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Abstract

Our aim is to give a simple estimate of the Bloch constant applying some fundamental facts on complex analysis. Our method is based on the Cauchy estimate, the maximum modulus principle, the Schwarz lemma and the Rouché theorem.

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1 Introduction

Given $a \in \mathbb{C}$ and $r > 0$ we write $B(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$. In particular the unit open disc is denoted by $\mathbb{D} := B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$. The following theorem is well known as the Bloch theorem.

Theorem 1.1 (The Bloch theorem)**.** *We can take a constant $B > 0$ so that for all holomorphic function f on $\mathbb{D}$ satisfying $f'(0) = 1$, there exists a point $z_f \in f(\mathbb{D})$ such that $B\left(z_f, B\right) \subset f(\mathbb{D})$.*

The supremum of B appearing in Theorem 1.1 is called the Bloch constant. The estimate of the constant is one of the most important problems in complex analysis. Ahlfors and Grunsky [2] have found the upper bound $B \le \dfrac{1}{\sqrt{1+\sqrt{3}}} \cdot \dfrac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)} < 0.4719$. They have also conjectured that the correct value equals to the upper bound above. It is known that the lower estimate $B > \sqrt{3}/4$ is obtained by using the Poincaré metric [1, 9, 12, 13]. Some better lower estimates are proved based on more developed methods [3, 4, 15].

The recent study on the Bloch theorem treats not only better estimates of the Bloch constant but also generalizations of the theorem to wider classes of functions. Some generalizations to the cases that functions of several variables, locally univalent functions and harmonic functions are known ([5, 6, 10]).

On the other hand, we note the method of the proof of the Bloch theorem. Recently Cortissoz and Montero [8] have proved it based on the Banach fixed point theorem. Of course an elementary complex analytic proof is known. For example we can find the proof with $B \ge \frac{1}{16\log 3}$ in [14]. An argument similar to [14] is found in [7, 11].

In the present paper we give a simple method of an estimate of the Bloch constant modifying the argument found in [7, 11, 14]. It suffices to use basic facts on complex analysis in order to a better estimate than the one obtained by [14]. More precisely we have only to use the Cauchy estimate, the maximum modulus principle, the Schwarz lemma and the Rouché theorem in order to prove the key lemma. We remark that our method gives not only a simple estimate of the Bloch constant but also a self-contained and elementary proof of the Bloch theorem.

2 The main result

In order to prove the Bloch theorem straightforwardly we use the next lemma. For readers' convenience we give a self-contained proof, however the lemma can be found in [7, 11, 14].

Lemma 2.1. *Let f be a holomorphic function on $B(0,R)$ satisfying $|f(z)| \le M$ for all $z \in B(0,R)$, where $M > 0$ is a constant independent of z, $f(0) = 0$ and $f'(0) = 1$. Then f satisfies $f\left(B\left(0, \frac{R^2}{4M}\right)\right) \supset B\left(0, \frac{R^2}{6M}\right)$.*

Proof. If the function $f(z) - z$ equals to a constant C on $B(0,R)$, then $f(0) = 0$ implies that $C = 0$. We easily see that the conclusion of the lemma is true in the case $f(z) = z$ on $B(0,R)$.

Below we suppose that $f(z)-z$ does not equal to any constant function on $B(0,R)$. Because f is analytic on $B(0,R)$, we obtain the Taylor expansion of f on $B(0,R)$, that is,

$$f(z)=\sum_{n=0}^{\infty} c_n z^n=\sum_{n=2}^{\infty} c_n z^n+z. \tag{2.1}$$

Here we remark that $c_0=f(0)=0$ and $c_1=f'(0)=1$. By virtue of the Schwarz lemma we have $|f'(0)|\le \frac{M}{R}$. Because of $f'(0)=1$, we additionally get $R\le M$. Below we write $r:=\frac{R^2}{4M}$. Then we have

$$r=\frac{R^2}{4M}\le\frac{R^2}{4R}=\frac{R}{4}<R\le M. \tag{2.2}$$

On the other hand, the function $f(z)-z$ is continuous on the bounded and closed set $\partial B(0,r)$. Thus the maximal value $M(r):=\max_{z\in\partial B(0,r)}|f(z)-z|$ does exist. Now we note that f is holomorphic on a domain including $\overline{B(0,r)}$. Hence for every $n\ge 2$, the Cauchy estimate implies $|c_n|=\left|\frac{f^n(0)}{n!}\right|\le\frac{M}{R^n}$. Thus we get $M(r)=\max_{z\in\partial B(0,r)}\left|\sum_{n=2}^{\infty}c_n z^n\right|\le\sum_{n=2}^{\infty}M\left(\frac{r}{R}\right)^n=\frac{M}{R^2}\cdot r^2\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n$. On the other hand, by virtue of (2.2) we get $\frac{r}{R}\le\frac{1}{4}$. Therefore we obtain

$$M(r)\le\frac{M}{R^2}\cdot\left(\frac{R^2}{4M}\right)^2\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n=\frac{R^2}{12M}.$$

Therefore we have

$$r=\frac{R^2}{4M}>M(r). \tag{2.3}$$

Now we remark that the function $f(z)-z$ is holomorphic and is not any constant function on the bounded domain $B(0,R)$. Additionally it is continuous on $\overline{B(0,r)}$. Thus by the maximum modulus principle we get $|f(z)-z|\le M(r)$ for all $z\in B(0,r)$. By virtue of the fact that $f(z)-z=0$ if $z=0$, the Schwarz lemma implies that $|f(z)-z|\le M(r)\cdot\frac{|z|}{r}$ holds for all $z\in B(0,r)$. By (2.3) we have that for all $z\in B(0,r)$,

$$|f(z)|\ge|z|-|f(z)-z|\ge|z|-M(r)\cdot\frac{|z|}{r}=|z|\left(1-\frac{M(r)}{r}\right)>0. \tag{2.4}$$

On the other hand, we have $|f(z)|\ge|z|-|f(z)-z|\ge r-M(r)\ge\dfrac{R^2}{6M}$ for all $z\in\partial B(0,r)$. Now we take $w\in B\left(0,\frac{R^2}{6M}\right)$ arbitrarily. Because $|f(z)|>|w|$ holds for all $z\in\partial B(0,r)$, the Rouché theorem implies that the number of the zero points included in $B(0,r)$ of f equals to the one of $f(z)-w$, where we take the order of each zero point into account. From (2.1) we see that the point $z=0$ is a zero point of order 1 of the function f. Combining the fact with (2.4), we see that f has the only one zero point $z=0$ on $B(0,r)$. Thus the function $f(z)-w$ has also only one zero point on $B(0,r)$, that is, there exists $z_w\in B\left(0,\frac{R^2}{4M}\right)$ uniquely such that $f(z_w)-w=0$. □

Applying the lemma above we begin the proof of the Bloch theorem. Reviewing the proofs of Lemma 2.1 and Theorem 1.1, we see that our method is elementary and simple.

Proof of Theorem 1.1. Fix $\varepsilon, \varepsilon_0 \in (0,1)$ arbitrarily. We will prove the theorem with $B = \frac{\varepsilon_0}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}$.

(i) We first consider the case that f is holomorphic on a domain including $\overline{\mathbb{D}}$ and satisfies $f'(0) = 1$.

Step 1: We estimate the function $(1-|z|^2)|f'(z)|$. Because this function is continuous on the bounded and closed set $\overline{\mathbb{D}}$, the maximal value $M := \max_{z\in\overline{\mathbb{D}}}(1-|z|^2)|f'(z)|$ does exist. Now we take $\alpha \in \overline{\mathbb{D}}$ so that $(1-|\alpha|^2)|f'(\alpha)| = M$ holds. We see that $M \geq (1-|0|^2)|f'(0)| = 1$. In addition, if $\beta \in \partial\mathbb{D}$, then $(1-|\beta|^2)|f'(\beta)| = 0$ holds. Namely we have $|\alpha| < 1$. Thus we can define a holomorphic function on $\mathbb{D}$ by

$$z(\zeta) := \frac{\alpha-\zeta}{1-\overline{\alpha}\zeta} \quad (\zeta \in \mathbb{D}). \tag{2.5}$$

The map $z : \mathbb{D} \to \mathbb{D}$ is bijective because it is the Möbius transform on $\mathbb{D}$.

Step 2: We estimate a function F defined by

$$F(\zeta) := f(z(\zeta)) = f\left(\frac{\alpha-\zeta}{1-\overline{\alpha}\zeta}\right) \quad (\zeta \in \mathbb{D}).$$

We remark that F is holomorphic on $\mathbb{D}$ and satisfies $F(\mathbb{D}) = f(\mathbb{D})$. For all $\zeta \in \mathbb{D}$, we have $z(\zeta) \in \mathbb{D}$. Hence we get

$$\begin{aligned}
(1-|\zeta|^2)|F'(\zeta)| &= (1-|\zeta|^2)\left|f'(z(\zeta))z'(\zeta)\right| \\
&= (1-|\zeta|^2)\left|\frac{|\alpha|^2-1}{(1-\bar{\alpha}\zeta)^2}\right||f'(z(\zeta))| \\
&= \frac{|1-\overline{\alpha}\zeta|^2-|\alpha-\zeta|^2}{|1-\overline{\alpha}\zeta|^2}\cdot|f'(z(\zeta))| \\
&= (1-|z(\zeta)|^2)|f'(z(\zeta))|.
\end{aligned}$$

Combing this result with Step 1 we obtain

$$(1-|\zeta|^2)|F'(\zeta)| = (1-|z(\zeta)|^2)|f'(z(\zeta))| \leq M. \tag{2.6}$$

Step 3: By virtue of $z(0) = \alpha$ and (2.6) we see that $|F'(0)| = M$, in particular $F'(0) \neq 0$. Thus we can define a holomorphic function g by

$$g(\zeta) := \frac{F(\zeta)-F(0)}{F'(0)} \quad (\zeta \in \mathbb{D}).$$

We prove that the inclusion relation $B\left(0, \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset g(\mathbb{D})$ holds. The function g satisfies $g(0) = 0$, $g'(0) = \frac{F'(0)}{F'(0)} = 1$ and $|g'(\zeta)| = \dfrac{|F'(\zeta)|}{M} \leq \dfrac{1}{1-|\zeta|^2}$ for every $\zeta \in \mathbb{D}$, where we have used (2.6). Take $\zeta_0 \in B(0,\varepsilon)$ arbitarily. Then there exist $r_0 \in [0,\varepsilon)$ and $\theta_0 \in \mathbb{R}$ such that $\zeta_0 = r_0e^{i\theta_0}$ $(0 \leq r \leq r_0)$. Let l be a line segment starting from 0 to ζ_0, namely $l : \zeta(r) =$

$re^{i\theta_0}$ $(0 \le r \le r_0)$. We note that $l \subset B(0,\varepsilon)$. Because g' is continuous and has the primitive function g on $B(0,\varepsilon)$, we have

$$\begin{aligned}|g(\zeta_0)| = |g(\zeta_0) - g(0)| &= \left|\int_l g'(\zeta)\,d\zeta\right| = \left|\int_0^{r_0} g'(re^{i\theta_0})e^{i\theta_0}\,dr\right| \\ &\le \int_0^{r_0} |g'(re^{i\theta_0})|\,dr \le \int_0^{r_0} \frac{1}{1-|re^{i\theta_0}|^2}\,dr \\ &= \int_0^{r_0} \frac{1}{1-r^2}\,dr = \frac{1}{2}\log\frac{1+r_0}{1-r_0} < \frac{1}{2}\log\frac{1+\varepsilon}{1-\varepsilon}\end{aligned}$$

Thus by applying Lemma 2.1 to g, for all $w \in B\left(0, \frac{\varepsilon^2}{6\cdot\frac{1}{2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) = B\left(0, \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right)$, there exists $\zeta_w \in B\left(0, \frac{\varepsilon^2}{4\cdot\frac{1}{2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) = B\left(0, \frac{1}{\frac{2}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right)$ uniquely such that $w = g(\zeta_w)$. This implies that

$$B\left(0, \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset g\left(B\left(0, \frac{1}{\frac{2}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right)\right) \subset g(\mathbb{D}).$$

Step 4: We prove $B\left(f(\alpha), \frac{M}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset f(\mathbb{D})$. Take $w \in B\left(f(\alpha), \frac{M}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right)$ arbitrarily. We easily obtain $\left|\frac{w-f(\alpha)}{F'(0)}\right| = \left|\frac{w-f(\alpha)}{M}\right| < \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}$, that is, $\frac{w-f(\alpha)}{F'(0)} \in B\left(0, \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right)$. By virtue of Step 3, $B\left(0, \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset g(\mathbb{D})$ holds. Thus we can take $\zeta_1 \in \mathbb{D}$ so that $\frac{w-f(\alpha)}{F'(0)} = g(\zeta_1)$. This implies that $w = f(z(\zeta_1)) \in f(\mathbb{D})$.

Step 5: We prove $B\left(f(\alpha), \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset f(\mathbb{D})$. Combing the fact that $M \ge 1$ with Step 4, we see that $B\left(f(\alpha), \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset B\left(f(\alpha), \frac{M}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset f(\mathbb{D})$. Consequently we have proved the theorem in the case (i).

(ii) We consider the case that f is holomorphic on $\mathbb{D}$ and satisfies $f'(0) = 1$. The function $g_0(z) := \frac{1}{\varepsilon_0} f(\varepsilon_0 z)$ is holomorphic on $B\left(0, \frac{1}{\varepsilon_0}\right)$ and satisfies $g_0'(0) = f'(0) = 1$. We also note that $\overline{\mathbb{D}} \subset B(0, \frac{1}{\varepsilon_0})$. Thus we can apply the result (i) to the function g_0, that is, there exists $z_{g_0} \in g_0(\mathbb{D})$ such that $B\left(z_{g_0}, \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset g_0(\mathbb{D})$. In addition, we can take $\alpha_0 \in \mathbb{D}$ so that $z_{g_0} = g_0(\alpha_0)$. Now we take $w \in B\left(f(\varepsilon_0\alpha_0), \frac{\varepsilon_0}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right)$ arbitrarily. Then we see that

$$\left|\frac{w}{\varepsilon_0} - \frac{1}{\varepsilon_0} f(\varepsilon_0\alpha_0)\right| < \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}},$$

that is, $\frac{w}{\varepsilon_0} \in B\left(g_0(\alpha_0), \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right)$. By virtue of $B\left(g_0(\alpha_0), \frac{1}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset g_0(\mathbb{D})$ there exists $z_1 \in \mathbb{D}$ such that $\frac{w}{\varepsilon_0} = g_0(z_1)$. This implies that $w = f(\varepsilon_0 z_1) \in f(\mathbb{D})$. Consequently we have proved $B\left(f(\varepsilon_0\alpha_0), \frac{\varepsilon_0}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}\right) \subset f(\mathbb{D})$ in the case (ii). □

Remark 2.2. We note that there do exist $\varepsilon, \varepsilon_0 \in (0,1)$ such that

$$\frac{1}{16\log 3} < \frac{\varepsilon_0}{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}. \tag{2.7}$$

In fact, if we take $\varepsilon = \frac{4}{5}$ and $\varepsilon_0 \in \left(\frac{75}{128}, 1\right)$, then we see that

$$\frac{\frac{3}{\varepsilon^2}\log\frac{1+\varepsilon}{1-\varepsilon}}{16\log 3} = \frac{\frac{75}{16}\log 9}{16\log 3} = \frac{75}{128} < \varepsilon_0,$$

that is, (2.7) holds. This implies that we have obtained a better estimate than $B \geq \frac{1}{16\log 3}$ due to Yoshida [14] applying only some fundamental facts on complex analysis.

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