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Existence and Global Stability Results for Volterra Type Fractional Hadamard Partial Integral Equations

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Abstract

This paper deals with the global existence and stability of solutions of a new class of partial integral equations of Hadamard fractional order.

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1 Introduction

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related

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fields of engineering and mathematical physics. We can find numerous applications of differential and integral equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. [17, 18, 21]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [5, 6], Kilbas *et al.* [20], Miller and Ross [22], Samko *et al.* [24], the papers of Abbas *et al.* [1, 2, 3], Banaś *et al.* [7, 8, 9, 10, 11, 12], Darwish *et al.* [16], and the references therein.

In [13], Butzer *et al.* investigate properties of the Hadamard fractional integral and derivative. In [14], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators. In [23], Pooseh *et al.* obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [24] and the references therein.

Recently, Abbas *et al.* [4] studied some existence and stability results for the nonlinear quadratic Volterra integral equation of Riemann-Liouville fractional order of the form

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \tfrac{1}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1} \\ &\quad \times g(t,x,s,u(s,x),u(\gamma(s),x))ds;\ \ (t,x)\in\mathbb{R}_+\times[0,b], \end{aligned} \tag{1.1}$$

where $b>0,\ \mathbb{R}_+=[0,\infty),\ r\in(0,\infty),\ \alpha,\ \beta,\ \gamma:\mathbb{R}_+\to\mathbb{R}_+,\ f:\mathbb{R}_+\times[0,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}_+\times[0,b]\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\zeta)=\int_0^\infty t^{\zeta-1}e^{-t}dt;\ \zeta>0.$$

This paper deals with the global existence and stability of solutions to the following nonlinear quadratic Volterra partial integral equation of Hadamard fractional order,

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left(\log\tfrac{\beta(t)}{s}\right)^{r_1-1}\left(\log\tfrac{x}{\xi}\right)^{r_2-1} \\ &\quad \times g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))\frac{d\xi ds}{s\xi};\ \ (t,x)\in J:=[1,\infty)\times[1,b], \end{aligned} \tag{1.2}$$

where $b>1,\ r_1,r_2\in(0,\infty),\ \alpha,\ \beta,\ \gamma:[1,\infty)\to[1,\infty),\ f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:J\times J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

Our existence results are based upon Schauder's fixed point theorem. Also, we obtain some results about the local asymptotic stability of solutions of the equation in question. Finally, we present an example illustrating the applicability of the imposed conditions.

This paper initiates the global existence and stability of such new class of fractional integral equations.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([1,a]\times[1,b])$; for $a,b>1$, we denote the space of Lebesgueintegrable functions $u : [1,a]\times[1,b] \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_1^a \int_1^b |u(t,x)|dxdt.$$

By $BC := BC(J)$ we denote the Banach space of all bounded and continuous functions from J into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC} = \sup_{(t,x)\in J} |u(t,x)|.$$

For $u_0 \in BC$ and $\eta \in (0,\infty)$, we denote by $B(u_0,\eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 2.1. [20] The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,a],\mathbb{R})$, is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\log\frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Example 2.2. The Hadamard fractional integral of order $q > 0$ for the function $w : [1,e] \to \mathbb{R}$, defined by $w(x) = (\log x)^{\beta-1}$ with $\beta > 0$, is

$$({}^H I_1^q w)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+q)} (\log x)^{\beta+q-1}.$$

Definition 2.3. Let $r_1,\ r_2 \geq 0,\ \sigma = (1,1)$ and $r = (r_1,r_2)$. For $w \in L^1(J,\mathbb{R})$, define the Hadamard partial fractional integral of order r by the expression

$$({}^H I_\sigma^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{w(s,t)}{st} dtds.$$

Let $\emptyset \neq \Omega \subset BC$, and let $G : \Omega \to \Omega$, and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{2.1}$$

Now we review the concept of attractivity of solutions for equation (1.2).

Definition 2.4. [5] Solutions of equation (2.1) are locally attractive if there exists a ball $B(u_0,\eta)$ in the space BC such that, for arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ of equations (2.1) belonging to $B(u_0,\eta)\bigcap\Omega$, we have that, for each $x \in [1,b]$,

$$\lim_{t\to\infty}\Big(v(t,x) - w(t,x)\Big) = 0. \tag{2.2}$$

When the limit (2.2) is uniform with respect to $B(u_0,\eta)$, solutions of equation (2.1) are said to be uniformly locally attractive (or equivalently that solutions of (2.1) are locally asymptotically stable).

Definition 2.5. [5] The solution $v = v(t,x)$ of equation (2.1) is said to be globally attractive if (2.2) holds for each solution $w = w(t,x)$ of (2.1). If condition (2.2) is satisfied uniformly with respect to the set Ω, solutions of equation (2.1) are said to be globally asymptotically stable (or uniformly globally attractive).

Lemma 2.6. *[15] Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*
(a) D is uniformly bounded in BC,
(b) The functions belonging to D are almost equicontinuous on $[1,\infty)\times[1,b]$,
i.e. equicontinuous on every compact of J,
(c) The functions from D are equiconvergent, that is, given $\epsilon > 0$, $x \in [1,b]$ there corresponds $T(\epsilon,x) > 0$ such that $|u(t,x) - \lim_{t\to\infty} u(t,x)| < \epsilon$ for any $t \geq T(\epsilon,x)$ and $u \in D$.

3 Existence and Global Stability Results

In this section, we are concerned with the existence and the asymptotic stability of solutions for the Hadamard partial integral equation (1.2).

The following hypotheses will be used in the sequel.

(H_1) The function $\alpha : [1,\infty) \to [1,\infty)$ satisfies $\lim_{t\to\infty} \alpha(t) = \infty$,

(H_2) There exist constants $M, L > 0$, and a nondecreasing function $\psi_1 : [0,\infty) \to (0,\infty)$ such that $M < \frac{L}{2}$,

$$|f(t,x,u_1,v_1) - f(t,x,u_2,v_2)| \leq \frac{M(|u_1-u_2| + |v_1-v_2|)}{(1+\alpha(t))(L+|u_1-u_2|+|v_1-v_2|)},$$

and

$$|f(t_1,x_1,u,v) - f(t_2,x_2,u,v)| \leq (|t_1-t_2| + |x_1-x_2|)\psi_1(|u|+|v|),$$

for each $(t,x),(t_1,x_1),(t_2,x_2) \in J$ and $u,v,u_1,v_1,u_2,v_2 \in \mathbb{R}$,

(H_3) The function $t \to f(t,x,0,0)$ is bounded on J with

$$f^* = \sup_{(t,x)\in[1,\infty)\times[1,b]} f(t,x,0,0)$$

and

$$\lim_{t\to\infty} |f(t,x,0,0)| = 0; \ x \in [1,b],$$

(H_4) There exist continuous functions $p,q,\varphi : J \to \mathbb{R}_+$, and a nondecreasing function $\psi_2 : [0,\infty) \to (0,\infty)$ such that

$$|g(t_1,x_1,s,\xi,u,v) - g(t_2,x_2,s,\xi,u,v)| \leq \varphi(s,\xi)(|t_1-t_2| + |x_1-x_2|)\psi_2(|u|+|v|),$$

and

$$|g(t,x,s,\xi,u,v)| \leq \frac{p(t,x)q(s,\xi)}{1+\alpha(t)+|u|+|v|},$$

for each $(t,x),(s,\xi),(t_1,x_1),(t_2,x_2) \in J$ and $u,v \in \mathbb{R}$. Moreover, assume that

$$\lim_{t\to\infty} p(t,x) \int_1^{\beta(t)} \int_1^x \left|\log\frac{\beta(t)}{s}\right|^{r_1-1} \left|\log\frac{x}{\xi}\right|^{r_2-1} q(s,\xi)d\xi ds = 0.$$

Theorem 3.1. *Assume that hypotheses $(H_1)-(H_4)$ hold. Then the integral equation (1.2) has at least one solution in the space BC. Moreover, solutions of equation (1.2) are globally asymptotically stable.*

Proof: Set $d^* := \sup_{(t,x)\in J} d(t,x)$ where

$$d(t,x)=\frac{p(t,x)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1}q(s,\xi)d\xi ds.$$

From hypothesis (H_4), we infer that d^* is finite. Let us define the operator N such that, for any $u\in BC$,

$$\begin{aligned}(Nu)(t,x) &= f(t,x,u(t,x),u(\alpha(t),x))+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left(\log\tfrac{\beta(t)}{s}\right)^{r_1-1}\left(\log\tfrac{x}{\xi}\right)^{r_2-1}\\ &\quad\times g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))\frac{d\xi ds}{s\xi},\quad (t,x)\in J.\end{aligned}\tag{3.1}$$

By considering the assumptions of this theorem, we infer that $N(u)$ is continuous on J. Now we prove that $N(u)\in BC$ for any $u\in BC$. For arbitrarily fixed $(t,x)\in J$ we have

$$\begin{aligned}|(Nu)(t,x)| &\le |f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,0,0)|+|f(t,x,0,0)|\\ &\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1}\\ &\quad\times|g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))|\frac{d\xi ds}{s\xi}|\\ &\le \frac{M(|u(t,x)|+|u(\alpha(t),x)|)}{(1+\alpha(t))(L+|u(t,x)|+|u(\alpha(t),x)|)}+|f(t,x,0,0)|\\ &\quad+\frac{p(t,x)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1}\\ &\quad\times\frac{q(s,\xi)}{1+\alpha(t)+|u(s,\xi))|+|u(\gamma(s),\xi))|}\frac{d\xi ds}{s\xi}\\ &\le \frac{M(|u(t,x)|+|u(\alpha(t),x)|)}{|u(t,x)|+|u(\alpha(t),x)|}+f^*+d^*.\end{aligned}$$

Thus

$$\|N(u)\|_{BC}\le M+f^*+d^*.\tag{3.2}$$

Hence $N(u)\in BC$. The equation (3.2) yields that N transforms the ball $B_\eta := B(0,\eta)$ into itself where $\eta = M+f^*+d^*$. We shall show that $N: B_\eta\to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [19]. The proof will be given in several steps and cases.

Step 1: *N is continuous.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then, for each $(t,x) \in J$, we have

$$\begin{aligned}
|(Nu_n)(t,x)-(Nu)(t,x)| &\leq |f(t,x,u_n(t,x),u_n(\alpha(t),x))-f(t,x,u(t,x),u(\alpha(t),x))| \\
&\quad +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x \left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \sup_{(s,\xi)\in J}|g(t,x,s,\xi,u_n(s,\xi),u_n(\gamma(s),\xi)) \\
&\quad -g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))|\frac{d\xi ds}{s\xi} \\
&\leq \frac{2M}{L}\|u_n-u\|_{BC} \\
&\quad +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x \left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times\|g(t,x,\cdot,\cdot,u_n(\cdot,\cdot),u_n(\gamma(\cdot),\cdot)) \\
&\quad -g(t,x,\cdot,\cdot,u(\cdot,\cdot),u(\gamma(\cdot),\cdot))\|_{BC}d\xi ds.
\end{aligned} \tag{3.3}$$

Case 1. If $(t,x) \in [1,T]\times[1,b],\ T>1$, then, since $u_n \to u$ as $n\to\infty$ and g,γ are continuous, then (3.3) gives

$$\|N(u_n)-N(u)\|_{BC} \to 0 \quad \text{as } n\to\infty.$$

Case 2. If $(t,x)\in(T,\infty)\times[1,b],\ T>1$, then from (H_4) and (3.3), for each $(t,x)\in J$, we have

$$\begin{aligned}
|(Nu_n)(t,x)-(Nu)(t,x)| &\leq \frac{2M}{L}\|u_n-u\|_{BC} \\
&\quad +\frac{2p(t,x)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x \left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1}\frac{q(s,\xi)}{s\xi}d\xi ds \\
&\leq \frac{2M}{L}\|u_n-u\|_{BC}+d(t,x).
\end{aligned}$$

Thus, we get

$$|(Nu_n)(t,x)-(Nu)(t,x)| \leq \frac{2M}{L}\|u_n-u\|_{BC}+d(t,x). \tag{3.4}$$

Since $u_n\to u$ as $n\to\infty$ and $t\to\infty$, then (3.4) gives

$$\|N(u_n)-N(u)\|_{BC} \to 0 \quad \text{as } n\to\infty.$$

Step 2: *$N(B_\eta)$ is uniformly bounded.*
This is clear since $N(B_\eta)\subset B_\eta$ and B_η is bounded.

Step 3: *$N(B_\eta)$ is equicontinuous on every compact subset $[1,a]\times[1,b]$ of J, $a>0$.*
Let $(t_1,x_1),(t_2,x_2)\in[1,a]\times[1,b]$, $t_1<t_2,\ x_1<x_2$ and let $u\in B_\eta$. Also without loss of gen-

erality suppose that $\beta(t_1) \le \beta(t_2)$. Then, we have

$$
\begin{aligned}
&|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)| \\
&\le |f(t_2,x_2,u(t_2,x_2),u(\alpha(t_2),x_2))-f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))| \\
&+|f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))-f(t_1,x_1,u(t_1,x_1),u(\alpha(t_1),x_1))| \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_1^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\
&\times|g(t_2,x_2,s,\xi,u(s,\xi),u(\gamma(s),\xi))-g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d\xi ds \\
&+\left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_1^{x_2}\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right. \\
&\times g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))d\xi ds \\
&-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1} \\
&\left.\times g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))d\xi ds\right| \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left|\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right. \\
&\left.-\left(\log\frac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\right||g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d\xi ds.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)| \\
&\le \frac{M}{L}(|u(t_2,x_2)-u(t_1,x_1)|+|u(\alpha(t_2),x_2)-u(\alpha(t_1),x_1)|) \\
&+(|t_2-t_1|+|x_2-x_1|)\psi_1(2\|u\|_{BC}) \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_1^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\
&\times\varphi(s,\xi)(|t_2-t_1|+|x_2-x_1|)\psi_2(2\|u\|_{BC})d\xi ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_1^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\
&\times|g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d\xi ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\
&\times|g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d\xi ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\
&\times|g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d\xi ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left|\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right. \\
&\left.-\left(\log\frac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\right||g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d\xi ds.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)|\\
&\le \tfrac{M}{L}(|u(t_2,x_2)-u(t_1,x_1)|+|u(\alpha(t_2),x_2)-u(\alpha(t_1),x_1)|)\\
&+(|t_2-t_1|+|x_2-x_1|)\psi_1(2\eta)\\
&+\tfrac{(|t_2-t_1|+|x_2-x_1|)\psi_2(2\eta)}{\Gamma(r_1)\Gamma(r_2)}\\
&\times\int_1^{\beta(t_2)}\int_1^{x_2}\left|\log\tfrac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\tfrac{x_2}{\xi}\right|^{r_2-1}\varphi(s,\xi)d\xi ds\\
&+\tfrac{p(t_1,x_1)}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_1^{x_2}\left|\log\tfrac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\tfrac{x_2}{\xi}\right|^{r_2-1}q(s,\xi)|d\xi ds\\
&+\tfrac{p(t_1,x_1)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\tfrac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\tfrac{x_2}{\xi}\right|^{r_2-1}q(s,\xi)|d\xi ds\\
&+\tfrac{p(t_1,x_1)}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\tfrac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\tfrac{x_2}{\xi}\right|^{r_2-1}q(s,\xi)|d\xi ds\\
&+\tfrac{p(t_1,x_1)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left|\left(\log\tfrac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\tfrac{x_2}{\xi}\right)^{r_2-1}\right.\\
&\left.-\left(\log\tfrac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\tfrac{x_1}{\xi}\right)^{r_2-1}\right|q(s,\xi)|d\xi ds.
\end{aligned}
$$

From continuity of α,β,f,g and as $t_1\to t_2$ and $x_1\to x_2$, the right-hand side of the above inequality tends to zero.

Step 4: *$N(B_\eta)$ is equiconvergent.*
Let $(t,x)\in J$ and $u\in B_\eta$, then we have

$$
\begin{aligned}
|u(t,x)|\le{}&\Big|f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,0,0)+f(t,x,0,0)\Big|\\
&+\left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\right.\\
&\left.\times g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))\frac{d\xi ds}{s\xi}\right|\\
\le{}&\frac{M(|u(t,x)|+|u(\alpha(t),x)|)}{(1+\alpha(t))(L+|u(t,x)|+|u(\alpha(t),x)|)}+|f(t,x,0,0)|\\
&+\frac{p(t,x)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\\
&\times\frac{q(s,\xi)}{1+\alpha(t)+|u(s,\xi)|+|u(\gamma(s),\xi)|}d\xi ds\\
\le{}&\frac{M}{1+\alpha(t)}+|f(t,x,0,0)|\\
&+\frac{p(t,x)}{\Gamma(r_1)\Gamma(r_2)(1+\alpha(t))}\int_1^{\beta(t)}\int_1^x\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}q(s,\xi)d\xi ds\\
\le{}&\frac{M}{1+\alpha(t)}+|f(t,x,0,0)|+\frac{d^*}{1+\alpha(t)}.
\end{aligned}
$$

Thus, for each $x\in[1,b]$, we get

$$|u(t,x)|\to 0,\ as\ t\to+\infty.$$

Hence,

$$|u(t,x)-u(+\infty,x)|\to 0,\ as\ t\to+\infty.$$

As a consequence of Steps 1 to 4 together with the Lemma 2.6, we can conclude that $N : B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's fixed point theorem [19], we deduce that N has a fixed point u which is a solution of the Hadamard integral equation (1.2).

Step 5: *The uniform global attractivity.*
Let us assume that u_0 is a solution of integral equation (1.2) with the conditions of this theorem. Consider the ball $B(u_0,\eta)$ with $\eta^* = \frac{LM^*}{L-2M}$, where

$$\begin{aligned} M^* \quad := \quad & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sup_{(t,x)\in J} \Big\{ \int_1^{\beta(t)} \int_1^x \left(\log \frac{\beta(t)}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \\ & \times |g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) \\ & -g(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))| d\xi ds;\ u\in BC \Big\}. \end{aligned}$$

Taking $u \in B(u_0,\eta^*)$. Then, we have

$$\begin{aligned} |(Nu)(t,x)-u_0(t,x)| \quad &= \quad |(Nu)(t,x)-(Nu_0)(t,x)| \\ &\leq \quad |f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,u_0(t,x),u_0(\alpha(t),x))| \\ &+ \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^x \left(\log \frac{\beta(t)}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \\ & \quad \times |g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) \\ & \quad -g(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))| \frac{d\xi ds}{s\xi} \\ &\leq \quad \frac{2M}{L||}||u-u_0||_{BC} + M^* \\ &\leq \quad \frac{2M}{L}\eta^* + M^* = \eta^*. \end{aligned}$$

Thus we observe that N is a continuous function such that $N(B(u_0,\eta^*)) \subset B(u_0,\eta^*)$. Moreover, if u is a solution of equation (1.2), then

$$\begin{aligned} |u(t,x)-u_0(t,x)| \quad &= \quad |(Nu)(t,x)-(Nu_0)(t,x)| \\ &\leq \quad |f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,u_0(t,x),u_0(\alpha(t),x))| \\ & \quad +\tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^x \left(\log \tfrac{\beta(t)}{s}\right)^{r_1-1} \left(\log \tfrac{x}{\xi}\right)^{r_2-1} \\ & \quad \times |g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) \\ & \quad -g(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))| d\xi ds. \end{aligned}$$

Thus

$$\begin{aligned} |u(t,x)-u_0(t,x)| \quad \leq \quad & \tfrac{M}{L}(|u(t,x)-u_0(t,x)|+|u(\alpha(t),x)-u_0(\alpha(t),x)|) \\ & +\tfrac{p(t,x)}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^x \left(\log \tfrac{\beta(t)}{s}\right)^{r_1-1} \left(\log \tfrac{x}{\xi}\right)^{r_2-1} q(s,\xi) d\xi ds. \end{aligned} \tag{3.5}$$

By using (3.5) and the fact that $\alpha(t) \to \infty$ as $t \to \infty$, we get

$$\begin{aligned} \lim_{t\to\infty} |u(t,x)-u_0(t,x)| \quad \leq \quad & \lim_{t\to\infty} \frac{L.p(t,x)}{\Gamma(r_1)\Gamma(r_2)(L-2M)} \int_1^{\beta(t)} \int_1^x \left(\log \frac{\beta(t)}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \\ \times \quad & q(s,\xi) d\xi ds = 0. \end{aligned}$$

Consequently, all solutions of the integral equation (1.2) are globally asymptotically stable.

4 An Example

As an application of our results we consider the following partial Hadamard integral equation of fractional order

$$\begin{aligned} u(t,x) &= \tfrac{tx}{10(1+t+t^2+t^3)}(1+2\sin(u(t,x)))+\tfrac{1}{\Gamma^2(\frac{1}{3})}\int_1^t\int_1^x\left(\log\tfrac{t}{s}\right)^{\frac{-2}{3}}\left(\log\tfrac{x}{\xi}\right)^{\frac{-2}{3}} \\ & \tfrac{\ln(1+2x(s\xi)^{-1}|u(s,\xi)|)}{(1+t+2|u(s,\xi)|)^2(1+x^2+t^4)}d\xi ds;\ (t,x)\in[1,\infty)\times[1,e], \end{aligned} \tag{4.1}$$

where $r_1 = r_2 = \frac{1}{3}$, $\alpha(t) = \beta(t) = \gamma(t) = t$,

$$f(t,x,u,v) = \frac{tx(1+\sin(u)+\sin(v))}{10(1+t)(1+t^2)},$$

and

$$g(t,x,s,\xi,u,v) = \frac{\ln(1+x(s\xi)^{-1}(|u|+|v|))}{(1+t+|u|+|v|)^2(1+x^2+t^4)};$$

for $(t,x),(s,\xi)\in[1,\infty)\times[1,e]$, and $u,v\in\mathbb{R}$.

We can easily check that the assumptions of Theorem 3.1 are satisfied. In fact, we have that the function f is continuous and satisfies assumption (H_2), where $M = \frac{1}{10}$, $L = 1$. Also f satisfies assumption (H_3), with $f^* = \frac{e}{10}$. Next, let us notice that the function g satisfies assumption (H_4), where $p(t,x) = \frac{1}{1+x^2+t^4}$ and $q(s,\xi) = (s\xi)^{-1}$. Also,

$$\begin{aligned} & \lim_{t\to\infty} p(t,x)\int_1^t\int_1^x\left|\log\frac{t}{s}\right|^{\frac{-2}{3}}\left|\log\frac{x}{\xi}\right|^{\frac{-2}{3}} q(s,\xi)d\xi ds \\ = & \lim_{t\to\infty}\frac{x}{1+x^2+t^4}\int_1^t\int_1^x\left|\log\frac{t}{s}\right|^{\frac{-2}{3}}\left|\log\frac{x}{\xi}\right|^{\frac{-2}{3}}\frac{d\xi ds}{s\xi} \\ = & \lim_{t\to\infty}\frac{9x(\log t)^{\frac{1}{3}}}{1+x^2+t^4} = 0. \end{aligned}$$

Hence by Theorem 3.1, the equation (4.1) has a solution defined on $[1,\infty)\times[1,e]$ and solutions of this equation are globally asymptotically stable.

References

[1] S. Abbas and M. Benchohra, Some stability concepts for Darboux problem for partial fractional differential equations on unbounded domain, *Fixed Point Theory* **16** (1) (2015), 3-14.

[2] S. Abbas and M. Benchohra, Nonlinear quadratic Volterra Riemann-Liouville integral equations of fractional order, *Nonlinear Anal. Forum* **17** (2012), 107-115.

[3] S. Abbas, M. Benchohra and M.A. Darwish. New stability results for partial fractional differential inclusions with not instantaneous impulses, *Frac. Calc. Appl. Anal.* **18** (1) (2015), 172-191.

[4] S. Abbas, M. Benchohra and J. Henderson, Global asymptotic stability of solutions of nonlinear quadratic Volterra integral equations of fractional order, *Comm. Appl. Nonlinear Anal.* **19** (2012), 79-89.

[5] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[6] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[7] J. Banaś and I. J. Cabrera, On existence and asymptotic behaviour of solutions of a functional integral equation, *Nonlinear Anal.* **66** (2007), 2246-2254.

[8] J. Banaś, J. Caballero, J. Rocha and K. Sadarangani, Monotonic solutions of a class of quadratic integral equations of Volterra type, *Comput. Math. Appl.* **49** (2005), 943-952.

[9] J. Banaś and B. Rzepka, On existence and asymptotic stability of solutions of a nonlinear integral equation, *J. Math. Anal. Appl.* **284** (2003), 165-173.

[10] J. Banaś and B. Rzepka, Monotonic solutions of a quadratic integral equation of fractional order, *J. Math. Anal. Appl.* **332** (2007), 1371-1379.

[11] J. Banaś and T. Zając, Solvability of a functional integral equation of fractional order in the class of functions having limits at infinity *Nonlinear Anal.* **71** (2009), 5491-5500.

[12] J. Banaś and T. Zając, A new approach to the theory of functional integral equations of fractional order, *J. Math. Anal. Appl.* **375** (2011), 375-387.

[13] P. L. Butzer, A. A. Kilbas, and J. J. Trujillo. Fractional calculus in the mellin setting and Hadamard-type fractional integrals. *J. Math. Anal. Appl.* **269** (2002), 1-27.

[14] P. L. Butzer, A. A. Kilbas, and J. J. Trujillo. Mellin transform analysis and integration by parts for Hadamard-type fractional integrals. *J. Math. Anal. Appl.* **270** (2002), 1-15.

[15] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[16] M. A. Darwish, J. Henderson and D. O'Regan, Existence and asymptotic stability of solutions of a perturbed fractional functional integral equations with linear modification of the argument, *Bull. Korean Math. Soc.* **48** (3) (2011), 539-553.

[17] L. Gaul, P. Klein and S. Kempfle, Damping description involving fractional operators, *Mech. Systems Signal Processing* **5** (1991), 81-88.

[18] W. G. Glockle and T. F. Nonnenmacher, A fractional calculus approach of selfsimilar protein dynamics, *Biophys. J.* **68** (1995), 46-53.

[19] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[20] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier Science B.V., Amsterdam, 2006.

[21] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180-7186.

[22] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[23] S. Pooseh, R. Almeida, and D. Torres. Expansion formulas in terms of integer-order derivatives for the hadamard fractional integral and derivative. *Numer. Funct. Anal. Optim.* **33** (3) (2012), 301-319.

[24] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.