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Abstract

We propose here a mathematical study of the small oscillations of a heavy viscous liquid in an arbitrary open rigid container supported by an elastic structure, i.e a spring-mass-damper system. From the equations of motion of the system, we deduce a variational formulation of the problem and after, an operatorial equation in a suitable Hilbert space. Then, we can study the spectrum of the problem. At first, we prove that it is formed by eigenvalues that are located in the right half-plane, so that the equilibrium position is stable. Besides, we show that the operator pencil of the problem is a well-known pencil, whose we prove by a simple method that it has two branches of real eigenvalues having as points of accumulation zero and the infinity and a number at most finite of complex eigenvalues. Finally, we prove the existence and the unicity of the solution of the associated evolution problem by means of Lions method. Afterwards, we consider the case where the damper is removed, that is very different. We prove in this case that the equilibrium position is stable, but the problem is reduced to the study of a Krein-Langer pencil, so that in particular, there exist always non oscillatory eigenmotions.

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1 Introduction

The problem of the small oscillations of a heavy inviscid liquid in an open rigid container by means of the methods of the functional analysis has been studied extensively [10, 7]. The more difficult case of a viscous liquid has been the subject of many papers, analyzed in [8]. The analogous problems, when the container is closed by an elastic cover or bottom, have been studied in the case of an inviscid liquid [7]. But for a viscous liquid, they have been treated by analytical methods and only for particular vessels [1, 2, 6]. Recently, we have studied the case of an arbitrary container closed by an elastic membrane [4].

In this work, we propose a mathematical study of the small oscillations of a heavy viscous liquid in an arbitrary open rigid container supported by an elastic structure, i.e a spring-mass-damper system. Such problems can appear in the study of the dynamics of liquid sloshing absorbers [6; ch.10].

From the equations of motion of the system, we deduce a variational formulation of the problem and after, an operatorial equation in a suitable Hilbert space. Then, we can study the spectrum of the problem. At first, we prove that it is formed by eigenvalues that are located in the right half-plane, so that the equilibrium position is stable in linear approximation.
Besides, we show that the operator pencil of the problem is a well-known pencil, whose we find aigan the properties by a very simple method: there are two branches of real eigenvalues having as points of accumulation zero and the infinity and a number at most finite of complex eigenvalues.

Finally, we prove the existence and the unicity of the solution of the associated evolution problem by means of a well known Lions method [3].
On the other hand, we consider the case where the damper is removed, case that is very different. We prove in this case that the equilibrium position is stable in linear approximation, but the problem is reduced to the study of a Krein-Langer pencil, so that, in particular, there exist always damped non oscillatory eigenmotions.

2 Position of the problem

The problem is treated in linear theory.

We consider an open arbitrary rigid container that partially filled by a heavy viscous liquid and that is supported by an elastic structure represented by a mass M, a spring stiffness k^2 and dashpot with damping coefficient K [6, ch.10].

In the equilibrium postion, we denote by (see Fig.1):

- Ω: domain occupied by the liquid;
- Γ: horizontal free surface (lies in the plane Ox_1x_2);

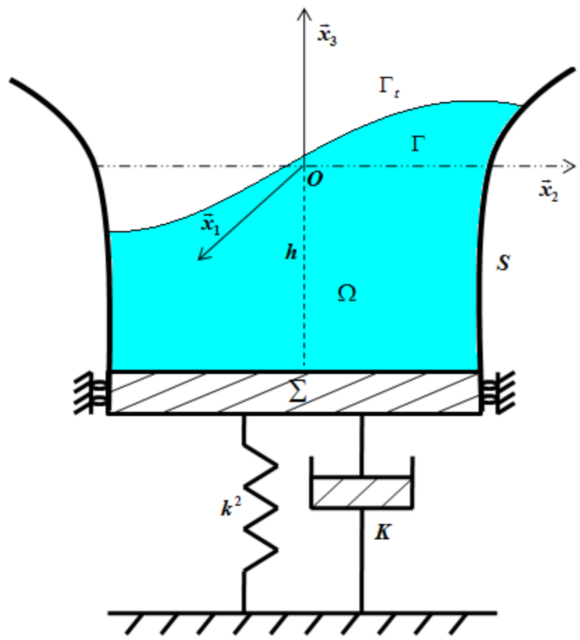

Figure 1. Model of the system.

- Ox_3; vertical upwards fixed axe (unit vector $\vec{x}_3$);
- Ox_1, Ox_2: horizontal fixed axes (unit vectors $\vec{x}_1, \vec{x}_2$);
- ρ: density of the liquid;
- μ: constant coefficient of viscosity of the liquid;
- p_a: constant external atmospheric pressure;
- Σ, M: area and masse of the structure.
- S : wetted wall of the container ;
- g: constant acceleration of the gravity;

and we denote by:

- Γ_t: position of the free surface at the instant t.

a) Let us study the equilibrium of the system.

The pressure of the liquid is

$$P_{\mathrm{e}} = p_{\mathrm{a}} - \rho g x_3 \ .$$

We denote by d the displacement of the spring from the natural state to the equilibrium position and by h the distance of O to the structure.
We have the equation

$$-Mg + k^2 d + \int_\Sigma [p_a - (p_a + \rho g h)]\, d\Sigma = 0$$

or

$$k^2 d = Mg + \rho g h \Sigma\,. \tag{2.1}$$

We remark that, since $h > d$, $k^2 - \rho g \Sigma$ is positive.

b) Let us study the motion of the viscous liquid.

Let $\vec{u}(x_1, x_2, x_3, t)$ and $s(t)\vec{x}_3$ the (small) displacements of a particle of the liquid and of the structure with respect to their equilibrium position, P the pressure, $p = P - P_e$ the dynamic pressure.

We have

$$\rho \ddot{\vec{u}} = -\overrightarrow{\text{grad}}\, p + \mu \Delta \dot{\vec{u}} \qquad \text{(Navier-Stokes equation)}\,, \tag{2.2}$$
$$\text{div}\dot{\vec{u}} = 0 \qquad \text{(incompressibility)}\,.$$

Integrating from the datum of the equilibrium to t, we obtain

$$\text{div}\vec{u} = 0\,. \tag{2.3}$$

The kinematic conditions are

$$\vec{u}|_S = 0\,, \tag{2.4}$$

$$\vec{u}|_\Sigma = s(t)\vec{x}_3\,. \tag{2.5}$$

We denote by

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right),$$

the components of the strain tensor.
Then, the components of the stress tensor are

$$\sigma_{ij} = -P\delta_{ij} + 2\mu\epsilon_{ij}\,.$$

On Γ_t, the equation of which is $x_3 = u_{n|\Gamma} + \cdots$, where $\vec{n}$ is the exterior normal unit vector to Γ_t, the dynamic conditions are

$$\left(-P\delta_{ij} + 2\mu\epsilon_{ij}\right) n_j + p_a n_i = 0 \quad (i, j = 1, 2, 3)\,,$$

or, in linear theory

$$(-p + \rho g u_{n|\Gamma}) n_i + 2\mu\epsilon_{ij} n_j = 0 \quad \text{on} \quad \Gamma\,,$$

i.e

$$\epsilon_{13} = \epsilon_{23} = 0 \quad \text{on} \quad \Gamma\,, \tag{2.6}$$

$$p_{|\Gamma} = \rho g u_{n|\Gamma} + 2\mu\epsilon_{33|\Gamma}\,. \tag{2.7}$$

c) Let us study the motion of the structure.
On the element $d\Sigma$ of the structure, the liquid exerts the force $\vec{T}d\Sigma$, $\vec{T}$ being the stress vector of components

$$T_i = \sigma_{ij} n_j \quad (i, j = 1,2,3)\,.$$

Since $n_1 = n_2 = 0$ on Σ, we have

$$T_1 = 2\mu\epsilon_{13} \quad ; \quad T_2 = 2\mu\epsilon_{23}\,,$$

that counterbalance the lateral stress exerted on the structure.

The equation of motion of the structure is

$$-k^2\,[s(t) - d] - Mg + \int_\Sigma [p_{\mathrm{a}} - P + 2\mu\epsilon_{33}]\,\mathrm{d}\Sigma - K\dot{s}(t) = 0\,,$$

Introducing the dynamic pressure p, using $x_{3|\Sigma} = -h + s(t)$ and the equation (2.1), we obtain

$$M\ddot{s} + K\dot{s} + (k^2 - \rho g\Sigma)s + \int_\Sigma (p - 2\mu\epsilon_{33})\,\mathrm{d}\Sigma = 0\,. \tag{2.8}$$

3 Variational formulation of the problem

We define the space of the kinematically admissible displacements by

$$U^{\mathrm{ad}} = \left\{ \tilde{U} = \begin{pmatrix} \tilde{\vec{u}} \\ \tilde{s} \end{pmatrix};\ \left\{ \tilde{s} \in \mathbb{C};\ \operatorname{div}\tilde{\vec{u}} = 0;\ \tilde{\vec{u}}_{|S} = 0;\ \tilde{\vec{u}}_{|\Sigma} = \tilde{s}\vec{x}_3 \right\} \right\}.$$

This space will be stated more precisely in what follows.
We have

$$\int_\Omega \rho\ddot{\vec{u}}\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega = -\int_\Omega \overrightarrow{\mathrm{grad}}p\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega + \mu\int_\Omega \Delta\dot{\vec{u}}\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega\,.$$

But, using the Green formula, we have

$$-\int_\Omega \overrightarrow{\mathrm{grad}}p\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega = -\int_\Gamma p_{|\Gamma}\,\bar{\tilde{u}}_{n|\Gamma}\,\mathrm{d}\Gamma + \int_\Sigma p_{|\Sigma}\,\bar{\tilde{u}}_{3|\Sigma}\,\mathrm{d}\Sigma\,.$$

On the other hand, the vectorial laplacian formula [7] gives

$$\begin{aligned}\int_\Omega \Delta\dot{\vec{u}}\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega &= -2\int_\Omega \epsilon_{ij}(\dot{\vec{u}})\epsilon_{ij}(\bar{\tilde{\vec{u}}})\,\mathrm{d}\Omega + 2\int_{\partial\Omega} \epsilon_{ij}(\dot{\vec{u}})n_j\bar{\tilde{u}}_i\,\mathrm{d}(\partial\Omega)\\ &= -2\int_\Omega \epsilon_{ij}(\dot{\vec{u}})\epsilon_{ij}(\bar{\tilde{\vec{u}}})\,\mathrm{d}\Omega + 2\int_\Gamma \epsilon_{33}(\dot{\vec{u}})\bar{\tilde{u}}_{n|\Gamma}\,\mathrm{d}\Gamma - 2\int_\Sigma \epsilon_{33}(\dot{\vec{u}})\bar{\tilde{u}}_{3|\Sigma}\,\mathrm{d}\Sigma\,,\end{aligned}$$

therefore, we have

$$\int_\Omega \rho \ddot{\vec{u}}\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega + 2\mu\int_\Omega \epsilon_{ij}(\dot{\vec{u}})\epsilon_{ij}(\bar{\tilde{\vec{u}}})\,\mathrm{d}\Omega - \int_\Sigma (p_{|\Sigma} - 2\mu\epsilon_{33|\Sigma})\bar{\tilde{u}}_{3|\Sigma}\,\mathrm{d}\Sigma - \int_\Gamma (-p_{|\Gamma} + 2\mu\epsilon_{33|\Gamma})\bar{\tilde{u}}_{n|\Gamma}\,\mathrm{d}\Gamma = 0\,.$$

The equation (2.8) gives

$$M\ddot{s}\,\bar{\tilde{s}} + K\dot{s}\,\bar{\tilde{s}} + (k^2 - \rho g\Sigma)s\,\bar{\tilde{s}} + \int_\Sigma (p_{|\Sigma} - 2\mu\epsilon_{33|\Sigma})\bar{\tilde{s}}\,\mathrm{d}\Sigma = 0\,.$$

Adding and taking into account the condition $\tilde{u}_{3|\Sigma} = \bar{\tilde{s}}$ and (2.7), we obtain

$$\int_\Omega \rho\ddot{\vec{u}}\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega + 2\mu\int_\Omega \epsilon_{ij}(\dot{\vec{u}})\epsilon_{ij}(\bar{\tilde{\vec{u}}})\,\mathrm{d}\Omega + \rho g\int_\Gamma u_{n|\Gamma}\,\bar{\tilde{u}}_{n|\Gamma}\,\mathrm{d}\Gamma + \left[M\ddot{s} + K\dot{s} + (k^2 - \rho g\Sigma)s\right]\bar{\tilde{s}} = 0\,, \tag{3.1}$$

for each admissible $\tilde{U}$.

Conversely, let U a function of t belonging to the space of admissible displacements and verifying the equation (3.1).
Then, U verifies the equation

$$\int_\Omega \rho\ddot{\vec{u}}\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega + 2\mu\int_\Omega \epsilon_{ij}(\dot{\vec{u}})\epsilon_{ij}(\bar{\tilde{\vec{u}}})\,\mathrm{d}\Omega + \rho g\int_\Gamma u_{n|\Gamma}\,\bar{\tilde{u}}_{n|\Gamma}\,\mathrm{d}\Gamma + \left[M\ddot{s} + K\dot{s} + (k^2 - \rho g\Sigma)s\right]\bar{\tilde{s}} + \int_\Omega X\,\mathrm{div}\,\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega = 0\,,$$

for each $\tilde{U}$ verifying only

$$\tilde{\vec{u}}_{|S} = 0; \quad \tilde{\vec{u}}_{|\Sigma} = \tilde{s}\vec{x}_3; \quad \tilde{s}\in\mathbb{C}\,,$$

where X is the multiplier associated to the constraint $\mathrm{div}\,\tilde{\vec{u}} = 0$.

Let us take $\tilde{\vec{u}} \in [\mathscr{D}(\Omega)]^3$; then $\tilde{\vec{u}}_{|\partial\Omega} = 0$, $\bar{\tilde{s}} = \tilde{u}_{3|\Sigma} = 0$, we obtain

$$\int_\Omega \left(\rho\ddot{\vec{u}} - \overrightarrow{\mathrm{grad}}X - \mu\Delta\dot{\vec{u}}\right)\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega = 0 \quad \forall\bar{\tilde{\vec{u}}}\in[\mathscr{D}(\Omega)]^3$$

or

$$\rho\ddot{\vec{u}} - \overrightarrow{\mathrm{grad}}X - \mu\Delta\dot{\vec{u}} = 0 \quad \text{in} \quad \left([\mathscr{D}(\Omega)]^3\right)'\,.$$

and

$$\left[M\ddot{s} + K\dot{s} + (k^2 - \rho g\Sigma)s - \int_\Sigma (X_{|\Sigma} + 2\mu\,\epsilon_{33|\Sigma})\,\mathrm{d}\Sigma\right]\bar{\tilde{s}} + \int_\Gamma \left(\rho g u_{n|\Gamma} + X_{|\Gamma} + 2\mu\,\epsilon_{33|\Gamma}\right)\bar{\tilde{u}}_{3|\Gamma}\,\mathrm{d}\Gamma = 0\,.$$

Taking $\tilde{s} = 0$, and $\tilde{u}_3$ arbitrary on Γ we obtain

$$\rho g u_{n|\Gamma} + X_{|\Gamma} + 2\mu\,\epsilon_{33|\Gamma} = 0\,.$$

Taking $\tilde{s}$ arbitrary, we obtain

$$M\ddot{s} + K\dot{s} + (k^2 - \rho g\Sigma)s - \int_\Sigma (X_{|\Sigma} + 2\mu\,\epsilon_{33|\Sigma})\ \mathrm{d}\Sigma = 0\ .$$

Setting $X = -p$, we obtain the equations of motion and the condition on Γ.

4 Operatorial equation of the problem

4.1. We precise the space U^{ad} by introducing the space:

$$\mathcal{V} = \left\{ U = \begin{pmatrix} \vec{u} \\ s \end{pmatrix};\right.$$

$$\left[\begin{array}{l} s \in \mathbb{C}; \\ \vec{u} \in J^1_{0,S}(\Omega) \overset{\text{def}}{=} \left\{ \vec{u} \in \chi^1(\Omega) \overset{\text{def}}{=} \left[H^1(\Omega)\right]^3; \quad \mathrm{div}\vec{u} = 0 \text{ in } \Omega; \quad \vec{u}_{|S} = 0 \right\}; \vec{u}_{|\Sigma} = s\vec{x}_3 \end{array}\right]\Bigg\}\,,$$

equipped with the norm defined by

$$\left\|\vec{u}\right\|^2_{\mathcal{V}} = 2\mu \int_\Omega \epsilon_{ij}(\vec{u})\epsilon_{ij}(\bar{\vec{u}})\,\mathrm{d}\Omega + K\,|s|^2\ .$$

We denote by χ the completion of $\mathcal{V}$ for the norm associated to the scalar product:

$$\left(U, \tilde{U}\right)_\chi = \int_\Omega \rho\vec{u}\cdot\bar{\tilde{\vec{u}}}\,\mathrm{d}\Omega + Ms\bar{\tilde{s}}\,.$$

The variational equation (3.1) takes the form

$$\left(\ddot{U}, \tilde{U}\right)_\chi + \left(\dot{U}, \tilde{U}\right)_{\mathcal{V}} + a\left(U, \tilde{U}\right) = 0, \quad \forall\,\tilde{U} \in \mathcal{V}\,, \tag{4.1}$$

where

$$a\left(U, \widetilde{U}\right) = \rho g \int_\Gamma u_{n|\Gamma}\,\bar{\tilde{u}}_{n|\Gamma}\,\mathrm{d}\Gamma + (k^2 - \rho g\Sigma)s\,\bar{\tilde{s}}\,.$$

By virtue of a trace theorem in $\chi^1(\Omega)$, we have

$$a\,(U,U) \leq c_0\,\|U\|^2_{\mathcal{V}} \quad \forall\,U \in \mathcal{V} \quad (c_0 > 0)\,.$$

Then, we can set

$$a\left(U, \widetilde{U}\right) = \left(\mathcal{B}U, \tilde{U}\right)_{\mathcal{V}}\,,$$

where $\mathcal{B}$ is a non negative, selfadjoint, bounded operator from $\mathcal{V}$ into $\mathcal{V}$.
On the other hand, let $\{U^p\}$ a weakly convergent sequence in $\mathcal{V}$; we have

$$(\mathcal{B}(U^p - U^q), U^p - U^q)_{\mathcal{V}} = \rho g \left\| u^p_{n|\Gamma} - u^q_{n|\Gamma} \right\|^2_{L^2(\Gamma)} + (k^2 - \rho g\Sigma)\left|s^p - s^q\right|^2\ .$$

By virtue of a trace theorem in $\chi^1(\Omega)$, the sequence $\left\{u_{n|\Gamma}^{p}\right\}$ converges strongly in $L^2(\Gamma)$. The sequence $\{s^p\}$ converges weakly, then strongly in $\mathbb{C}$.
Therefore, we have

$$(\mathcal{B}(U^p - U^q), U^p - U^q)_{\mathcal{V}} \to 0 \quad \text{when} \quad p, q \to +\infty ,$$

so that $\mathcal{B}$ is compact from $\mathcal{V}$ into $\mathcal{V}$ [11, p.204].

The variational equation (4.1) can be written

$$\left(\ddot{U}, \tilde{U}\right)_{\chi} + \left(\dot{U}, \tilde{U}\right)_{\mathcal{V}} + \left(\mathcal{B}U, \tilde{U}\right)_{\mathcal{V}} = 0, \quad \forall \tilde{U} \in \mathcal{V}. \tag{4.2}$$

4.2. The embedding $\mathcal{V} \subset \chi$ is classically dense, continuous and compact.
Let A the unbounded operator of χ associated to the pair $(\mathcal{V}, \chi)$ and to the scalar product $\left(U, \tilde{U}\right)_{\mathcal{V}}$.
The variational equation (4.2) is equivalent [9] to the operatorial equation

$$\ddot{U} + A\left(\dot{U} + \mathcal{B}U\right) = 0, \quad \forall U \in \mathcal{V}, \tag{4.3}$$

we obtain an equation with bounded coefficients by setting

$$A^{1/2}U = W \in \chi .$$

Carrying out in (4.3) and applying the operator $A^{-1/2}$, we obtain

$$A^{-1}\ddot{W} + \dot{W} + A^{1/2}\mathcal{B}A^{-1/2}W = 0, \quad W \in \chi . \tag{4.4}$$

A^{-1} is classically a compact, positive definite, self adjoint operator from χ into χ and $A^{1/2}\mathcal{B}A^{-1/2}$ is compact, not negative and selfadjoint from χ into χ.

5 Study of the spectrum of the problem

Seeking the solutions of (4.4) depending on the time by the law $e^{-\lambda t}$, $\lambda \in \mathbb{C}$, we obtain

$$\mathscr{L}(\lambda)W \overset{\text{def}}{=} \left(\lambda^2 A^{-1} - \lambda I_{\chi} + A^{1/2}\mathcal{B}A^{-1/2}\right)W = 0 .$$

5.1. $\lambda = 0$ is an eigenvalue of infinite multiplicity:
Indeed, for $\lambda = 0$, we have $A^{1/2}\mathcal{B}A^{-1/2}W = 0$, so that $W \in \text{Ker}\mathcal{B}A^{-1/2}$.
We precise the space:

$$\left(A^{1/2}\mathcal{B}A^{-1/2}W, W\right)_{\chi} = 0 \text{ is equivalent to } (\mathcal{B}U, U)_{\mathcal{V}} = 0,$$

i.e

$$\left\{U = \begin{pmatrix}\vec{u}\\ s\end{pmatrix}, \quad \vec{u} \in J^1_{0,S}(\Omega), \quad \vec{u}_{|\Sigma} = 0, \quad u_{n|\Gamma} = 0; \quad s = 0, \text{and} \quad W = A^{1/2}U\right\}$$

5.2. Discarding $\lambda = 0$, we have

$$\left[I_{\mathcal{X}} - \left(\lambda A^{-1} + \lambda^{-1} A^{1/2} \mathcal{B} A^{-1/2}\right)\right] W = 0\,.$$

The pencil is a Fredholm pencil in $\mathbb{C} - \{0\} - \{\infty\}$, that is regular since, for λ real negative, the bracket is strongly positive and, consequently, has an inverse.
Then, we have a discrete spectrum formed by eigenvalues of finite multiplicity, having $\lambda = 0, \lambda = \infty$ as possible points of accumulation.

5.3. The pencil $\mathscr{L}(\lambda)$ being selfadjoint, its spectrum is symmetrical with respect to the real axis.

5.4. If λ is an eigenevalue and $W \neq 0$ a corresponding eigenelement, we have

$$\lambda \left(A^{-1} W, W\right)_{\mathcal{X}} + \lambda^{-1} \left(A^{1/2} \mathcal{B} A^{-1/2} W, W\right)_{\mathcal{X}} = \|W\|_{\mathcal{X}}^2\,.$$

Taking the real part, we obtain

$$\Re\lambda \left[\left(A^{-1} W, W\right)_{\mathcal{X}} + |\lambda|^{-2} \left(A^{1/2} \mathcal{B} A^{-1/2} W, W\right)_{\mathcal{X}}\right] = \|W\|_{\mathcal{X}}^2\,,$$

so that we have

$$\Re\lambda \geq 0\,.$$

The mechanical system is stable in linear approximation.

5.5. Taking the imaginary parts, we have

$$\Im\lambda \left[\left(A^{-1} W, W\right)_{\mathcal{X}} - |\lambda|^{-2} \left(A^{1/2} \mathcal{B} A^{-1/2} W, W\right)_{\mathcal{X}}\right] = 0\,,$$

so that, for the non real eigenvalues, if they exist, we have

$$\left(A^{-1} W, W\right)_{\mathcal{X}} = |\lambda|^{-2} \left(A^{1/2} \mathcal{B} A^{-1/2} W, W\right)_{\mathcal{X}}$$

and consequently, using a precedent result:

$$2\Re\lambda \left(A^{-1} W, W\right)_{\mathcal{X}} = \|W\|_{\mathcal{X}}^2$$

or

$$\Re\lambda = \frac{1}{2} \frac{\|W\|_{\mathcal{X}}^2}{(A^{-1} W, W)_{\mathcal{X}}}$$

and finally

$$\Re\lambda \geq \frac{1}{2\left\|A^{-1}\right\|}\,.$$

We have still

$$2\Re\lambda |\lambda|^{-2} \left(A^{1/2} \mathcal{B} A^{-1/2} W, W\right)_{\mathcal{X}} = \|W\|_{\mathcal{X}}^2$$

or

$$|\lambda| \leq \frac{|\lambda|^2}{\Re\lambda} = \frac{2\left(A^{1/2}\mathcal{B}A^{-1/2}W, W\right)_\chi}{||W||_\chi^2}$$

and

$$|\lambda| \leq \frac{2\,(\mathcal{B}U, U)_\mathcal{V}}{||U||_\mathcal{V}^2} \leq 2\,||\mathcal{B}||\,.$$

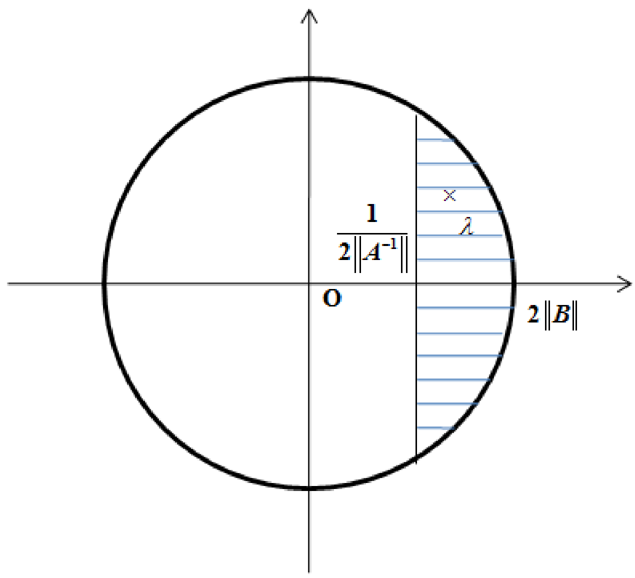

Figure 2. Localisation of the eigenvalues.

Consequently, if $4\left\|A^{-1}\right\|\,||B|| < 1$, i.e if the coefficient of viscosity μ is sufficiently small, the possible non real eigenvalues are located in the hatched domain (see Fig.2), and then, there is an at most finite number of such eigenvalues.
$\mathscr{L}(\lambda)$ is a pencil of type studied in the book [8]. But we are going to obtain easily two important results by using a theorem of the theory of selfadjoint operators pencils [7; p.74].

5.6. There exists a set of positive real eigenvalues having zero as point of accumulation.
$\mathscr{L}(\lambda)$ is an operatorial function holomorphic in the vicinity of $\lambda = 0$.
we have

$$\mathscr{L}(0) = A^{1/2}\mathcal{B}A^{-1/2} \quad \text{compact, non negative},$$

$$\mathscr{L}'(0) = -I_\chi \qquad \text{strongly negative}.$$

Then [7, p 74], for each $\varepsilon > 0$ sufficiently small, there exists in $]-\varepsilon, \varepsilon[$ an infinity of positive real eigenvalues λ_n having zero as point of accumulation. The corresponding eigenelements and an orthogonal basis of Ker$BA^{-1/2}$ form a Riesz basis in a subspace of χ with finite defect.

5.7. There exists a set of positive real eigenvalues having the infinity as point of accumulation.

We set $\lambda' = \lambda^{-1}$ and we consider the selfadjoint operators pencil:

$$\hat{\mathscr{L}}(\lambda') \stackrel{\text{def}}{=} \lambda'^2 \mathscr{L}(\lambda'^{-1}) = A^{-1} - \lambda' I_\chi + \lambda'^2 A^{1/2} \mathcal{B} A^{-1/2} .$$

$\hat{\mathscr{L}}(\lambda')$ is an operatorial function holomorphic in the vicinity of $\lambda' = 0$.
We have

$$\hat{\mathscr{L}}(0) = A^{-1} \quad \text{compact, positive definite},$$

$$\hat{\mathscr{L}}'(0) = -I_\chi \quad \text{strongly negative}.$$

Then, for each $\varepsilon > 0$ sufficiently small, there exists in $]0,\varepsilon[$ an infinity of positive real eigenvalues λ'_n having zero as point of accumulation and the corresponding eigenelements form a Riesz basis in a subspace of χ with finite defect.
For our problem, there is an infinity of eigenvalues $\hat{\lambda}_n = \lambda_n'^{-1}$ having the infinity as point of accumulation.
We can obtain an asymptotic formula for the $\hat{\lambda}_n$.
Indeed, we have

$$-\lambda^{-1} \mathscr{L}(\lambda) = I_\chi - \lambda A^{-1} - \lambda^{-1} A^{1/2} \mathcal{B} A^{-1/2} .$$

This pencil has the form indicated in [7, pp 71–72], so that we have

$$\hat{\lambda}_n = \frac{1}{\lambda_n(A^{-1})} \left[1 + o(1)\right] .$$

6 Existence and unicity of the solution of the associated evolution problem

Setting $U = \mathcal{U} e^t$ in the variational equation (4.2), we obtain:

$$\left(\ddot{\mathcal{U}}, \tilde{\mathcal{U}}\right)_\chi + \left[\left(2\dot{\mathcal{U}}, \tilde{\mathcal{U}}\right)_\chi + \left(\dot{\mathcal{U}}, \tilde{\mathcal{U}}\right)_\mathcal{V}\right] + \left[\left(\mathcal{U}, \tilde{\mathcal{U}}\right)_\mathcal{V} + \left(\mathcal{B}\mathcal{U}, \tilde{\mathcal{U}}\right)_\mathcal{V} + \left(\mathcal{U}, \tilde{\mathcal{U}}\right)_\chi\right] = 0 .$$

We have

$$\begin{cases} (\mathcal{U},\mathcal{U})_\mathcal{V} + (\mathcal{B}\mathcal{U},\mathcal{U})_\mathcal{V} + (\mathcal{U},\mathcal{U})_\chi \geq \|\mathcal{U}\|_\mathcal{V}^2 \\ (\mathcal{U},\mathcal{U})_\mathcal{V} = \|\mathcal{U}\|_\mathcal{V}^2 \\ \left(2\mathcal{U}, \tilde{\mathcal{U}}\right)_\chi \leq 2\|\mathcal{U}\|_\chi \left\|\tilde{\mathcal{U}}\right\|_\chi \leq c\|\mathcal{U}\|_\mathcal{V} \left\|\tilde{\mathcal{U}}\right\|_\chi \quad (c > 0) \end{cases} .$$

The conditions of the problem denoted by P_1 in the book [3, Vol 8, p 666] are satisfied.
Then, let

$$\mathcal{U}^0 = \mathcal{U}(0) \in \mathcal{V}, \quad \mathcal{U}^1 = \dot{\mathcal{U}}(0) \in \chi .$$

The problem for $\mathcal{U}$ hase one and only one solution such that

$$\mathcal{U}(t) \in L^2([0,T]; \mathcal{V}) \quad ; \quad \dot{\mathcal{U}} \in L^2([0,T]; \mathcal{V}) \quad (T > 0) \; .$$

Since

$$U(0) = \mathcal{U}(0) \in \mathcal{V} \quad ; \quad \dot{U}(0) = \dot{\mathcal{U}}(0) + \mathcal{U}(0) \in \chi \; ,$$

the problem for U has one and only one solution in the same spaces.

7 The case where the dashpot is removed $(K = 0)$

We are going to see that the results are very different.
We keep the same notations $\mathcal{V}, \chi, A, \cdots$ although it is matter of different spaces, operators, …
The variational equation (3.1) is now:

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\vec{\tilde{u}}} \, d\Omega + 2\mu \int_\Omega \epsilon_{ij}(\dot{\vec{u}}) \epsilon_{ij}(\bar{\vec{\tilde{u}}}) \, d\Omega + \rho g \int_\Gamma u_{n|\Gamma} \bar{\tilde{u}}_{n|\Gamma} \, d\Gamma + \left[M\ddot{s} + (k^2 - \rho g \Sigma) s \right] \bar{\tilde{s}} = 0 \; . \quad (7.1)$$

7.1. We introduce the space:

$$\mathcal{V} = \left\{ U = \begin{pmatrix} \vec{u} \\ s \end{pmatrix}; \quad s \in \mathbb{C}; \quad \vec{u} \in J^1_{0,S}(\Omega); \quad \vec{u}_{|\Sigma} = s \vec{x}_3 \right\} ,$$

equipped with the norm defined by

$$\|U\|^2_{\mathcal{V}} = 2\mu \int_\Omega \epsilon_{ij}(\vec{u}) \epsilon_{ij}(\bar{\vec{u}}) \, d\Omega + (k^2 - \rho g \Sigma) |s|^2$$

and the space χ completion of $\mathcal{V}$ for the norm associated to the scalar product:

$$\left(U, \tilde{U}\right)_\chi = \int_\Omega \rho \vec{u} \cdot \bar{\vec{\tilde{u}}} \, d\Omega + M s \bar{\tilde{s}} \; .$$

Since

$$(k^2 - \rho g \Sigma) |s|^2 \le \|U\|^2_{\mathcal{V}} \; ,$$

we can write

$$(k^2 - \rho g \Sigma) s \bar{\tilde{s}} = \left(CU, \tilde{U}\right)_{\mathcal{V}} \; ,$$

where C is an operator from $\mathcal{V}$ into $\mathcal{V}$, bounded, selfadjoint and not negative.

C is also compact. Indeed, let $\{U^p\}$ a weakly convergent sequence in $\mathcal{V}$; we have

$$(C(U^p - U^q), U^p - U^q)_{\mathcal{V}} = (k^2 - \rho g \Sigma) \left| s^p - s^q \right|^2 \to 0 \; .$$

By virtue of a trace theorem in $\chi^1(\Omega)$, we have

$$\left\| u_{n|\Gamma} \right\|_{L^2(\Gamma)} \le c_0 \left\| \vec{u} \right\|_{J^1_{0,S}(\Omega)} \quad (c_0 > 0) \; ,$$

so that there exists an operator D from $\mathcal{V}$ into $\mathcal{V}$, bounded, selfadjoint, not negative, such that

$$\rho g \int_\Gamma u_{n|\Gamma}\, \bar{\tilde{u}}_{n|\Gamma} \mathrm{d}\Gamma = \left(DU, \tilde{U}\right)_{\mathcal{V}} .$$

D is also compact because

$$(D(U^p - U^q), U^p - U^q)_{\mathcal{V}} = \rho g \left\| u^p_{n|\Gamma} - u^q_{n|\Gamma} \right\|^2_{L^2(\Gamma)} \to 0 ,$$

by virtue of another trace theorem.

We have

$$2\mu \int_\Omega \epsilon_{ij}(\vec{u})\epsilon_{ij}\left(\bar{\vec{\tilde{u}}}\right) \mathrm{d}\Omega = \left((I_{\mathcal{V}} - C)\, U, \tilde{U}\right)_{\mathcal{V}}$$

and the variational equation (7.1) takes the form

$$\left(\ddot{U}, \tilde{U}\right)_\chi + \left((I_{\mathcal{V}} - C)\, \dot{U}, \tilde{U}\right)_{\mathcal{V}} + \left((C + D)\, \dot{U}, \tilde{U}\right)_{\mathcal{V}} = 0, \quad \forall \tilde{U} \in \mathcal{V} . \tag{7.2}$$

The embedding $\mathcal{V} \subset \chi$ is continuous, dense and compact. We denote by A the unbounded operator of χ associated to the pair $(\mathcal{V},\chi)$ and to the scalar product $(\cdot,\cdot)_{\mathcal{V}}$.

The variational equation (7.2) is equivalent to the operatorial equation [9]

$$\ddot{U} + A\left[(I_{\mathcal{V}} - C)\, \dot{U} + (C + D)\, U\right] = 0; \quad \forall U \in \mathcal{V} . \tag{7.3}$$

Setting

$$A^{1/2} U = W \in \chi$$

and applying $A^{-1/2}$, we obtain the operatorial equation with bounded coefficients

$$A^{-1}\ddot{W} + \left(I_\chi - A^{1/2} C A^{-1/2}\right)\dot{W} + A^{1/2}(C + D)\, A^{-1/2} W = 0; \quad \forall\, W \in \chi . \tag{7.4}$$

A^{-1} and $A^{1/2}(C+D)A^{-1/2}$ are compact, but $I_\chi - A^{1/2}CA^{-1/2}$ is not compact.

7.2. Seeking the solutions of (7.4) depending on the time according to the law $e^{\lambda t}$, $\lambda \in \mathbb{C}$, we have

$$\left(\lambda^2 A^{-1} + \lambda\left(I_\chi - A^{1/2} C A^{-1/2}\right) + A^{1/2}(C + D)\, A^{-1/2}\right) W = 0 .$$

a) $\lambda = 0$ **is an eigenvalue with infinite multiplicity**

Indeed, we have

$$\left(A^{1/2}(C + D)\, A^{-1/2} W, W\right)_\chi = ((C + D)\, U, U)_{\mathcal{V}} = \rho g \int_\Gamma \left|u_{n|\Gamma}\right|^2 \mathrm{d}\Gamma + (k^2 - \rho g \Sigma)\, |s|^2 ,$$

so that the eigenspace associated to $\lambda = 0$ is defined by

$$\left\{\vec{u} \in J^1_{0,S}(\Omega); \quad u_{n|\Gamma} = 0, \quad s = 0\right\} .$$

b) Discarding $\lambda = 0$ and dividing by λ, we obtain the pencil

$$L_1(\lambda) = I_\chi - A^{1/2} C A^{-1/2} + \lambda A^{-1} + \lambda^{-1} A^{1/2}(C + D)\, A^{-1/2} .$$

$L_1(\lambda)$ is a Fredholm pencil in $\mathbb{C}-\{0\}-\{\infty\}$ since the opertaors are compact, except the first. It is regular in this domain, then $L_1(1)=I_\chi+A^{-1}+A^{1/2}DA^{-1/2}$ is strongly positive.

We have a discrete spectrum formed by eigenvalues of finite multiplicity, having $\lambda=0$, $\lambda=\infty$ as possible points of accumulation.

c) The pencil being selfadjoint, its spectrum is symmetrical with respect to the real axis.

d) If λ **is an eigenvalue** and $W\neq 0$ a corresponding eigenelement, we have

$$\lambda\left(A^{-1}W,W\right)_\chi+\lambda^{-1}\left(A^{1/2}(C+D)A^{-1/2}W,W\right)_\chi=\left(\left(I_\chi-A^{1/2}CA^{-1/2}\right)W,W\right)_\chi$$

and then

$$\Re\lambda\left[\left(A^{-1}W,W\right)_\chi+|\lambda|^{-2}\left(A^{1/2}(C+D)A^{-1/2}W,W\right)_\chi\right]=-\left(\left(I_\chi-A^{1/2}CA^{-1/2}\right)W,W\right)_\chi\ ,$$

so that

$$\Re\lambda\leq 0\ .$$

The mechanical system is stable in linear approximation.

e) Let us prove that the pencil $L_1(\lambda)$ can be reduced to a Krein-Langer pencil [5, pp 295-309].

Setting

$$\lambda=\lambda'+1\ ,$$

we obtain

$$L_2(\lambda')W=\left(\lambda'^2A^{-1}+\lambda'G+F\right)W\ ,$$

with

$$F=I_\chi+A^{-1}+A^{1/2}DA^{-1/2}\quad\text{selfadjoint and strongly positive}\ ,$$
$$G=2A^{-1}+I_\chi+A^{1/2}CA^{-1/2}\quad\text{selfadjoint, positive definite, not compact}\ .$$

Setting

$$F^{1/2}W=Z\in\chi$$

and applying $F^{-1/2}$, we obtain the selfadjoint pencil

$$L_3(\lambda')=I_\chi+\lambda'F^{-1/2}GF^{-1/2}+\lambda'^2F^{-1/2}A^{-1}F^{-1/2}\ ,$$

that is a Krein-Langer pencil, since $F^{-1/2}A^{-1}F^{-1/2}$ is positive definite and compact and $F^{-1/2}GF^{-1/2}$ positive definite.

The theory of such pencil can be found in [5, pp 295-309].

For our problem, we mention only two new results:

α) since $G=G^*$ is not compact, the spectrum contains always real points; therefore, there exists always damped not oscillatory eigenmotions,

β) the possible not real eigenvalues can have only the infinity as point of accumulation.

7.3. We are going to prove the existence and the unicity of the solution of the associated evolution problem.

a) For $U \in \mathcal{V}$, we have

$$((I_{\mathcal{V}} - C)\, U, U)_{\mathcal{V}} = 2\mu \int_{\Omega} \epsilon_{ij}(\vec{u})\epsilon_{ij}\left(\bar{\vec{u}}\right) \mathrm{d}\Omega \,.$$

Consequently, we have

$$(CU, U)_{\mathcal{V}} < \|U\|^2_{\mathcal{V}} \quad \text{for } U \neq 0 \,.$$

Since C is selfadjoint and compact, $\|C\|$ is its larger eigenvalue, so that $\|C\| < 1$ and then

$$((I_{\mathcal{V}} - C)\, U, U)_{\mathcal{V}} \geq (1 - \|C\|)\, \|U\|^2_{\mathcal{V}} \quad \forall U \in \mathcal{V} \,.$$

b) In the variational equation (7.2), we set

$$U = \mathcal{U} e^t$$

and we obtain

$$\left(\ddot{\mathcal{U}}, \tilde{\mathcal{U}}\right)_{\chi} + \left[\left(2\dot{\mathcal{U}}, \tilde{\mathcal{U}}\right)_{\chi} + \left((I_{\mathcal{V}} - C)\, \dot{\mathcal{U}}, \tilde{\mathcal{U}}\right)_{\mathcal{V}}\right] + \left[\left(\mathcal{U}, \tilde{\mathcal{U}}\right)_{\chi} + \left((I_{\mathcal{V}} + D)\, \mathcal{U}, \tilde{\mathcal{U}}\right)_{\mathcal{V}}\right] = 0 \,.$$

We have

$$\begin{cases} (\mathcal{U}, \mathcal{U})_{\chi} + ((I_{\mathcal{V}} + D)\, \mathcal{U}, \mathcal{U})_{\mathcal{V}} \geq \|\mathcal{U}\|^2_{\mathcal{V}} \\ ((I_{\mathcal{V}} - C)\, \mathcal{U}, \mathcal{U})_{\mathcal{V}} \geq (1 - \|C\|)\, \|\mathcal{U}\|^2_{\mathcal{V}} \\ \left(2\mathcal{U}, \tilde{\mathcal{U}}\right)_{\chi} \leq C_0\, \|\mathcal{U}\|_{\mathcal{V}} \left\|\tilde{\mathcal{U}}\right\|_{\chi} \quad (C_0 > 0) \end{cases} .$$

Applying the same theorem as previously, we obtain the following analogous results:

Let

$$\mathcal{U}^0 \in \mathcal{V}, \quad \mathcal{U}^1 \in \chi \,.$$

The problem has one and only one solution verifying:

$$\mathcal{U} \in L^2([0, T]; \mathcal{V}) \quad ; \quad \dot{\mathcal{U}} \in L^2([0, T]; \mathcal{V}) \quad (T > 0) \,,$$

$$\mathcal{U}(0) = \mathcal{U}^0; \quad \dot{\mathcal{U}}(0) = \mathcal{U}^1 \,.$$

The problem for $U(\cdot)$ has one and only one solution in the same spaces.

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