\prod_{k=1}^{p}(n+k)}{n^p}\right) = 1$ and $\left(\frac{(\Delta^p s)(n)}{n}\right)_{n\in\mathbb{Z}_+}$ is bounded from above, we obtain the desired result.
Now we prove (4.7.5). We prove that $\lim_{n\to+\infty} \frac{(\Delta s)(n)}{s(n)} = 0$, which is equivalent to proving that $\lim_{n\to+\infty} \frac{s(n+1)}{s(n)} = 1$ by Remark 4.6. If $p = 0$, then s is concave by hypothesis. Furthermore, s is

strictly increasing and $\lim_{n\to+\infty} s(n) = +\infty$. Since $\varDelta s$ converges by Lemma 4.5, it follows that $\lim_{n\to+\infty} \frac{(\varDelta s)(n)}{s(n)} = 0$. Now assume $p \in \mathbb{Z}_+$ (and consequently $p-1 \in \mathbb{N}$). From Theorem 4.3 we derive that, for each $n \in \mathbb{N}$, we have

$$s(n) = (\Sigma^p(\varDelta^p s))(n) \geq \frac{1}{p+1}\binom{n+p}{p}(\varDelta^p s)(n) \qquad (\text{as } (\varDelta^p s)(0) = s(0) \geq 0)$$

and

$$(\varDelta s)(n) = (\Sigma^{p-1}(\varDelta^p s))(n) \leq \binom{n+p-1}{p-1}(\varDelta^p s)(n).$$

Since both $s(n)$ and $(\varDelta s)(n)$ are strictly positive for every $n \in \mathbb{Z}_+$ (see (4.7.1) and (4.7.2)), from the two inequalities above we conclude that

$$0 < \frac{(\varDelta s)(n)}{s(n)} \leq \frac{\binom{n+p-1}{p-1}(\varDelta^p s)(n)}{\frac{1}{p+1}\binom{n+p}{p}(\varDelta^p s)(n)} = \frac{\frac{(n+p-1)!}{(p-1)!\,n!}}{\frac{1}{p+1}\cdot\frac{(n+p)!}{p!\,n!}} = \frac{(p+1)!\,(n+p-1)!}{(p-1)!\,(n+p)!} = \frac{p(p+1)}{n+p}$$

for every $n \in \mathbb{Z}_+$. Now $\lim_{n\to+\infty} \frac{p(p+1)}{n+p} = 0$ yields $\lim_{n\to+\infty} \frac{(\varDelta s)(n)}{s(n)} = 0$, which is the desired result. Finally, (4.7.6) is a consequence of Corollary 3.10 and Lemma 4.5. The proof is now complete. □

Remark 4.8. Under the hypotheses of Theorem 4.7, for each $j = 0, \ldots, p$ we have $\varDelta^{p-j}(\varDelta^j s) = \varDelta^p s$. Since $(\varDelta^j s)(0) = s(0) \geq 0$, we are enabled to apply Theorem 4.7 to the sequence $\varDelta^j s$. Hence:

(4.8.1) $(\varDelta^j s)(n) > 0$ for every $n \in \mathbb{Z}_+$;

(4.8.2) $\varDelta^j s$ is strictly increasing;

(4.8.3) $\lim_{n\to+\infty} \frac{(\varDelta^j s)(n)}{n^{p-j}} = +\infty$ (and consequently $\lim_{n\to+\infty} (\varDelta^j s)(n) = +\infty$);

(4.8.4) the sequence $\left(\frac{(\varDelta^j s)(n)}{n^{p+1-j}}\right)_{n\in\mathbb{Z}_+}$ is bounded;

(4.8.5) $\lim_{n\to+\infty} \frac{(\varDelta^j s)(n+1)}{(\varDelta^j s)(n)} = 1$ (or, equivalently, $\lim_{n\to+\infty} \frac{(\varDelta^{j+1} s)(n)}{(\varDelta^j s)(n)} = 0$).

If in addition $j < p$, then the sequence $\varDelta^j s$ is also convex.

We conclude this section with an example of an important sequence satisfying the hypotheses of Theorem 4.7, that is, the sequence of the Cesàro numbers of order α for $\alpha > 0$.

Example 4.9. Fix $\alpha \in (0, +\infty)$, and consider the sequence $A_\alpha : \mathbb{N} \to \mathbb{R}$ of the Cesàro numbers of order α. Then $A_\alpha(0) = 1 > 0$. Also, from (2.1) it follows that $A_\alpha = \Sigma A_{\alpha-1}$, or equivalently $\varDelta A_\alpha = A_{\alpha-1}$. Hence $\varDelta^k A_\alpha = A_{\alpha-k}$ for each $k \in \mathbb{N}$. Now we set

$$p = \max\{k \in \mathbb{N} : k < \alpha\}. \tag{4.1}$$

Then

$$p = \begin{cases} [\alpha] & \text{if } \alpha \notin \mathbb{Z}_+ \\ \alpha - 1 & \text{if } \alpha \in \mathbb{Z}_+ \end{cases} \qquad \text{and} \qquad 0 < \alpha - p \leq 1.$$

Since $\alpha - p > 0$, from (2.2) we derive that the sequence $\Delta^p A_\alpha = A_{\alpha-p}$ is unbounded from above. Besides, since $-1 < \alpha - p - 1 \le 0$, it follows that the sequence $\Delta(\Delta^p A_\alpha) = A_{\alpha-p-1}$ is nonincreasing (see [13], III, (1-17)), and consequently $\Delta^p A_\alpha$ is concave. Hence A_α satisfies the hypotheses of Theorem 4.7 if p is as in (4.1).

Notice that a real sequence need not be infinite of order α for some $\alpha \in (0, +\infty)$ in order to satisfy the hypotheses of Theorem 4.7: for instance, the sequence $(\log(n+1))_{n\in\mathbb{N}}$ of nonnegative real numbers, being concave and unbounded from above, satisfies the hypotheses of Theorem 4.7 for $p = 0$. Nevertheless, it is infinite of order less than α for each $\alpha \in (0, +\infty)$.

5 An index of unboundedness from above for a real sequence

Definition 5.1. For each real sequence $a : \mathbb{N} \to \mathbb{R}$, we set

$$\mathcal{H}(a) = \inf\Big\{m \in \mathbb{N} : \text{ the sequence } \Big(\tfrac{a(n)}{n^m}\Big)_{n\in\mathbb{Z}_+} \text{ is bounded from above}\Big\}.$$

Remark 5.2. Let $a : \mathbb{N} \to \mathbb{R}$ be a real sequence. We observe that $\mathcal{H}(a) \in \mathbb{N} \cup \{+\infty\}$ and the infimum above is attained if and only if $\mathcal{H}(a) < +\infty$. Also, $\mathcal{H}(a) < +\infty$ if and only if the sequence $(\frac{a(n)}{n^\alpha})_{n\in\mathbb{Z}_+}$ is bounded from above for some $\alpha \in [0, +\infty)$, in which case $\mathcal{H}(a)$ is the minimum nonnegative integer m for which the sequence $(\frac{a(n)}{n^m})_{n\in\mathbb{Z}_+}$ is bounded from above. Moreover, if $\mathcal{H}(a) < +\infty$, then clearly $(\frac{a(n)}{n^m})_{n\in\mathbb{Z}_+}$ is bounded from above for every $m \in \mathbb{N}_{\mathcal{H}(a)}$ (indeed, for every $m \in [\mathcal{H}(a), +\infty)$). Notice also that a is bounded from above if and only if $\mathcal{H}(a) = 0$.
Finally, we remark that if $s : \mathbb{N} \to \mathbb{R}$ is a real sequence satisfying the hypotheses of Theorem 4.7, then $\mathcal{H}(s) = p + 1$. Thus, if $s : \mathbb{N} \to \mathbb{R}$ is a real sequence such that $s(0) \ge 0$, there exists at most one $p \in \mathbb{N}$ for which the hypotheses of Theorem 4.7 are satisfied.

Theorem 5.3. *Let $b : \mathbb{N} \to \mathbb{R}$ be a real sequence, which is not bounded from above and satisfies $\mathcal{H}(b) < +\infty$. Then $\mathcal{H}(b) \in \mathbb{Z}_+$. Besides, if we set $p = \mathcal{H}(b) - 1$, then $p \in \mathbb{N}$ and b has a majorant $s : \mathbb{N} \to \mathbb{R}$ such that $s(0) \ge 0$ and $\Delta^p s$ is concave and is not bounded from above (which implies that s satisfies* (4.7.1)–(4.7.6)*, besides being convex if $p \in \mathbb{Z}_+$—equivalently, if $\mathcal{H}(b) \in \mathbb{N}_2$), and moreover* $\limsup\limits_{n\to+\infty} \frac{b(n)}{s(n)} \in [\frac{1}{p+1}, 1] = [\frac{1}{\mathcal{H}(b)}, 1]$.

Proof. Let us first notice that $\mathcal{H}(b) \in \mathbb{Z}_+$—and consequently $p \in \mathbb{N}$—by Remark 5.2, being b unbounded from above. By going to the sequence

$$\tilde{b} : \mathbb{N} \ni n \longmapsto \begin{cases} b(n) & \text{if } n \in \mathbb{Z}_+ \\ 0 & \text{if } n = 0 \end{cases} \in \mathbb{R}$$

if $b(0) < 0$, it is not restrictive to assume that $b(0) \ge 0$. Now let $a : \mathbb{N} \to \mathbb{R}$ be the sequence defined by

$$a(n) = \frac{(p+1)b(n)}{\binom{n+p}{p}} - pb(0) \qquad \text{for every } n \in \mathbb{N}.$$

Then $a(0) = b(0)$. Furthermore, since $\lim_{n\to+\infty} \frac{\binom{n+p}{p}}{n^p} = \lim_{n\to+\infty} \left(\frac{\prod_{k=1}^{p}(n+k)}{p!n^p} \right) = \frac{1}{p!}$, and the sequence $(\frac{b(n)}{n^{p+1}})_{n\in\mathbb{Z}_+}$ is bounded from above, whereas $(\frac{b(n)}{n^p})_{n\in\mathbb{Z}_+}$ is not (as $\mathcal{H}(b) = p+1$), it follows that the sequence $(\frac{a(n)}{n})_{n\in\mathbb{Z}_+}$ is bounded from above, and a is not. Hence a has a concave majorant by Proposition 3.3, and consequently (see Remark 3.5) has a least concave majorant. Let $c : \mathbb{N} \to \mathbb{R}$ denote the least concave majorant of a. Then from (3.6.1) we obtain $c(0) = a(0) = b(0)$. Moreover, from Theorem 3.9 we conclude that c is strictly increasing, $\lim_{n\to+\infty} c(n) = +\infty$, and $\limsup_{n\to+\infty} \frac{a(n)}{c(n)} = 1$. Hence $\lim_{n\to+\infty} \frac{pb(0)}{c(n)} = 0$, and consequently

$$\limsup_{n\to+\infty} \frac{b(n)}{\frac{1}{p+1}\binom{n+p}{p}c(n)} = 1. \tag{5.1}$$

Now let $s \in \mathbb{R}^{\mathbb{N}}$ be defined by $s = \Sigma^p c$. We prove that s is a majorant of b. Since c is concave and nondecreasing, from Theorem 4.3 we derive that, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
s(n) = (\Sigma^p c)(n) &\geq \frac{1}{p+1}\binom{n+p}{p}c(n) + \frac{p}{p+1}\binom{n+p}{p}c(0) \\
&= \frac{1}{p+1}\binom{n+p}{p}c(n) + \frac{p}{p+1}\binom{n+p}{p}b(0) \\
&\geq \frac{1}{p+1}\binom{n+p}{p}a(n) + \frac{p}{p+1}\binom{n+p}{p}b(0) \\
&= \frac{1}{p+1}\binom{n+p}{p}\left(\frac{(p+1)b(n)}{\binom{n+p}{p}} - pb(0)\right) + \frac{p}{p+1}\binom{n+p}{p}b(0) \\
&= b(n) - \frac{p}{p+1}\binom{n+p}{p}b(0) + \frac{p}{p+1}\binom{n+p}{p}b(0) = b(n),
\end{aligned}$$

which is the desired result. We also remark that $s(0) = c(0) = b(0) \geq 0$, and $\Delta^p s = c$ is concave and is not bounded from above.

Now it remains to prove that $\limsup_{n\to+\infty} \frac{b(n)}{s(n)} \in [\frac{1}{p+1}, 1]$. From Theorem 4.7 it follows that $(\Sigma^p c)(n) = s(n) > 0$ for every $n \in \mathbb{Z}_+$. Then, since s is a majorant of b, we clearly have $\limsup_{n\to+\infty} \frac{b(n)}{s(n)} \leq 1$. Finally, we prove that $\limsup_{n\to+\infty} \frac{b(n)}{s(n)} \geq \frac{1}{p+1}$. Since c is strictly increasing, $c(0) \geq 0$ yields $c(n) > 0$ for every $n \in \mathbb{Z}_+$. Then

$$\frac{b(n)}{s(n)} = \left(\frac{b(n)}{\frac{1}{p+1}\binom{n+p}{p}c(n)}\right)\left(\frac{\frac{1}{p+1}\binom{n+p}{p}c(n)}{(\Sigma^p c)(n)}\right) \qquad \text{for every } n \in \mathbb{Z}_+. \tag{5.2}$$

Since c is concave and nondecreasing, from Theorem 4.3 we conclude that

$$\frac{1}{p+1} \leq \frac{\frac{1}{p+1}\binom{n+p}{p}c(n)}{(\Sigma^p c)(n)} \qquad \text{for every } n \in \mathbb{Z}_+. \tag{5.3}$$

Now the desired result is a consequence of (5.1), (5.2) and (5.3). The proof is thus complete. □

We remark that, by virtue of Theorem 5.3, any real sequence b which is unbounded from above and such that $\mathcal{H}(b) < +\infty$, has a majorant s which enjoys the good properties of Theorem 4.7 for $p = \mathcal{H}(b) - 1$, satisfies $\mathcal{H}(s) = p + 1 = \mathcal{H}(b)$ (see Remark 5.2) and is not infinite of higher order than b (equivalently, has a subsequence which is infinite of the same order as the corresponding subsequence of b). Thus, in some sense, s is not "too far" from b.

Proposition 5.4. *Let $a : \mathbb{N} \to \mathbb{R}$ be a real sequence, and $q \in \mathbb{Z}_+$ be such that $\Delta^q a \in \ell_1$. Then $\mathcal{H}(a) \leq q - 1$.*

Proof. If we set $M = \|\Delta^q a\|_{\ell_1}$, for each $n \in \mathbb{N}$ we have

$$(\Delta^{q-1}a)(n) \leq |(\Delta^{q-1}a)(n)| = |(\Sigma\Delta^q a)(n)| = \left|\sum_{k=0}^{n}(\Delta^q a)(k)\right|$$
$$\leq \sum_{k=0}^{n}|(\Delta^q a)(k)| \leq M = MA_0(n).$$

Since the linear operator Σ preserves inequalities, and consequently Σ^{q-1} also does, from (2.1) we conclude that

$$a(n) = (\Sigma^{q-1}\Delta^{q-1}a)(n) \leq M(\Sigma^{q-1}A_0)(n) = MA_{q-1}(n) = M\binom{n+q-1}{n}$$

for each $n \in \mathbb{N}$.
Since $\lim\limits_{n\to+\infty} \frac{1}{n^{q-1}}\binom{n+q-1}{n} = \frac{1}{(q-1)!}$, and consequently the sequence $\left(\frac{1}{n^{q-1}}\binom{n+q-1}{n}\right)_{n\in\mathbb{Z}_+}$ is bounded, it follows that the sequence $\left(\frac{a(n)}{n^{q-1}}\right)_{n\in\mathbb{Z}_+}$ is bounded from above. Hence $\mathcal{H}(a) \leq q - 1$. □

Remark 5.5. Let $a : \mathbb{N} \to \mathbb{R}$ be a real sequence. If $\Delta^q a \in \ell_1$ for some $q \in \mathbb{N}$, since $\Delta(\ell_1) \subseteq \ell_1$ it follows that $\Delta^k a \in \ell_1$ for each $k \in \mathbb{N}_q$.

Remark 5.6. If a real sequence a is unbounded from above, and $q \in \mathbb{Z}_+$ is such that $\Delta^q a \in \ell_1$, then from Proposition 5.4 it follows that $q \geq 2$ (as $\mathcal{H}(a) \geq 1$).

6 A uniform ergodic theorem for Nörlund means

We begin with a result relating several properties of the sequence of the norms of the iterates of a bounded linear operator.

Theorem 6.1. *Let X be a complex nonzero Banach space, and $T \in L(X)$. Then the following conditions are equivalent:*

(6.1.1) $\mathcal{H}((\|T^n\|_{L(X)})_{n\in\mathbb{N}}) < +\infty$;

(6.1.2) *there exists a sequence b of strictly positive real numbers such that $\mathcal{H}(b) < +\infty$, $\lim\limits_{n\to+\infty} b(n) = +\infty$, and $\lim\limits_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{b(n)} = 0$;*

(6.1.3) *there exists a sequence s of strictly positive real numbers such that $\Delta^p s$ is concave and unbounded from above for some $p \in \mathbb{N}$ (which implies that s satisfies* (4.7.2)–(4.7.6)*, besides being convex if $p \in \mathbb{Z}_+$), and* $\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{s(n)} = 0$*;*

(6.1.4) *there exists a nondecreasing sequence s of strictly positive real numbers such that* $\lim_{n\to+\infty} s(n) = +\infty$, $\lim_{n\to+\infty} \frac{s(n+1)}{s(n)} = 1$, *$\Delta^q s \in \ell_1$ for some $q \in \mathbb{N}_2$, and* $\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{s(n)} = 0$.

Furthermore, if $b : \mathbb{N} \to \mathbb{R}$ is a sequence of strictly positive real numbers satisfying (6.1.2)*, then $\mathcal{H}(b) \in \mathbb{Z}_+$, and a sequence $s : \mathbb{N} \to \mathbb{R}$ of strictly positive real numbers can be chosen so that* (6.1.3) *is satisfied for $p = \mathcal{H}(b) - 1$, $s(n) \geq b(n)$ for each $n \in \mathbb{N}$, and moreover* $\limsup_{n\to+\infty} \frac{b(n)}{s(n)} \in [\frac{1}{\mathcal{H}(b)}, 1]$.
Finally, the equivalent conditions (6.1.1)–(6.1.4) *imply the following:*

(6.1.5) $r(T) \leq 1$.

Proof. We begin by proving that (6.1.1) implies (6.1.2). If $\mathcal{H}((\|T^n\|_{L(X)})_{n\in\mathbb{N}}) < +\infty$, it suffices to define $b : \mathbb{N} \to \mathbb{R}$ as follows: $b(n) = (n+1)^{\mathcal{H}((\|T^k\|_{L(X)})_{k\in\mathbb{N}})+1}$ for every $n \in \mathbb{N}$. Indeed, $b(n) > 0$ for every $n \in \mathbb{N}$, $\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{b(n)} = \lim_{n\to+\infty} \left(\frac{\|T^n\|_{L(X)}}{(n+1)^{\mathcal{H}((\|T^k\|_{L(X)})_{k\in\mathbb{N}})}} \right) (\frac{1}{n+1}) = 0$, and $\mathcal{H}(b) = \mathcal{H}((\|T^n\|_{L(X)})_{n\in\mathbb{N}}) + 1 < +\infty$.
Now let us assume that condition (6.1.2) is satisfied by a sequence b of strictly positive real numbers. Since $\mathcal{H}(b) < +\infty$ and b is unbounded from above, from Theorem 5.3 it follows that $\mathcal{H}(b) \in \mathbb{Z}_+$. Furthermore, if we set $p = \mathcal{H}(b) - 1$ (which gives $p \in \mathbb{N}$), then b has a majorant s such that $\Delta^p s$ is concave and is not bounded from above, and besides $\limsup_{n\to+\infty} \frac{b(n)}{s(n)} \in [\frac{1}{\mathcal{H}(b)}, 1]$. Notice also that $s(n) \geq b(n) > 0$ for each $n \in \mathbb{N}$. Finally, since $\frac{\|T^n\|_{L(X)}}{s(n)} \leq \frac{\|T^n\|_{L(X)}}{b(n)}$ for each $n \in \mathbb{N}$, we derive that $\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{s(n)} = 0$. Hence condition (6.1.3) is satisfied by s for $p = \mathcal{H}(b) - 1$.
From Theorem 4.7 it follows that (6.1.3) implies (6.1.4). Now we prove that (6.1.4) implies (6.1.1). Let $s : \mathbb{N} \to \mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers which satisfies (6.1.4). Then $\mathcal{H}(s) \leq q - 1$ by Proposition 5.4. Also, the sequence $\left(\frac{\|T^n\|_{L(X)}}{s(n)} \right)_{n\in\mathbb{N}}$ is bounded, and consequently $\mathcal{H}((\|T^n\|_{L(X)})_{n\in\mathbb{N}}) \leq \mathcal{H}(s) < +\infty$.

We have thus proved equivalence of conditions (6.1.1)–(6.1.4), as well as the subsequent claim. It remains to prove that if the equivalent conditions (6.1.1)–(6.1.4) are satisfied, then $r(T) \leq 1$, which follows from Remark 2.5. □

Remark 6.2. If T is a bounded linear operator on a complex nonzero Banach space X, such that $r(T) < 1$, then $\lim_{n\to+\infty} \|T^n\|_{L(X)} = 0$. Consequently, the sequence $(\|T^n\|_{L(X)})_{n\in\mathbb{N}}$ is bounded, which gives $\mathcal{H}((\|T^n\|_{L(X)})_{n\in\mathbb{N}}) = 0$.
However, condition (6.1.5) is not equivalent to (6.1.1)–(6.1.4). Indeed, the following example shows that a bounded linear operator T on a complex nonzero Banach space X, such that $r(T) = 1$, need not satisfy $\mathcal{H}((\|T^n\|_{L(X)})_{n\in\mathbb{N}}) < +\infty$.

Example 6.3. Let us consider the complex Hilbert space ℓ_2 and the unilateral weighted shift operator $T : \ell_2 \to \ell_2$ defined by

$$Tx = \sum_{n=0}^{+\infty} e^{\frac{1}{\sqrt{(n+1)}}}\, x(n)\, e_{n+1} \qquad \text{for every } x \in \ell_2,$$

where $\{e_n : n \in \mathbb{N}\}$ denotes the canonical orthonormal basis of ℓ_2. Then $T \in L(\ell_2)$. Besides (see [7], Solution 77), for each $k \in \mathbb{Z}_+$ we have

$$\|T^k\|_{L(\ell_2)} = \sup\left\{ \prod_{j=1}^{k} e^{\frac{1}{\sqrt{n+j}}} : n \in \mathbb{N} \right\} = \sup\left\{ e^{\sum_{j=1}^{k} \frac{1}{\sqrt{n+j}}} : n \in \mathbb{N} \right\} = e^{\sum_{j=1}^{k} \frac{1}{\sqrt{j}}}.$$

Since $\frac{1}{\sqrt{j}} \to 0$ as $j \to +\infty$, from the classical Cesàro means theorem we conclude that

$$\sqrt[k]{\|T^k\|_{L(\ell_2)}} \;=\; e^{\frac{1}{k}\sum_{j=1}^{k} \frac{1}{\sqrt{j}}} \xrightarrow[k\to+\infty]{} e^0 = 1,$$

and consequently $r(T) = 1$.
Now fix $\alpha \in (0,+\infty)$. Since for each $j \in \mathbb{Z}_+$ we have $\frac{1}{\sqrt{j}} \geq \frac{1}{\sqrt{x}}$ for every $x \in [j, j+1]$, and consequently $\frac{1}{\sqrt{j}} \geq \int_j^{j+1} \frac{1}{\sqrt{x}}\, dx$, it follows that

$$\sum_{j=1}^{k} \frac{1}{\sqrt{j}} \;\geq \sum_{j=1}^{k} \int_j^{j+1} \frac{1}{\sqrt{x}}\, dx \;= \int_1^{k+1} \frac{1}{\sqrt{x}}\, dx \;=\; 2\sqrt{k+1} - 2 \qquad \text{for every } k \in \mathbb{Z}_+.$$

Then

$$\frac{\|T^k\|_{L(\ell_2)}}{k^\alpha} \;=\; e^{\sum_{j=1}^{k} \frac{1}{\sqrt{j}} - \alpha \log k} \;\geq\; e^{2\sqrt{k+1} - \alpha \log k - 2} \xrightarrow[k\to+\infty]{} +\infty.$$

Hence the sequence $\left(\frac{\|T^n\|_{L(X)}}{n^\alpha}\right)_{n\in\mathbb{Z}_+}$ is bounded from above for no $\alpha \in (0,+\infty)$, that is, $\mathcal{H}((\|T^n\|_{L(X)})_{n\in\mathbb{N}}) = +\infty$.

Lemma 6.4. *Let $\mathcal{A}$ be an algebra with identity $\mathbf{1}_{\mathcal{A}}$ over $\mathbb{K}$, $\tau \in \mathcal{A}$, $a \in \mathbb{K}^{\mathbb{N}}$. Then for each $m \in \mathbb{Z}_+$ and each $n \in \mathbb{N}$ we have*

$$\left(\sum_{k=0}^{n} a(n-k)\, \tau^k \right) (\mathbf{1}_{\mathcal{A}} - \tau)^m$$
$$= (-1)^m \sum_{k=0}^{n+m} (\varDelta^m a)(n+m-k)\, \tau^k + \sum_{j=0}^{m-1} (-1)^j\, (\varDelta^j a)(n+j+1)\, (\mathbf{1}_{\mathcal{A}} - \tau)^{m-1-j}.$$

Proof. We proceed by induction on m. We set

$$S = \Big\{ m \in \mathbb{Z}_+ : \Big(\sum_{k=0}^{n} a(n-k)\, \tau^k \Big) (\mathbf{1}_{\mathcal{A}} - \tau)^m = (-1)^m \sum_{k=0}^{n+m} (\varDelta^m a)(n+m-k)\, \tau^k$$
$$+ \sum_{j=0}^{m-1} (-1)^j\, (\varDelta^j a)(n+j+1)\, (\mathbf{1}_{\mathcal{A}} - \tau)^{m-1-j} \text{ for every } n \in \mathbb{N} \Big\}.$$

Since for each $n \in \mathbb{N}$ we have

$$\left(\sum_{k=0}^{n} a(n-k)\,\tau^k\right)(\mathbf{1}_{\mathcal{A}}-\tau) = \sum_{k=0}^{n} a(n-k)\,\tau^k - \sum_{k=0}^{n} a(n-k)\,\tau^{k+1}$$

$$= \sum_{k=0}^{n} a(n-k)\,\tau^k - \sum_{k=1}^{n+1} a(n+1-k)\,\tau^k$$

$$= -\sum_{k=0}^{n} a(n+1-k)\,\tau^k + \sum_{k=0}^{n} a(n-k)\,\tau^k - a(0)\,\tau^{n+1} + a(n+1)\,\mathbf{1}_{\mathcal{A}}$$

$$= -\left(\sum_{k=0}^{n} (\mathit{\Delta} a)(n+1-k)\,\tau^k + (\mathit{\Delta} a)(0)\,\tau^{n+1}\right) + (\mathit{\Delta}^0 a)(n+1)\,\mathbf{1}_{\mathcal{A}}$$

$$= -\left(\sum_{k=0}^{n+1} (\mathit{\Delta} a)(n+1-k)\,\tau^k\right) + (\mathit{\Delta}^0 a)(n+1)\,(\mathbf{1}_{\mathcal{A}}-\tau)^0,$$

it follows that $1 \in S$.
Now let $m \in S$. Then, since $(\mathit{\Delta}^m a)(0) = a(0) = (\mathit{\Delta}^{m+1} a)(0)$, for each $n \in \mathbb{N}$ we have

$$\left(\sum_{k=0}^{n} a(n-k)\,\tau^k\right)(\mathbf{1}_{\mathcal{A}}-\tau)^{m+1} = \left(\left(\sum_{k=0}^{n} a(n-k)\,\tau^k\right)(\mathbf{1}_{\mathcal{A}}-\tau)^m\right)(\mathbf{1}_{\mathcal{A}}-\tau)$$

$$= \left((-1)^m \sum_{k=0}^{n+m} (\mathit{\Delta}^m a)(n+m-k)\,\tau^k + \sum_{j=0}^{m-1} (-1)^j (\mathit{\Delta}^j a)(n+j+1)\,(\mathbf{1}_{\mathcal{A}}-\tau)^{m-1-j}\right)(\mathbf{1}_{\mathcal{A}}-\tau)$$

$$= (-1)^m \sum_{k=0}^{n+m} (\mathit{\Delta}^m a)(n+m-k)\,\tau^k + (-1)^{m+1} \sum_{k=0}^{n+m} (\mathit{\Delta}^m a)(n+m-k)\,\tau^{k+1}$$

$$+ \sum_{j=0}^{m-1} (-1)^j (\mathit{\Delta}^j a)(n+j+1)\,(\mathbf{1}_{\mathcal{A}}-\tau)^{m-j}$$

$$= (-1)^{m+1} \left(\sum_{k=1}^{n+m+1} (\mathit{\Delta}^m a)(n+m+1-k)\,\tau^k - \sum_{k=0}^{n+m} (\mathit{\Delta}^m a)(n+m-k)\,\tau^k\right)$$

$$+ \sum_{j=0}^{m-1} (-1)^j (\mathit{\Delta}^j a)(n+j+1)\,(\mathbf{1}_{\mathcal{A}}-\tau)^{m-j}$$

$$= (-1)^{m+1} \left(\sum_{k=0}^{n+m} (\mathit{\Delta}^{m+1} a)(n+m+1-k)\,\tau^k + (\mathit{\Delta}^m a)(0)\,\tau^{n+m+1} - (\mathit{\Delta}^m a)(n+m+1)\,\mathbf{1}_{\mathcal{A}}\right)$$

$$+ \sum_{j=0}^{m-1} (-1)^j (\mathit{\Delta}^j a)(n+j+1)\,(\mathbf{1}_{\mathcal{A}}-\tau)^{m-j}$$

$$= (-1)^{m+1} \left(\sum_{k=0}^{n+m} (\mathit{\Delta}^{m+1} a)(n+m+1-k)\,\tau^k + (\mathit{\Delta}^{m+1} a)(0)\,\tau^{n+m+1}\right)$$

$$+ (-1)^m (\mathit{\Delta}^m a)(n+m+1)\,\mathbf{1}_{\mathcal{A}} + \sum_{j=0}^{m-1} (-1)^j (\mathit{\Delta}^j a)(n+j+1)\,(\mathbf{1}_{\mathcal{A}}-\tau)^{m-j}$$

$$= (-1)^{m+1}\Big(\sum_{k=0}^{n+m+1}(\varDelta^{m+1}a)(n+m+1-k)\,\tau^k\Big) + \sum_{j=0}^{m}(-1)^j(\varDelta^j a)(n+j+1)\,(\mathbf{1}_{\mathcal{A}}-\tau)^{m-j},$$

from which we conclude that $m+1 \in S$. The proof is now complete. □

Lemma 6.5. *Let $s \in \mathbb{K}^{\mathbb{N}}$ be an eventually nonzero sequence, such that $\lim_{n\to+\infty} \frac{s(n+1)}{s(n)} = 1$. Then for each $k \in \mathbb{Z}_+$ we have $\lim_{n\to+\infty} \frac{(\varDelta^k s)(n)}{s(n)} = 0$ and $\lim_{n\to+\infty} \frac{s(n+k)}{s(n)} = 1$.*

Proof. We begin by proving that $\lim_{n\to+\infty} \frac{(\varDelta^k s)(n)}{s(n)} = 0$ for each $k \in \mathbb{Z}_+$, proceeding by induction. From Remark 4.6 it follows that $\lim_{n\to+\infty} \frac{(\varDelta s)(n)}{s(n)} = 0$. Now let $k \in \mathbb{Z}_+$ be such that $\lim_{n\to+\infty} \frac{(\varDelta^k s)(n)}{s(n)} = 0$. Since for each $n \in \mathbb{N}$ such that $s(n) \neq 0$—and therefore for sufficiently large n—we have

$$\frac{(\varDelta^{k+1}s)(n)}{s(n)} = \frac{(\varDelta^k s)(n)}{s(n)} - \frac{(\varDelta^k s)(n-1)}{s(n-1)} \cdot \frac{s(n-1)}{s(n)},$$

we conclude that $\lim_{n\to+\infty} \frac{(\varDelta^{k+1}s)(n)}{s(n)} = 0$, which gives the desired result.
Now, in order to finish the proof of the lemma, it suffices to observe that for each $k \in \mathbb{Z}_+$ we have

$$\frac{s(n+k)}{s(n)} = \prod_{j=0}^{k-1} \frac{s(n+j+1)}{s(n+j)} \xrightarrow[n\to+\infty]{} 1.$$

□

Definition 6.6. If X is a normed space and $T \in L(X)$, let $\mathcal{M}_T : \mathbb{N} \to \mathbb{R}$ be the real sequence defined by

$$\mathcal{M}_T(n) = \max\{\|T^k\|_{L(X)} : k = 0,\dots,n\} \qquad \text{for every } n \in \mathbb{N}.$$

Theorem 6.7. *Let X be a complex nonzero Banach space, $T \in L(X)$, and $b : \mathbb{N} \to \mathbb{R}$ be a sequence of strictly positive real numbers, such that $\mathcal{H}(b) < +\infty$, $\lim_{n\to+\infty} b(n) = +\infty$ and $\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{b(n)} = 0$. Then $r(T) \leq 1$. Furthermore, if $s : \mathbb{N} \to \mathbb{R}$ is any nondecreasing sequence of strictly positive real numbers, such that $\lim_{n\to+\infty} s(n) = +\infty$, $\lim_{n\to+\infty} \frac{s(n+1)}{s(n)} = 1$, $\varDelta^q s \in \ell_1$ for some $q \in \mathbb{N}_2$, and the sequence $\left(\frac{b(n)}{s(n)}\right)_{n\in\mathbb{N}}$ is bounded[4], then $\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{s(n)} = 0$, and the following conditions are equivalent:*

(6.7.1) *the sequence* $\left(\frac{\sum_{k=0}^{n}(\varDelta s)(n-k)\,T^k}{s(n)}\right)_{n\in\mathbb{N}}$ *converges in $L(X)$;*

(6.7.2) 1 *is either in $\rho(T)$, or a simple pole of $\mathfrak{R}_T$;*

(6.7.3) $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$;

(6.7.4) *$\mathcal{R}(I_X - T)$ is closed in X and $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$.*

[4]Notice that, by virtue of Theorem 6.1, such a sequence s exists, and can be chosen so that it is not infinite of higher order than b.

Finally, if the equivalent conditions (6.7.1)–(6.7.4) *are satisfied and* $P \in L(X)$ *is such that* $\frac{\sum_{k=0}^{n}(\Delta s)(n-k)T^k}{s(n)} \longrightarrow P$ *in* $L(X)$ *as* $n \to +\infty$*, then* P *is the projection of* X *onto* $\mathcal{N}(I_X - T)$ *along* $\mathcal{R}(I_X - T)$*.*

Proof. We begin by remarking that $r(T) \le 1$ by Theorem 6.1. Now let $s : \mathbb{N} \to \mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers, such that $\lim_{n\to+\infty} s(n) = +\infty$, $\lim_{n\to+\infty} \frac{s(n+1)}{s(n)} = 1$, $\Delta^q s \in \ell_1$ for some $q \in \mathbb{N}_2$, and the sequence $\left(\frac{b(n)}{s(n)}\right)_{n\in\mathbb{N}}$ is bounded. Clearly, $\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{b(n)} = 0$ yields $\lim_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{s(n)} = 0$. We prove that conditions (6.7.1)–(6.7.4) are equivalent.

We first observe that conditions (6.7.2)–(6.7.4) are equivalent by Theorem 2.1. Now suppose that the equivalent conditions (6.7.2)–(6.7.4) are satisfied, and let P denote the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$. Then $P \in L(X)$. We prove that $\frac{\sum_{k=0}^{n}(\Delta s)(n-k)T^k}{s(n)} \longrightarrow P$ in $L(X)$ as $n \to +\infty$.

Since $Tx = x$ for every $x \in \mathcal{N}(I_X - T)$, it follows that $TP = P$, and consequently $T^k P = P$ for every $k \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we have

$$\left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)T^k}{s(n)}\right)P = \frac{\sum_{k=0}^{n}(\Delta s)(n-k)P}{s(n)} = \left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)}{s(n)}\right)P$$
$$= \left(\frac{\sum_{j=0}^{n}(\Delta s)(j)}{s(n)}\right)P = \left(\frac{(\Sigma\Delta s)(n)}{s(n)}\right)P = \left(\frac{s(n)}{s(n)}\right)P = P. \tag{6.1}$$

Now we prove that $\left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)T^k}{s(n)}\right)(I_X - P) \longrightarrow 0_{L(X)}$ in $L(X)$ as $n \to +\infty$. Since T satisfies the equivalent conditions (6.7.2)–(6.7.4), from Theorem 2.1 it follows that $\mathcal{N}((I_X - T)^n) = \mathcal{N}(I_X - T)$ and $\mathcal{R}((I_X - T)^n) = \mathcal{R}(I_X - T)$ for every $n \in \mathbb{Z}_+$. Then the bounded linear operator

$$A : \mathcal{R}(I_X - T) \ni x \longmapsto (I_X - T)^{q-1}x \in \mathcal{R}(I_X - T)$$

is bijective: indeed, since $q \in \mathbb{N}_2$ (and so $q - 1 \in \mathbb{Z}_+$), we have

$$\mathcal{N}(A) = \mathcal{N}((I_X - T)^{q-1}) \cap \mathcal{R}(I_X - T) = \mathcal{N}(I_X - T) \cap \mathcal{R}(I_X - T) = \{0_X\},$$

and

$$\mathcal{R}(A) = \mathcal{R}((I_X - T)^q) = \mathcal{R}(I_X - T).$$

Since $\mathcal{R}(I_X - T)$ is a closed subspace of X, and consequently a Banach space, it follows that the linear map $A^{-1} : \mathcal{R}(I_X - T) \longrightarrow \mathcal{R}(I_X - T)$ is bounded. Since $I_X - P$ is the projection of X onto $\mathcal{R}(I_X - T)$ along $\mathcal{N}(I_X - T)$, and consequently $\mathcal{R}(I_X - P) = \mathcal{R}(I_X - T)$, that is the domain of A^{-1}, we can define the linear operator

$$B : X \ni x \longmapsto A^{-1}(I_X - P)x \in X.$$

We remark that $B \in L(X)$ and

$$(I_X - T)^{q-1} B = I_X - P. \tag{6.2}$$

By virtue of Lemma 6.4, for each $n \in \mathbb{N}$ we have

$$\left(\frac{\sum_{k=0}^{n} (\Delta s)(n-k)\, T^k}{s(n)} \right) (I_X - T)^{q-1} =$$

$$\frac{(-1)^{q-1} \sum_{k=0}^{n+q-1} (\Delta^q s)(n+q-1-k)\, T^k + \sum_{j=0}^{q-2} (-1)^j (\Delta^{j+1} s)(n+j+1)\, (I_X - T)^{q-2-j}}{s(n)},$$

from which we conclude that, for each $n \in \mathbb{N}$,

$$\left\| \left(\frac{\sum_{k=0}^{n} (\Delta s)(n-k)\, T^k}{s(n)} \right) (I_X - T)^{q-1} \right\|$$

$$\le \sum_{j=0}^{q-2} \frac{|(\Delta^{j+1} s)(n+j+1)|}{s(n)} \|I_X - T\|_{L(X)}^{q-2-j} + \frac{\sum_{k=0}^{n+q-1} |(\Delta^q s)(n+q-1-k)| \|T^k\|_{L(X)}}{s(n)}$$

$$\le \sum_{j=0}^{q-2} \frac{|(\Delta^{j+1} s)(n+j+1)|}{s(n)} \|I_X - T\|_{L(X)}^{q-2-j} + \frac{\mathcal{M}_T(n+q-1)}{s(n)} \sum_{k=0}^{n+q-1} |(\Delta^q s)(n+q-1-k)| \tag{6.3}$$

$$= \sum_{j=0}^{q-2} \frac{|(\Delta^{j+1} s)(n+j+1)|}{s(n)} \|I_X - T\|_{L(X)}^{q-2-j} + \frac{\mathcal{M}_T(n+q-1)}{s(n)} \sum_{h=0}^{n+q-1} |(\Delta^q s)(h)|$$

$$\le \sum_{j=0}^{q-2} \frac{|(\Delta^{j+1} s)(n+j+1)|}{s(n)} \|I_X - T\|_{L(X)}^{q-2-j} + \frac{\mathcal{M}_T(n+q-1)}{s(n)} \|\Delta^q s\|_{\ell_1}.$$

For each $j \in \{0, \ldots, q-2\}$, we have

$$\frac{(\Delta^{j+1} s)(n+j+1)}{s(n)} = \frac{(\Delta^{j+1} s)(n+j+1)}{s(n+j+1)} \cdot \frac{s(n+j+1)}{s(n)} \quad \text{for each } n \in \mathbb{N}. \tag{6.4}$$

From Lemma 6.5 it follows that

$$\frac{(\Delta^{j+1} s)(n+j+1)}{s(n+j+1)} \xrightarrow[n\to+\infty]{} 0 \qquad \text{and} \qquad \frac{s(n+j+1)}{s(n)} \xrightarrow[n\to+\infty]{} 1. \tag{6.5}$$

Now from (6.4) and (6.5) we obtain

$$\lim_{n\to+\infty} \frac{(\Delta^{j+1} s)(n+j+1)}{s(n)} = 0 \qquad \text{for all } j = 0, \ldots, q-2,$$

from which we derive that

$$\sum_{j=0}^{q-2} \frac{|(\Delta^{j+1} s)(n+j+1)|}{s(n)} \|I_X - T\|_{L(X)}^{q-2-j} \xrightarrow[n\to+\infty]{} 0. \tag{6.6}$$

By hypothesis, s is nondecreasing and $s(n) > 0$ for each $n \in \mathbb{N}$. Then the sequence $\left(\frac{1}{s(n)}\right)_{n\in\mathbb{N}}$ is nonincreasing. Since $\lim\limits_{n\to+\infty} \frac{\|T^n\|_{L(X)}}{s(n)} = 0 = \lim\limits_{n\to+\infty} \frac{1}{s(n)}$ (as $\lim\limits_{n\to+\infty} s(n) = +\infty$), from [2], 2.3 we conclude that $\lim\limits_{n\to+\infty} \frac{\mathcal{M}_T(n)}{s(n)} = 0$. Since $\lim\limits_{n\to+\infty} \frac{s(n+q-1)}{s(n)} = 1$ by Lemma 6.5, we derive that

$$\frac{\mathcal{M}_T(n+q-1)}{s(n)} = \frac{\mathcal{M}_T(n+q-1)}{s(n+q-1)} \cdot \frac{s(n+q-1)}{s(n)} \xrightarrow[n\to+\infty]{} 0.$$

This, together with (6.6) and (6.3), gives

$$\left(\frac{\sum\limits_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)}\right)(I_X - T)^{q-1} \xrightarrow[n\to+\infty]{} 0_{L(X)} \qquad \text{in } L(X).$$

Consequently, by (6.2),

$$\begin{aligned}\left(\frac{\sum\limits_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)}\right)&(I_X - P)\\ &= \left(\frac{\sum\limits_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)}\right)(I_X - T)^{q-1} B \xrightarrow[n\to+\infty]{} 0_{L(X)} \qquad \text{in } L(X). \end{aligned} \tag{6.7}$$

Now from (6.1) and (6.7) we conclude that

$$\begin{aligned}\frac{\sum\limits_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)} &= \left(\frac{\sum\limits_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)}\right) P + \left(\frac{\sum\limits_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)}\right)(I_X - P)\\ &= P + \left(\frac{\sum\limits_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)}\right)(I_X - P) \xrightarrow[n\to+\infty]{} P \qquad \text{in } L(X).\end{aligned}$$

We have thus proved that if the equivalent conditions (6.7.2)–(6.7.4) are satisfied, then the sequence $\left(\frac{\sum_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)}\right)_{n\in\mathbb{N}}$ converges in $L(X)$ to the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$. It remains to prove that if the sequence $\left(\frac{\sum_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)}\right)_{n\in\mathbb{N}}$ converges in $L(X)$, then the equivalent conditions (6.7.2)–(6.7.4) hold.

Let $P \in L(X)$ be such that $\frac{\sum_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)} \longrightarrow P$ in $L(X)$ as $n \to +\infty$. Since

$$\frac{\sum\limits_{k=0}^{n+1} (\varDelta s)(n+1-k)\, T^k}{s(n+1)} \xrightarrow[n\to+\infty]{} P \quad \text{in } L(X) \qquad \text{and} \qquad \frac{s(n+1)}{s(n)} \xrightarrow[n\to+\infty]{} 1,$$

it follows that

$$\frac{\sum_{k=0}^{n+1}(\Delta s)(n+1-k)\,T^k}{s(n)} \xrightarrow[n\to+\infty]{} P \quad \text{in } L(X),$$

which in turn yields the following limit in $L(X)$.

$$\begin{aligned} 0_{L(X)} &= \lim_{n\to+\infty}\left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)\,T^k}{s(n)} - \frac{\sum_{k=0}^{n+1}(\Delta s)(n+1-k)\,T^k}{s(n)}\right) \\ &= \lim_{n\to+\infty}\left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)\,T^k}{s(n)} - \frac{(\Delta s)(n+1)\,I_X + \sum_{k=1}^{n+1}(\Delta s)(n+1-k)\,T^k}{s(n)}\right) \\ &= \lim_{n\to+\infty}\left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)\,T^k}{s(n)} - \frac{(\Delta s)(n+1)\,I_X + \sum_{k=0}^{n}(\Delta s)(n-k)\,T^{k+1}}{s(n)}\right) \\ &= \lim_{n\to+\infty}\left(\frac{(I_X-T)\sum_{k=0}^{n}(\Delta s)(n-k)\,T^k}{s(n)} - \frac{(\Delta s)(n+1)}{s(n)}\,I_X\right) \\ &= \lim_{n\to+\infty}(I_X-T)\left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)\,T^k}{s(n)}\right) \end{aligned} \tag{6.8}$$

(as $\lim\limits_{n\to+\infty}\frac{(\Delta s)(n+1)}{s(n+1)} = 0$ by Remark 4.6, being $\lim\limits_{n\to+\infty}\frac{s(n+1)}{s(n)} = 1$, and consequently $\frac{(\Delta s)(n+1)}{s(n)} = \frac{(\Delta s)(n+1)}{s(n+1)}\cdot\frac{s(n+1)}{s(n)} \longrightarrow 0$ as $n\to+\infty$).

Now, for each $n\in\mathbb{N}$, let $f_n:\mathbb{C}\longrightarrow\mathbb{C}$ be the polynomial defined by

$$f_n(z) = \frac{\sum_{k=0}^{n}(\Delta s)(n-k)\,z^k}{s(n)} \qquad \text{for each } z\in\mathbb{C}.$$

Since $f_n(1) = \frac{(\Sigma\Delta s)(n)}{s(n)} = \frac{s(n)}{s(n)} = 1$ and $f_n(T) = \frac{\sum_{k=0}^{n}(\Delta s)(n-k)\,T^k}{s(n)}$ for every $n\in\mathbb{N}$, (6.8) enables us to apply Theorem 2.2 (together with Remark 2.3) to the sequence $(f_n)_{n\in\mathbb{N}}$, and to conclude that the equivalent conditions (6.7.2)–(6.7.4) are satisfied. This finishes the proof. □

Remark 6.8. Let X be a complex nonzero Banach space, $T\in L(X)$, and $s:\mathbb{N}\to\mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers, such that $\lim\limits_{n\to+\infty}s(n) = +\infty$, $\lim\limits_{n\to+\infty}\frac{s(n+1)}{s(n)} = 1$, $\Delta^q s\in\ell_1$ for some $q\in\mathbb{N}_2$ (which implies $\mathcal{H}(s)\le q-1$ by Proposition 5.4), and $\lim\limits_{n\to+\infty}\frac{\|T^n\|_{L(X)}}{s(n)} = 0$. Then $r(T)\le 1$ by Theorem 6.1. Furthermore, from Theorem

6.7 and Theorem 2.4 we derive that, given any $E \in L(X)$, the following two conditions are equivalent:

(6.8.1) $$\lim_{n\to+\infty}\left\|\frac{\sum_{k=0}^{n}(\Delta s)(n-k)T^k}{s(n)}-E\right\|_{L(X)}=0;$$

(6.8.2) $$\lim_{\lambda\to 1^+}\|(\lambda-1)\Re_T(\lambda)-E\|_{L(X)}=0.$$

Also, if the equivalent conditions (6.8.1) and (6.8.2) are satisfied, then 1 is either in $\rho(T)$, or a simple pole of $\Re_T$ (so that $\mathcal{R}(I_X-T)$ is closed in X and $X=\mathcal{N}(I_X-T)\oplus\mathcal{R}(I_X-T)$), and E is the projection of X onto $\mathcal{N}(I_X-T)$ along $\mathcal{R}(I_X-T)$.

The following is a consequence of Theorem 6.7 and Theorem 4.7.

Corollary 6.9. *Let X be a complex nonzero Banach space, $T \in L(X)$, and $b:\mathbb{N}\to\mathbb{R}$ be a sequence of strictly positive real numbers, such that $\mathcal{H}(b)<+\infty$, $\lim_{n\to+\infty}b(n)=+\infty$ and $\lim_{n\to+\infty}\frac{\|T^n\|_{L(X)}}{b(n)}=0$ (so that $r(T)\le 1$ by Theorem* 6.1*). If $s:\mathbb{N}\to\mathbb{R}$ is any sequence of strictly positive real numbers, such that $\Delta^p s$ is concave and unbounded from above for some $p\in\mathbb{N}$, and the sequence $\left(\frac{b(n)}{s(n)}\right)_{n\in\mathbb{N}}$ is bounded, then $\lim_{n\to+\infty}\frac{\|T^n\|_{L(X)}}{s(n)}=0$, and each of conditions* (6.7.2)–(6.7.4) *is equivalent to the following:*

(6.9.1) *the sequence* $\left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)T^k}{s(n)}\right)_{n\in\mathbb{N}}$ *converges in $L(X)$.*

Finally, if $P\in L(X)$ is such that $\frac{\sum_{k=0}^{n}(\Delta s)(n-k)T^k}{s(n)}\longrightarrow P$ in $L(X)$ as $n\to+\infty$ (so that conditions (6.7.2)–(6.7.4) *are also satisfied), then P is the projection of X onto $\mathcal{N}(I_X-T)$ along $\mathcal{R}(I_X-T)$.*

Let $\alpha\in(0,+\infty)$. By applying Theorem 6.7 or Corollary 6.9 to the sequences $b=((n+1)^\alpha)_{n\in\mathbb{N}}$ and $s=A_\alpha$ (see Example 4.9, (2.2) and Theorem 4.7; see also (2.1)), we derive that if T is a bounded linear operator on a complex nonzero Banach space X, such that $\lim_{n\to+\infty}\frac{\|T^n\|_{L(X)}}{n^\alpha}=0$, then the sequence $\left(\frac{\sum_{k=0}^{n}A_{\alpha-1}(n-k)T^k}{A_\alpha(n)}\right)_{n\in\mathbb{N}}$ converges in $L(X)$ if and only if 1 is either in $\rho(T)$, or a simple pole of $\Re_T$. From this and from the result by E. Hille mentioned in the Introduction ([8], Theorem 6), together with Remark 6.8, Theorem 2.6 can be deduced. We remark that, however, Theorem 6.7 does not completely extend Theorem 2.6 to a larger class of sequences than the one of the sequences of Cesàro numbers (that is, the class of divergent nondecreasing sequences s of strictly positive real numbers for which $\lim_{n\to+\infty}\frac{s(n+1)}{s(n)}=1$ and $\Delta^q s\in\ell_1$ for some $q\in\mathbb{N}_2$), and neither does Corollary 6.9 relative to the class of all sequences s of strictly positive real numbers for which $\Delta^p s$ is concave and unbounded from above for some $p\in\mathbb{N}$. Indeed, if X is a complex nonzero Banach space, $T\in L(X)$, and s is a nondecreasing sequence of strictly positive real numbers, such that $\lim_{n\to+\infty}s(n)=+\infty$, $\lim_{n\to+\infty}\frac{s(n+1)}{s(n)}=1$, $\Delta^q s\in\ell_1$ for some $q\in\mathbb{N}_2$, and the sequence $\left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)T^k}{s(n)}\right)_{n\in\mathbb{N}}$ converges in $L(X)$, then $\frac{\|T^n\|_{L(X)}}{s(n)}$ need not converge to zero as $n\to+\infty$, even if $\varDelta^p s$ is concave and unbounded from above for some $p \in \mathbb{N}$. The following is an example.

Example 6.10. Let us consider the complex Banach space $\mathbb{C}^2$, endowed with the infinity norm (that is, $\|(z_1, z_2)\|_{\mathbb{C}^2} = \max\{|z_1|, |z_2|\}$ for all $(z_1, z_2) \in \mathbb{C}^2$). If $A \in L(\mathbb{C}^2)$ is the operator represented by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (with respect to the canonical basis of $\mathbb{C}^2$), then $\sigma(A) = \{1\}$. Furthermore, for each $n \in \mathbb{N}$, A^n is represented by the matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.
Now let $T \in L(\mathbb{C}^2)$ be defined by $T = -A$. Then $\sigma(T) = \{-1\}$, which gives $r(T) = 1$ and $1 \in \rho(T)$.
We define a sequence $a : \mathbb{N} \to \mathbb{R}$ of strictly positive real numbers as follows:

$$a(n) = \begin{cases} 1 & \text{if } n = 0 \\ \frac{5}{2} & \text{if } n = 1 \\ \frac{1}{n-1} + \frac{2}{n} + \frac{1}{n+1} & \text{if } n \in \mathbb{N}_2. \end{cases}$$

Since $a(2) = \frac{7}{3} < \frac{5}{2} = a(1)$, it follows that the sequence $(a(n))_{n \in \mathbb{Z}_+}$ is strictly decreasing. Now let $s : \mathbb{N} \to \mathbb{R}$ be the sequence defined by $s = \varSigma a$. Then $\varDelta s = a$. We also remark that $s(n) > 0$ for each $n \in \mathbb{N}$, and $\lim\limits_{n \to +\infty} s(n) = +\infty$. Furthermore, since the sequence $(s(n+1) - s(n))_{n \in \mathbb{N}} = (a(n+1))_{n \in \mathbb{N}}$ is strictly decreasing, it follows that s is concave. Then s satisfies the hypotheses of Theorem 4.7 for $p = 0$ (so that $\lim\limits_{n \to +\infty} \frac{s(n+1)}{s(n)} = 1$ and $\varDelta^2 s \in \ell_1$).
We prove that

$$\frac{\sum\limits_{k=0}^{n} (\varDelta s)(n-k)\, T^k}{s(n)} \xrightarrow[n \to +\infty]{} 0_{L(\mathbb{C}^2)} \qquad \text{in } L(\mathbb{C}^2). \tag{6.9}$$

We remark that, for each $k \in \mathbb{N}$, T^k is represented by the matrix $\begin{pmatrix} (-1)^k & (-1)^k k \\ 0 & (-1)^k \end{pmatrix}$. Hence proving (6.9) is equivalent to proving that

$$\frac{\sum\limits_{k=0}^{n} (-1)^k (\varDelta s)(n-k)}{s(n)} \xrightarrow[n \to +\infty]{} 0 \tag{6.10}$$

and

$$\frac{\sum\limits_{k=0}^{n} (-1)^k k\, (\varDelta s)(n-k)}{s(n)} \xrightarrow[n \to +\infty]{} 0. \tag{6.11}$$

We begin by proving (6.10). We observe that for each $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n} (-1)^k (\varDelta s)(n-k) = \sum_{k=0}^{n} (-1)^{n-k} (\varDelta s)(k) = (-1)^n \sum_{k=0}^{n} (-1)^k a(k). \tag{6.12}$$

Since a is eventually nonincreasing and $\lim\limits_{n \to +\infty} a(n) = 0$, we conclude that the series $\sum\limits_{n=0}^{+\infty} (-1)^n a(n)$ converges, and consequently the sequence $\left(\sum\limits_{k=0}^{n} (-1)^k a(k) \right)_{n \in \mathbb{N}}$ is bounded.

Since $\lim\limits_{n\to+\infty} s(n) = +\infty$, the desired result now follows from (6.12).
Now we prove (6.11). We first remark that, since for each $t \in (-1,1)$ we have $\frac{1}{(1-t)^2} = \sum\limits_{n=1}^{+\infty} n\,t^{n-1}$, we obtain

$$\sum_{n=0}^{+\infty}(-1)^n n\,t^n = -t\sum_{n=1}^{+\infty} n(-t)^{n-1} = -\frac{t}{(t+1)^2} \qquad \text{for each } t \in (-1,1). \tag{6.13}$$

Also, since $\sum\limits_{n=0}^{+\infty} \frac{t^n}{n+1} = \frac{1}{t}\sum\limits_{n=1}^{+\infty} \frac{t^n}{n} = -\frac{\log(1-t)}{t}$ for each $t \in (0,1)$, we conclude that

$$\begin{aligned} -\frac{(t+1)^2\log(1-t)}{t} &= (t^2+2t+1)\sum_{n=0}^{+\infty}\frac{t^n}{n+1} \\ = \sum_{n=0}^{+\infty}\frac{t^{n+2}}{n+1} + \sum_{n=0}^{+\infty}\frac{2\,t^{n+1}}{n+1} + \sum_{n=0}^{+\infty}\frac{t^n}{n+1} &= 1 + (\tfrac{1}{2}+2)\,t + \sum_{n=2}^{+\infty}(\tfrac{1}{n-1}+\tfrac{2}{n}+\tfrac{1}{n+1})\,t^n \\ = 1 + \tfrac{5}{2}\,t + \sum_{n=2}^{+\infty}(\tfrac{1}{n-1}+\tfrac{2}{n}+\tfrac{1}{n+1})\,t^n &= \sum_{n=0}^{+\infty} a(n)\,t^n \qquad \text{for each } t \in (0,1). \end{aligned} \tag{6.14}$$

Since

$$-\frac{t}{(t+1)^2}\left(-\frac{(t+1)^2\log(1-t)}{t}\right) = \log(1-t) = -\sum_{n=1}^{+\infty}\frac{t^n}{n} \qquad \text{for each } t \in (0,1),$$

from (6.13) and (6.14) it follows that

$$\sum_{k=0}^{n}(-1)^k k\,a(n-k) = -\frac{1}{n} \qquad \text{for each } n \in \mathbb{Z}_+. \tag{6.15}$$

Since $\lim\limits_{n\to+\infty} s(n) = +\infty$, from (6.15) we conclude that

$$\frac{\sum\limits_{k=0}^{n}(-1)^k k(\varDelta s)(n-k)}{s(n)} = \frac{\sum\limits_{k=0}^{n}(-1)^k k\,a(n-k)}{s(n)} = -\frac{1}{n\,s(n)} \xrightarrow[n\to+\infty]{} 0,$$

which is the desired result. We have thus proved (6.10) and (6.11), and consequently (6.9).
Hence $r(T) = 1$, $1 \in \rho(T)$, and the sequence $\left(\frac{\sum_{k=0}^{n}(\varDelta s)(n-k)T^k}{s(n)}\right)_{n\in\mathbb{N}}$ converges in $L(\mathbb{C}^2)$ to $0_{L(\mathbb{C}^2)}$, which is, by the way, the projection of $\mathbb{C}^2$ onto $\{0_{\mathbb{C}^2}\} = \mathcal{N}(I_{\mathbb{C}^2} - T)$ along $\mathbb{C}^2 = \mathcal{R}(I_{\mathbb{C}^2} - T)$. Nevertheless, we prove that the sequence $\left(\frac{\|T^n\|_{L(\mathbb{C}^2)}}{s(n)}\right)_{n\in\mathbb{N}}$ does not converge to zero as $n \to +\infty$.
Since for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \|T^n(z_1,z_2)\|_{\mathbb{C}^2} &= \|(-1)^n(z_1+nz_2,z_2)\|_{\mathbb{C}^2} = \max\{|z_1+nz_2|,|z_2|\} \le \max\{|z_1|+n|z_2|,|z_2|\} \\ &\le (n+1)\max\{|z_1|,|z_2|\} = (n+1)\|(z_1,z_2)\|_{\mathbb{C}^2} \qquad \text{for each } (z_1,z_2) \in \mathbb{C}^2 \end{aligned}$$

and

$$\|T^n(1,1)\|_{\mathbb{C}^2} = \|(n+1,1)\|_{\mathbb{C}^2} = n+1,$$

being $\|(1,1)\|_{\mathbb{C}^2} = 1$ it follows that $\|T^n\|_{L(\mathbb{C}^2)} = n+1$ for each $n \in \mathbb{N}$. On the other hand, since s is concave, by Remark 3.4 there exists $M \in (0,+\infty)$ such that $\frac{s(n)}{n+1} \leq M$ for each $n \in \mathbb{N}$. Since $s(n) > 0$ for each $n \in \mathbb{N}$, it follows that

$$\frac{\|T^n\|_{L(\mathbb{C}^2)}}{s(n)} = \frac{n+1}{s(n)} \geq \frac{1}{M} \qquad \text{for each } n \in \mathbb{N}.$$

Hence the sequence $\left(\frac{\|T^n\|_{L(\mathbb{C}^2)}}{s(n)}\right)_{n\in\mathbb{N}}$ does not converge to zero as $n \to +\infty$.
Actually, we can see that $\lim\limits_{n\to+\infty} \frac{\|T^n\|_{L(\mathbb{C}^2)}}{s(n)} = +\infty$. Indeed, since for each $k \in \mathbb{N}_2$ we have $\frac{1}{k} \leq \frac{1}{x}$ for each $x \in [k-1,k]$, and consequently $\frac{1}{k} \leq \int_{k-1}^{k} \frac{1}{x}\, dx$, it follows that for each $n \in \mathbb{N}_3$ we have

$$\begin{aligned} s(n) = 1 + \tfrac{5}{2} + \sum_{k=2}^{n}\left(\tfrac{1}{k-1} + \tfrac{2}{k} + \tfrac{1}{k+1}\right) &\leq \tfrac{7}{2} + 4\sum_{k=2}^{n} \tfrac{1}{k-1} = \tfrac{7}{2} + 4 + 4\sum_{k=3}^{n} \tfrac{1}{k-1} = \tfrac{15}{2} + 4\sum_{k=2}^{n-1} \tfrac{1}{k} \\ &\leq \tfrac{15}{2} + 4\sum_{k=2}^{n-1} \int_{k-1}^{k} \tfrac{1}{x}\, dx = \tfrac{15}{2} + 4\int_{1}^{n-1} \tfrac{1}{x}\, dx = \tfrac{15}{2} + 4\log(n-1). \end{aligned}$$

Hence

$$\frac{\|T^n\|_{L(\mathbb{C}^2)}}{s(n)} = \frac{n+1}{s(n)} \geq \frac{n+1}{\frac{15}{2} + 4\log(n-1)} \qquad \text{for each } n \in \mathbb{N}_3,$$

which gives $\lim\limits_{n\to+\infty} \frac{\|T^n\|_{L(\mathbb{C}^2)}}{s(n)} = +\infty$.

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