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Hardy Classes and Symbols of Toeplitz Operators

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Abstract

The purpose of this paper is to study functions in the unit disk $\mathbb{D}$ through the family of Toeplitz operators $\{T_{\varphi d\sigma_t}\}_{t\in[0,1)}$, where $T_{\varphi d\sigma_t}$ is the Toeplitz operator acting the Bergman space of $\mathbb{D}$ and where $d\sigma_t$ is the Lebesgue measure in the circle tS^1. In particular for $1 \le p < \infty$ we characterize the harmonic functions φ in the Hardy space $h^p(\mathbb{D})$ by the growth in t of the p-Schatten norms of $T_{\varphi d\sigma_t}$. We also study the dependence in t of the norm operator of $T_{a d\sigma_t}$ when $a \in H^p_{at}$, the atomic Hardy space in the unit circle with $1/2 < p \le 1$.

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1 Introduction and notation

For $0 < p < \infty$, let $\mathcal{A}^p$ be the Bergman space of all the holomorphic functions on the open unit disk $\mathbb{D}$, such that

$$\|f\|_p = \left(\int_{\mathbb{D}} |f|^p dA\right)^{1/p} < \infty,$$

where dA is the normalized Lebesgue measure on $\mathbb{D}$.

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Denote by $\mathcal{L}(\mathcal{A}^p)$ the bounded operators in $\mathcal{A}^p$ and for $p > 0$, let S_p be the Schatten classes in the Bergman space $\mathcal{A}^2$. For a complex Borel measure μ on $\mathbb{D}$ the Toeplitz operator $T_\mu : \mathcal{A}^2 \to hol(\mathbb{D})$ is defined by

$$T_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2} d\mu(w),$$

where $hol(\mathbb{D})$ is the space of all holomorphic functions on $\mathbb{D}$. The measure μ is called the symbol of T_μ. When $d\mu = \varphi dA$ with $\varphi \in L^1(\mathbb{D})$ we write T_φ.

Consider the case when μ is a Borel measure supported at $tS^1 := \{z \in \mathbb{D} : |z| = t\}$ given by

$$\mu(A) = \int_{tS^1} a(w) d\sigma_t(w)$$

where $a \in L^1(tS^1)$ and $d\sigma_t$ is the arc length measure on tS^1. We will write in this case $d\mu = ad\sigma_t$. Thus we have

$$T_{ad\sigma_t} f(z) = t \int_0^{2\pi} \frac{a(\theta) f(te^{i\theta})}{(1 - zte^{-i\theta})^2} d\theta. \tag{1.1}$$

If φ is measurable in $\mathbb{D}$ we can formally split the Toeplitz operator T_φ as

$$T_\varphi = \int_0^1 T_{\varphi d\sigma_t} dt. \tag{1.2}$$

In previous work [3], the authors obtained the precise dependence on t of the operator norm and the p-Schatten norm of $T_{ad\sigma_t}$ when a is a positive density in $L^1(tS^1)$. It is proved that for $a \geq 0$ in $L^1(tS^1)$ and $0 < t < 1$, the norm operator $\left\|T_{ad\sigma_t}\right\|_{\mathcal{L}(\mathcal{A}^2)}$ satisfies

$$\left\|T_{ad\sigma_t}\right\|_{\mathcal{L}(\mathcal{A}^2)} \sim \frac{1}{(1-t)^2} \sup \int_\Gamma a(\xi) d\sigma_t(\xi), \tag{1.3}$$

uniformly in $[0, 1)$, where the supremum is taken over all the arcs Γ contained in tS^1 such that $\sigma_t(\Gamma) \leq (1 - t)$. For the Schatten norms it was proved precise estimates for $\|T_{ad\sigma_t}\|_{S_p}$, $1 \leq p < \infty$ (see (1.16) and (1.17) below). In view of (1.2) this allowed to study new classes of Toeplitz operators T_φ with finite mixed norms involving $\|T_{\varphi d\sigma_t}\|_{S_p}$ and a weighted L^q norm in the variable $t \in [0, 1)$.

The purpose of this paper is twofold. First we want to study functions φ in $\mathbb{D}$ through the family of operators $\{T_{\varphi d\sigma_t}\}_{t\in[0,1)}$. In concrete we characterize the membership of a harmonic function to the Hardy space $h^p = h^p(\mathbb{D})$, $1 \leq p < \infty$ by the $p-$Schatten norms of the operators $T_{\varphi d\sigma_t}$. On the other hand, extending the results in [3] we will study the behavior as t tends to 1 of the norm of $T_{ad\sigma_t}$ in Bergman spaces $\mathcal{A}^q$, when $a \in H^p_{at}$ the atomic Hardy space in the unit circle for $1/2 < p \leq 1$.

We start in Section 2 by extending to complex functions, one side of the estimate (1.3) for $\|T_{ad\sigma_t}\|_{\mathcal{L}(\mathcal{A}^p)}$.

Theorem 1.1. *Let $a \in L^1(tS^1)$, $0 < t < 1$.*

a) For every $p > 1$ there exists a constant $C_p > 0$ such that

$$\|T_{ad\sigma_t}\|_{\mathcal{L}(\mathcal{A}^p)} \le \frac{C_p}{(1-t)^2} \sup \left| \int_\Gamma a(\xi) d\sigma_t(\xi) \right|,$$

where the supremum is taken over all the arcs Γ contained in tS^1 such that $\sigma_t(\Gamma) \le (1-t)$.

b) There exists a constant $C > 0$ such that

$$\|T_{ad\sigma_t}\|_{\mathcal{L}(\mathcal{A}^1)} \le \frac{C \log(1/(1-t))}{(1-t)^2} \sup_{\sigma_t(\Gamma)<1-t} \left| \int_\Gamma a(\xi) d\sigma_t(\xi) \right|,$$

where the supremum is taken over all the arcs Γ contained in tS^1 such that $\sigma_t(\Gamma) \le (1-t)$.

Next in Section 3 we characterize those functions u in $\mathbb{D}$ that belong to the Hardy space h^p by the growth of the Schatten norms $\|T_{|u|d\sigma_t}\|_{S_p}$.

Theorem 1.2. *Let $u : \mathbb{D} \to \mathbb{C}$ be a harmonic function and $1 \le p < \infty$. Then the following statements are equivalent*

a) $u \in h^p$.

b) $L = \sup_{t_0<t<1} \|T_{(1-t)^{1+1/p}|u|d\sigma_t}\|_{S_p} < \infty$, *for some* $0 < t_0 < 1$.

c) $\sup_{0\le t<1} \|T_{(1-t)^{1+1/p}|u|d\sigma_t}\|_{S_p} < \infty$.

Some work has been done about Toeplitz operators with distributional symbols, see for example [4, 5]. For $h \in \mathcal{D}'(S^1)$ we define the Toeplitz operator

$$T_{hd\sigma_t} f(z) = \left\langle h, \frac{f(te^{i\cdot})}{(1-zte^{-i\cdot})} \right\rangle, \; f \in hol(\mathbb{D}), \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing of $\mathcal{D}'(S^1)$-$C^\infty(S^1)$. We end the paper giving in Theorem 1.3 estimates for the growth in t of Schatten norms $\|T_{hd\sigma_t}\|_{S_q}$ when $h \in \mathcal{D}'(S^1)$ belongs to the atomic Hardy space H^p_{at}. We have

Theorem 1.3. *Let $h \in H^p_{at}$ with $1/2 < p \le 1$ and $q \ge p$ then*

a) For $q > p$ given, there exists $C > 0$ such that

$$\|T_{hd\sigma_t}\|_{\mathcal{L}(\mathcal{A}^q)} \le \frac{C\|h\|_{H^p_{at}}}{(1-t)^{1/q+1/p+1}}.$$

b) There exists $C > 0$ such that

$$\|T_{hd\sigma_t}\|_{\mathcal{L}(\mathcal{A}^p)} \le \frac{C}{(1-t)^{2/p+1}} \left(\log(1/(1-t))\right)^{1/p} \|h\|_{H^p_{at}}.$$

The following notations and facts will be used in the paper: We denote by $D(z,r)$ the hyperbolic disk in $\mathbb{D}$, namely the disk centered at z and radius $r>0$ with respect to the Bergman metric

$$\beta(z,w)=\frac{1}{2}\log\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}.$$

The following inequalities will be useful.

$$\int_0^1 \frac{ds}{(1-\alpha s)^{1+\beta}} \le \frac{C}{(1-\alpha)^{\beta}},\ \alpha\in[0,1), \beta>0, \tag{1.5}$$

$$\int_0^1 \frac{ds}{1-\alpha s} = \frac{1}{\alpha}\log\left(\frac{1}{1-\alpha}\right),\ \alpha\in(0,1), \tag{1.6}$$

If $0\le t<1, \beta>0$ then (see [2, Theorem 1.7])

$$\int_0^{2\pi} \frac{1}{|1-te^{i\theta}|^{1+\beta}}d\theta \sim \frac{1}{(1-t)^{\beta}}. \tag{1.7}$$

The following estimate holds (see [6, Proposition 4.13]):

For each $r>0$, $q>0$, $\alpha>-1$, there exists a constant $C_r>0$ such that

$$|f(z)|^q \le \frac{C_r}{(1-|z|^2)^{2+\alpha}}\int_{D(z,r)} |f(w)|^q dA_\alpha(w) \tag{1.8}$$

for every holomorphic function f on $\mathbb{D}$ where $dA_\alpha(z)=(1-|z|^2)^{\alpha}dA(z)$.
In particular, for each $q>0$ we have that

$$|f'(z)|^q \le \frac{C_r}{(1-|z|^2)^{2+q}}\int_{D(z,r)} (1-|w|^2)^q|f'(w)|^q dA(w) \tag{1.9}$$

for every holomorphic function f on $\mathbb{D}$.
Since for holomorphic functions is $\|(1-|z|)f'\|_q \le C_q\|f\|_q$, we have for $f\in\mathcal{A}^q$ and $q>0$,

$$|f(te^{i\theta})| \le \frac{C}{(1-t)^{2/q}}\|f\|_q, \tag{1.10}$$

$$|f'(te^{i\theta})| \le \frac{C}{(1-t)^{2/q+1}}\|f\|_q. \tag{1.11}$$

If $c>0$, $t>-1$, then (see [6, Lemma 3.10])

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t+c}}dA(w) \sim \frac{1}{(1-|z|^2)^c} \quad \text{as } |z|\to 1^-, \tag{1.12}$$

and

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t}}dA(w) \sim \log\frac{1}{1-|z|^2} \quad \text{as } |z|\to 1^-. \tag{1.13}$$

For a function f defined and integrable on S^1 and $\tau>0$ we let the averaging operator

$$\mathcal{E}_\tau f(\theta)=\frac{1}{2\tau}\int_{\theta-\tau}^{\theta+\tau} f(e^{i\eta})d\eta, \tag{1.14}$$

and denote the Hardy Littlewood maximal function as

$$\mathcal{M}f(\theta) = \sup_{\tau>0} \frac{1}{2\tau} \int_{\theta-\tau}^{\theta+\tau} f(e^{i\eta})d\eta = \sup_{\tau>0} \mathcal{E}_\tau f(\theta).$$

Notice that for $\tau > 0$,

$$\mathcal{E}_\tau f * g(\theta) = f * \mathcal{E}_\tau g(\theta),\ \theta \in [0, 2\pi),$$

where $f * g$ denotes the convolution in S^1. If φ is defined in $\mathbb{D}$ we will also write $\mathcal{E}_\tau\varphi$ meaning

$$\mathcal{E}_\tau\varphi(te^{i\theta}) = \frac{1}{2\tau} \int_{\theta-\tau}^{\theta+\tau} \varphi(te^{i\eta})d\eta.$$

Consider $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(s) = \frac{1}{c} \frac{s^{1+p}}{(1+s)^{1/2}} \chi_{[0,1)}(s), \tag{1.15}$$

with c chosen so that $\int_0^1 \psi(s)ds = 1$ and where $\chi_{[0,1)}$ stands for the characteristic function of $[0, 1)$.

For $\lambda > 0$, denote by $\psi_\lambda(s) = \lambda^{-1}\psi(\lambda^{-1}s)$, so that ψ_λ acts as an approximate identity, namely ψ_λ tends to the Dirac δ as λ goes to 0. If f is a nonnegative function in $L^1(tS^1)$, $0 \le t < 1$, it was proved in [3] the two sided estimate of the Schatten norm of the Toeplitz operator $T_{fd\sigma_t}$, valid for $1 < p < \infty$:

$$\|T_{fd\sigma_t}\|_{S_p} \sim t^{1/p'}(1-t)^{-(1+1/p)} \left\{ \int_0^{1-t} \|\mathcal{E}_\tau f\|^p_{L^p(tS^1)} \psi_{1-t}(\tau)d\tau \right\}^{1/p}, \tag{1.16}$$

and

$$\|T_{fd\sigma_t}\|_{S_1} \sim (1-t)^{-2} \int_0^{1-t} \|f\|_{L^1(tS^1)} \psi_{1-t}(\tau)d\tau, \tag{1.17}$$

with constants independent of f and t. Here p' denotes the conjugate exponent of $p > 1$.

2 Norm estimates for $T_{ad\sigma_t}$ on the Bergman spaces $\mathcal{A}^p$ with $a \in L^1(tS^1)$

Proof of Theorem 1.1. Fix $r > 0$ and let K_z be the reproducing kernel of $\mathcal{A}^2$ at the point $z \in \mathbb{D}$. By (1.12) we have

$$\|K_z\|_p \le C(1-|z|)^{-2/p'} \tag{2.1}$$

for all $z \in \mathbb{D}$, whenever $p > 1$.

We choose a partition $\{x_0, \ldots x_N\}$ of the interval $[0, 2\pi]$ such that the corresponding arcs in tS^1 have lenght less than $1 - t$.

For $j = 0,\dots,N-1$ we set

$$\mathcal{D}_j := \bigcup_{x_j \le \theta < x_{j+1}} D(te^{i\theta}, r).$$

Let $f \in hol(\mathbb{D})$, we integrate by parts to rewrite

$$T_{ad\sigma_t} f(z) = \sum_{j=1}^{N} \Big(I_j(z) + J_j(z) + L_j(z) \Big),$$

where

$$I_j(z) = t \left(\int_{x_j}^{x_{j+1}} a(te^{i\varrho}) d\varrho \right) \frac{f(te^{ix_{j+1}})}{(1 - zte^{-ix_{j+1}})^2},$$

$$J_j(z) = -it^2 \int_{x_j}^{x_{j+1}} \left(\int_{x_j}^{\theta} a(te^{i\varrho}) d\varrho \right) \frac{e^{i\theta} f'(te^{i\theta})}{(1 - zte^{-i\theta})^2} d\theta,$$

$$L_j(z) = 2it^2 z \int_{x_j}^{x_{j+1}} \left(\int_{x_j}^{\theta} a(te^{i\varrho}) d\varrho \right) \frac{e^{-i\theta} f(te^{i\theta})}{(1 - zte^{-i\theta})^3} d\theta.$$

Moreover, we set

$$\gamma = \gamma_{a,t} := \sup \left| \int_{\Gamma} a(\xi) d\sigma_t(\xi) \right|,$$

where the supremum is taken over all the arcs Γ contained in tS^1 with $\sigma_t(\Gamma) \le 1 - t$. We use (1.8) with $\alpha = 0$, and (2.1) to get

$$\begin{aligned} \|I_j\|_p &\le \frac{C_p \gamma t}{(1-t)^{2/p}} \|K_{te^{ix_{j+1}}}\|_p \left(\int_{D(te^{ix_{j+1}}, r)} |f(w)|^p dA(w) \right)^{1/p} \\ &\le \frac{C_p \gamma}{(1-t)^2} \left(\int_{\mathcal{D}_j} |f(w)|^p dA(w) \right)^{1/p}. \end{aligned} \tag{2.2}$$

We use the Minkowski integral inequality, the inequality (1.9), and the assumption about the points x_j to obtain

$$\begin{aligned} \|J_j\|_p &\le \gamma t^2 \int_{x_j}^{x_{j+1}} |f'(te^{i\theta})| \|K_{te^{i\theta}}\|_p d\theta \\ &\le \frac{C_p \gamma t^2}{(1-t)^3} \int_{x_j}^{x_{j+1}} \left(\int_{D(te^{i\theta}, r)} (1-|w|^2)^p |f'(w)|^p dA(w) \right)^{1/p} d\theta \\ &\le \frac{C_p \gamma}{(1-t)^2} \left(\int_{\mathcal{D}_j} (1-|w|^2)^p |f'(w)|^p dA(w) \right)^{1/p}. \end{aligned} \tag{2.3}$$

In a similar way, we use (1.12) to get

$$\begin{aligned} \|L_j\|_p &\le \gamma t^2 \int_{x_j}^{x_{j+1}} |f(te^{i\theta})| \|(1 - \cdot te^{i\theta})^{-3}\|_p d\theta \\ &\le \frac{C_p \gamma t^2}{(1-t)^3} \int_{x_j}^{x_{j+1}} \left(\int_{D(te^{i\theta}, r)} |f(w)|^p dA(w) \right)^{1/p} d\theta \\ &\le \frac{C_p \gamma}{(1-t)^2} \left(\int_{\mathcal{D}_j} |f(w)|^p dA(w) \right)^{1/p}. \end{aligned} \tag{2.4}$$

For small enough $r > 0$, we notice that the sets $\mathcal{D}_j$ overlap each other at most twice. Thus $\|T_{ad\sigma_t} f\|_p \le C_p \gamma (1-t)^{-2} \|f\|_p$ for all $f \in \mathcal{A}^p$.

For $p = 1$ the Fubini theorem and (1.13) imply that

$$\begin{aligned} \|I_j\|_1 &\le t \left| \int_{x_j}^{x_{j+1}} a(te^{i\varrho}) d\varrho \right| \frac{\|K_{te^{ix_{j+1}}}\|_1}{(1-t)^2} \int_{D(te^{ix_{j+1}}, r)} |f(w)| dA(w) \\ &\le C\gamma \frac{\log(1/(1-t))}{(1-t)^2} \int_{\mathcal{D}_j} |f(w)| dA(w), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \|J_j\|_1 &\le \gamma \int_{x_j}^{x_{j+1}} |f'(te^{i\theta})| \|K_{te^{i\theta}}\|_1 d\theta \\ &\le C\gamma \frac{\log(1/(1-t))}{(1-t)^2} \int_{\mathcal{D}_j} (1-|w|^2) |f'(w)| dA(w). \end{aligned} \tag{2.6}$$

Using (1.12) we get that

$$\begin{aligned} \|L_j\|_1 &\le 2\gamma \int_{x_j}^{x_{j+1}} |f(te^{i\theta})| \|(1 - \cdot te^{i\theta})^{-3}\|_1 d\theta \\ &\le \frac{C\gamma}{(1-t)^2} \int_{\mathcal{D}_j} |f(w)| dA(w). \end{aligned} \tag{2.7}$$

Therefore $\|T_{ad\sigma_t} f\|_1 \le C\gamma (1-t)^{-2} \log(1/(1-t)) \|f\|_1$ for all $f \in \mathcal{A}^1$. □

3 Hardy classes and Toeplitz operators

Recall that the Hardy space h^p, $1 \le p < \infty$ consists of all the harmonic functions u in $\mathbb{D}$ such that

$$\sup_{t \in [0,1)} \left\{ \int_0^{2\pi} |u(te^{i\theta})|^p \frac{d\theta}{2\pi} \right\}^{1/p} < \infty.$$

If u is any function defined in $\mathbb{D}$, we denote by u_t the function given in $\overline{\mathbb{D}}$ as $u_t(z) = u(tz), t \in [0,1)$, $z \in \overline{\mathbb{D}}$.

Proposition 3.1. *Let $p \ge 1$. There exists $C > 0$ such that if $f \in L^p(S^1)$ and u is the harmonic extension of f on $\mathbb{D}$, then*

$$\sup_{0 \le t < 1} \|T_{(1-t)^{1+1/p} u d\sigma_t}\|_{S_p} \le C \|f\|_{L^p(S^1)}. \tag{3.1}$$

Moreover, if $f \ge 0$ then for t close to 1,

$$\|T_{(1-t)^{1+1/p} u d\sigma_t}\|_{S_p} \sim \|f\|_{L^p(S^1)}. \tag{3.2}$$

Proof. Let $p > 1$. To prove (3.1) assume first that $f \geq 0$. Notice that the continuity of $\mathcal{M}$ in $L^p(S^1)$ implies that

$$\|\mathcal{E}_\tau u\|_{L^p(tS^1)} \leq C\|f\|_{L^p(S^1)},\ \tau > 0, \tag{3.3}$$

hence (3.1) follows by (1.16). Indeed,

$$\begin{aligned}\|\mathcal{E}_\tau u\|_{L^p(tS^1)} &= t^{1/p}\|\mathcal{E}_\tau u_t\|_{L^p(S^1)}\| \leq t^{1/p}\|\mathcal{M}u_t\|_{L^p(S^1)} \\ &\leq C\|t^{1/p}u_t\|_{L^p(S^1)} \leq C\|f\|_{L^p(S^1)}.\end{aligned}$$

Hence by (1.16), $\|T_{(1-t)^{1+1/p}ud\sigma_t}\| \leq C\|f\|_{L^p(S^1)}$.

To prove (3.1) for a complex-valued function f, we write $f = f_1 - f_2 + i(f_3 - f_4)$, where each f_i is nonnegative, then we have $u = u_1 - u_2 - i(u_3 - u_4)$, where u_i is the Poisson integral of f_i. We have then

$$\sup_{0\leq t<1} \|T_{(1-t)^{1+1/p}ud\sigma_t}\|_{S_p} \leq C\sum_{i=1}^{4}\|f_i\|_{L^p(S^1)} \leq C\|f\|_{L^p(S^1)}.$$

To prove (3.2) notice that

$$\begin{aligned}\Big|\|\mathcal{E}_\tau u\|_{L^p(tS^1)} - \|\mathcal{E}_\tau u_t\|_{L^p(S^1)}\Big| &= (1-t^{1/p})\|\mathcal{E}_\tau u_t\|_{L^p(S^1)} \leq (1-t^{1/p})\|\mathcal{M}u_t\|_{L^p(S^1)} \\ &\leq C(1-t^{1/p})\|u_t\|_{L^p(S^1)} \leq C(1-t^{1/p})\|u\|_{h^p}.\end{aligned} \tag{3.4}$$

Thus,

$$\lim_{t\to 1}\Big|\|\mathcal{E}_\tau u\|_{L^p(tS^1)} - \|\mathcal{E}_\tau u_t\|_{L^p(S^1)}\Big| \to 0$$

uniformly in τ. Also

$$\begin{aligned}\|f - \mathcal{E}_\tau u_t\|_{L^p(S^1)} &\leq \|f - \mathcal{E}_\tau f\|_{L^p(S^1)} + \|\mathcal{E}_\tau(u_t - f)\|_{L^p(S^1)} \\ &\leq \|f - \mathcal{E}_\tau f\|_{L^p(S^1)} + C\|u_t - f\|_{L^p(S^1)},\end{aligned} \tag{3.5}$$

and since $|f - \mathcal{E}_\tau f| \leq |f| + \mathcal{M}f$ we see by the Lebesgue's dominated convergence theorem that $\|f - \mathcal{E}_\tau u_t\|_{L^p(S^1)}$ tends to 0 as $t \to 0$.

We conclude from (3.4) and (3.5) and the fact that $\|\mathcal{E}_\tau u\|_{L^p(tS^1)}$ is bounded by $C\|f\|_{L^p(S^1)}$ that

$$\begin{aligned}\lim_{t\to 1}\Big|\|\mathcal{E}_\tau u\|^p_{L^p(tS^1)} - \|f\|^p_{L^p(S^1)}\Big| &\leq C\lim_{t\to 1}\Big|\|\mathcal{E}_\tau u_t\|_{L^p(S^1)} - \|f\|_{L^p(S^1)}\Big| \\ &\leq C\lim_{t\to 1}\|\mathcal{E}_\tau u_t - f\|_{L^p(S^1)} = 0,\end{aligned}$$

uniformly in $\tau \in (0, 1-t)$. Since ψ_{1-t} is an approximate identity we finally have that

$$\lim_{t\to 1} t^{1/p'}\int_0^{1-t}\|\mathcal{E}_\tau u\|^p_{L^p(tS^1)}\psi_{1-t}(\tau)d\tau = \|f\|^p_{L^p(S^1)}.$$

Then (3.2) follows from (1.16). The case $p = 1$ can by handled in a similar way, easier, using (1.17) instead. □

Proof of Theorem 1.2. Consider $p > 1$ first. $a) \Rightarrow c)$ follows from Proposition 3.1. Next $c) \Rightarrow b)$ is obvious.

Now suppose that $b)$ holds. Then the same is true for u_s uniformly for $0 < s < 1$. In fact, we write

$$u_s(te^{i\theta}) = P_s * u_t(\theta), \tag{3.6}$$

where P_s is the Poisson kernel in $\mathbb{D}$. Then

$$\begin{aligned}\|\mathcal{E}_\tau(|u_s|)\|_{L^p(tS^1)} &\le \|\mathcal{E}_\tau(t^{1/p}|P_s * u_t|)\|_{L^p(S^1)}\\ &\le \|t^{1/p}P_s * \mathcal{E}_\tau(|u_t|)\|_{L^p(S^1)}\\ &\le \|t^{1/p}\mathcal{E}_\tau(|u_t|)\|_{L^p(S^1)}\\ &= \|\mathcal{E}_\tau(|u|)\|_{L^p(tS^1)}.\end{aligned}$$

Thus,

$$t^{p/p'}\int_0^{1-t}\|\mathcal{E}_\tau(|u_s|)\|^p_{L^p(tS^1)}\psi_{1-t}(\tau)d\tau \le L^p, \text{ for all } 0 < s < 1. \tag{3.7}$$

Now write

$$\begin{aligned}t^{p/p'}\int_0^{1-t}\|u_s\|^p_{L^p(tS^1)}\psi_{1-t}(\tau)d\tau &= t^{p/p'}\int_0^{1-t}\|\mathcal{E}_\tau(|u_s|)\|^p_{L^p(tS^1)}\psi_{1-t}(\tau)d\tau\\ &\quad + t^{p/p'}\int_0^{1-t}H(t,\tau)\psi_{1-t}(\tau)d\tau\end{aligned}$$

where $H(t,\tau) = \|u_s\|^p_{L^p(tS^1)} - \|\mathcal{E}_\tau(|u_s|)\|^p_{L^p(tS^1)}$, so that

$$t^{p/p'}\int_0^{1-t}\|u_s\|^p_{L^p(tS^1)}\psi_{1-t}(\tau)d\tau \le L^p + t^{p/p'}\int_0^{1-t}H(t,\tau)\psi_{1-t}(\tau)d\tau. \tag{3.8}$$

For s fixed, u_s and ∇u_s are bounded in $\mathbb{D}$ and so is $\mathcal{E}_\tau(|u_s|)$ uniformly in τ. Thus,

$$\begin{aligned}|\mathcal{E}_\tau(|u_s|)(\theta) - |u_s|(\theta)| &\le \frac{1}{2\tau}\int_{\theta-\tau}^{\theta+\tau}||u_s(te^{i\eta})| - |u_s(te^{i\theta})||d\eta\\ &\le \frac{1}{2\tau}\int_{\theta-\tau}^{\theta+\tau}|u_s(te^{i\eta}) - u_s(te^{i\theta})|d\eta\\ &\le \frac{C}{2\tau}\int_{\theta-\tau}^{\theta+\tau}|\eta-\theta|d\eta \le C\tau,\end{aligned}$$

and for $0 < \tau < 1-t$ we have

$$\begin{aligned}|H(t,\tau)| &\le C|\|\mathcal{E}_\tau(|u_s|)\|_{L^p(tS^1)} - \|u_s\|_{L^p(tS^1)}|\\ &\le C\|\mathcal{E}_\tau(|u_s|) - |u_s|\|_{L^p(tS^1)} \le C(1-t).\end{aligned}$$

Thus $H(t,\tau)$ tends to zero uniformly as $t \to 1^-$ and $0 < \tau < 1-t$. Taking the limit in (3.8) as $t \to 1$ we obtain by (1.16) that $\|u_s\|_{L^p(S^1)} \le L$ and hence $\|u\|_{h^p} \le L$. Hence $b) \Rightarrow a)$ and the proof of the theorem is complete for $p > 1$. For $p = 1$ the result is obvious by (1.17). □

We recall the definition of p-atom, see [1].

Definition 3.2. For $1/2 < p \leq 1$ we say that $a : S^1 \to \mathbb{C}$ is an p-atom in S^1 if either $a(t) \equiv 1/(2\pi)$ or

a) a is supported in an interval $I \subset S^1$,

b) $\|a\|_\infty \leq 1/|I|^{1/p}$, where $|I|$ is the arc length measure of I,

c) $\int_{S^1} a d\sigma = 0$.

Notice that if a is a p-atom with $1/2 < p \leq 1$, and $a \neq 1/(2\pi)$ then for every $\varphi \in C^\infty(S^1)$

$$\left|\int_I a\varphi d\theta\right| \leq \int_I \left|a(e^{i\theta})\left(\varphi(e^{i\theta}) - \varphi(\zeta)\right)\right| d\theta \leq |I|^2 \|a\|_\infty \|\varphi'\|_\infty \leq 2\pi \|\varphi'\|_\infty, \tag{3.9}$$

where ζ is a point in I.
Denote by H^p_{at} the space of distributions in $\mathcal{D}'(S^1)$ of the form

$$h = \sum_{i=1}^{\infty} \lambda_i a_i, \tag{3.10}$$

where the complex sequence (λ_i) satisfies $\sum |\lambda_i|^p < \infty$, and each a_i is a p-atom, and we assume $1/2 \leq p \leq 1$. By (3.9) we have that the series converges in $\mathcal{D}'(S^1)$. Denote

$$\|h\|_{H^p_{at}} = \inf \left\{ \left(\sum_{i=1}^{\infty} |\lambda_i|^p \right)^{1/p} \right\},$$

where the infimum is taken over the representations (3.10).

If $h = \sum_{i=1}^{\infty} \lambda_i a_i \in H^p_{at}$, by (1.4) we have that

$$T_{hd\sigma_t} f(z) = \sum_{i=1}^{\infty} \lambda_i T_{a_i d\sigma_t} f(z). \tag{3.11}$$

Notice that the convergence of the series in $\mathcal{D}'(S^1)$ implies the pointwise convergence of $\sum_{i=1}^{\infty} \lambda_i T_{a_i d\sigma_t} f(z)$ and in particular the series in (3.11) does not depend on the representation of h.

Proof of the Theorem 1.3. It suffices to prove that the estimates hold with the same constant for any p-atom. Consider first a p-atom a different to the constant $1/(2\pi)$. Let I an interval containing its support and satisfying (b) in the definition of p-atom. Let ζ be the center of I, so we can write

$$\begin{aligned} T_{ad\sigma_t} f(z) =& t \int_0^{2\pi} a(\theta) \left[\frac{f(te^{i\theta})}{(1 - zte^{-i\theta})^2} - \frac{f(t\zeta)}{(1 - zt\overline{\zeta})^2} \right] d\theta \\ =& S_1 f(z) + S_2 f(z), \end{aligned}$$

where

$$S_1 f(z) = t \int_I a(\theta) \frac{f(te^{i\theta}) - f(t\zeta)}{(1 - zte^{-i\theta})^2} d\theta,$$

$$S_2f(z) = tf(t\zeta)\int_I a(\theta)\left[\frac{1}{(1-zte^{-i\theta})^2} - \frac{1}{(1-z\overline{t\zeta})^2}\right]d\theta.$$

To estimate $S_1f(z)$ we notice that for $\theta \in I$, $|f(te^{i\theta}) - f(t\zeta)| \le t|I||f'(te^{i\xi})|$, where ξ lies between θ and the argument of ζ. Using (1.11) we obtain

$$|S_1f(z)| \le \frac{Ct^2|I|^{1-1/p}\|f\|_q}{(1-t)^{2/q+1}}\int_0^{2\pi}\frac{\chi_I(\theta)}{|1-zte^{i\theta}|^2}d\theta. \tag{3.12}$$

Suppose first that $1/2 < p < 1$ so that $1/p = 1+s$, with $s \in (0,1)$. Then by Holder's inequality for the conjugate exponents $1/s$ and $1/(1-s)$, and using (1.7) we obtain

$$\begin{aligned}|S_1f(z)| &\le \frac{Ct^2|I|^{-s}\|f\|_q}{(1-t)^{2/q+1}}|I|^s\left(\int_0^{2\pi}\frac{1}{|1-zte^{i\theta}|^{2/(1-s)}}d\theta\right)^{1-s} \\ &\le \frac{Ct^2}{(1-t)^{2/q+1}}\frac{1}{(1-|z|t)^{1/p}}\|f\|_q.\end{aligned} \tag{3.13}$$

Now by (1.5) it follows that

$$\|S_1f\|_q \le \frac{C}{(1-t)^{1/q+1/p+1}}\|f\|_q, \quad q > p \tag{3.14}$$

and

$$\|S_1f\|_p \le \frac{C}{(1-t)^{2/p+1}}\left(\log(1/(1-t))\right)^{1/p}\|f\|_p. \tag{3.15}$$

On the other hand

$$\left|\frac{1}{(1-z\overline{te^{i\theta}})^2} - \frac{1}{(1-zt\overline{\zeta})^2}\right| = \left|\int_\Gamma \frac{d}{dw}\frac{1}{(1-\overline{z}w)^2}dw\right| = \left|\int_\theta^{\theta_0}\frac{2t\overline{z}i}{(1-\overline{z}te^{i\varphi})^3}d\varphi\right|, \tag{3.16}$$

where Γ es the arc in tS^1 connecting $te^{i\theta}$ and $\zeta = te^{i\theta_0}$.

Then using Holder's inequality as in the previous estimate, Jensen's inequality and (1.11) we have

$$\begin{aligned}|S_2f(z)| \le& t\|f\|_{L^\infty(tS^1)}\|a\|_\infty\int_I\left|\frac{1}{(1-zte^{-i\theta})^2} - \frac{1}{(1-z\overline{\zeta})^2}\right|d\theta \\ \le&\frac{t\|f\|_q|I|^{-1/p+s}}{(1-t)^{2/q}}\left(\int_I\left|\frac{1}{(1-zte^{-i\theta})^2} - \frac{1}{(1-zt\overline{\zeta})^2}\right|^{1/(1-s)}d\theta\right)^{1-s} \\ \le&\frac{t^2\|f\|_q\||I|^{-1/p+s}}{(1-t)^{2/q}}\left(\int_I\left|\int_\theta^{\theta_0}\frac{2\overline{z}i}{(1-\overline{z}te^{i\varphi})^3}d\varphi\right|^{1/(1-s)}d\theta\right)^{1-s} \\ \le&\frac{t^2\|f\|_q\||I|^{-1/p+s}}{(1-t)^{2/q}}\left(\int_I|\theta-\theta_0|^{1/(1-s)-1}\int_0^{2\pi}\frac{1}{|1-\overline{z}te^{i\varphi}|^{3/(1-s)}}d\varphi d\theta\right)^{1-s} \\ \le&\frac{Ct^2\|f\|_q}{(1-t)^{2/q}(1-|z|t)^{3-(1-s)}} = \frac{Ct^2\|f\|_q}{(1-t)^{2/q}(1-|z|t)^{1+1/p}}.\end{aligned} \tag{3.17}$$

Hence using (1.5) we obtain for $1/2 < p < 1$,

$$\|S_2 f\|_q \leq \frac{Ct^2\|f\|_q}{(1-t)^{1/p+1/q+1}}, \tag{3.18}$$

that together with (3.14) and (3.15) proves that there exist $C > 0$ such that

$$\|T_{ad\sigma_t}\|_{\mathcal{L}(\mathcal{A}^q)} \leq \frac{C}{(1-t)^{1/p+1/q+1}}$$

and

$$\|T_{ad\sigma_t}\|_{\mathcal{L}(\mathcal{A}^p)} \leq \frac{C}{(1-t)^{2/p+1}} \left(\log(1/(1-t))\right)^{1/p}$$

for every p-atom a different to the constant $1/2\pi$ and constant independent of a.

For the case $p = 1$ we have

$$|T_{ad\sigma_t} f(z)| \leq \frac{t}{|I|} \int_I \frac{|f(te^{i\theta})|}{|1 - zte^{i\theta}|^2} d\theta \leq \frac{Ct}{|I|(1-t)^{2/q}} \int_I \frac{\|f\|_q}{|1 - zte^{i\theta}|^2} d\theta.$$

When $q > 1$ the Minkowski integral inequality implies that

$$\|T_{ad\sigma_t} f\|_q \leq \frac{Ct|I|\|f\|_q}{|I|(1-t)^{2/q}(1-t)^{2-2/q}} = \frac{Ct\|f\|_q}{(1-t)^2}$$

for all $f \in \mathcal{A}^q$.

When $q = 1 = p$ Fubini's theorem and (1.13) yield

$$\|T_{ad\sigma_t} f\|_1 \leq \frac{Ct\log(1/(1-t))\|f\|_1}{(1-t)^2}.$$

The proof of the theorem is complete for non-constant atoms.

Now for $a = 1/2\pi$ we have by Theorem 1.1 that

$$\|T_{(1/2\pi)d\sigma_t}\|_{\mathcal{L}(\mathcal{A}^q)} \leq \frac{C}{1-t}, \quad q > 1,$$

and

$$\|T_{(1/2\pi)d\sigma_t}\|_{\mathcal{L}(\mathcal{A}^1)} \leq \frac{C}{1-t} \log(1/(1-t)).$$

which imply the estimates of the theorem. The theorem follows expanding any $h \in H^p_{at}$ as $h = \sum_{i=1}^\infty \lambda_i a_i$, with a_i a p-atom and $\sum_{i=1}^\infty |\lambda_i|^p < \infty$.

□

For $0 < p \leq 1$, we say that a harmonic function u in $\mathbb{D}$ belongs to the Hardy space $H^p(\mathbb{D})$ if the maximal function $Mu(\theta) = \sup_{z\in\Gamma_\theta} |u(z)| \in L^p(S^1)$, where Γ_θ is the Stoltz region in $\mathbb{D}$ with vertex in $e^{i\theta}$. For $1/2 < p \leq 1$ a function u belongs to $H^p(\mathbb{D})$ if and only if it is the Poisson integral of some $f \in H^p_{at}$, moreover in this case $u_t \in H^p_{at}$ for every $0 \leq t < 1$. Then from Theorem 1.3 we have

Corollary 3.3. *If* $1/2 < p \leq 1$*, there exists* $C > 0$ *such that*

$$\|T_{u_t}\| \leq \frac{C\|u\|_{H^p(\mathbb{D})}}{(1-t)^{3/2+1/p}},$$

for every $u \in h^p(\mathbb{D})$*.*

Example 3.4. The function

$$f(z) = \frac{1}{(1-z)^C}, \, C < 1,$$

belongs to the Hardy space h^1 and it is known that $h^1 \cap hol(\mathbb{D}) \subset H^1(\mathbb{D})$. Then $\{f(te^{i(\cdot)}) : 0 \leq t < 1)\} \subset H^1_{at}$.
We consider the symbol $\varphi = f(te^{i(\cdot)})$, so we have

$$\begin{aligned}
T_{\varphi d\sigma_t}(1)(z) &= \int_0^{2\pi} \frac{td\theta}{(1-zte^{-i\theta})^2(1-te^{i\theta})^C} \\
&= \int_0^{2\pi} \left(\sum_{n=0}^{\infty}(n+1)t^n z^n e^{-in\theta}\right)\left(\sum_{n=0}^{\infty} \frac{\Gamma(m+C)}{m!\Gamma(C)} t^m e^{im\theta}\right) td\theta \\
&= t\sum_{n=0}^{\infty} t^{2n}(n+1)\frac{\Gamma(n+C)}{\Gamma(C)n!} z^n \\
&= t\sum_{n=0}^{\infty} t^{2n}(n+1)^{3/2}\frac{\Gamma(n+C)}{\Gamma(C)n!} e_n(z),
\end{aligned}$$

where $e_n(z) = (n+1)^{-1/2}z^n$.
The Stirling's formula implies that

$$\begin{aligned}
\|T_{\varphi d\sigma_t}\|^2_{\mathcal{L}(A^2)} &\geq t^2 \sum_{n=0}^{\infty} t^{4n}(n+1)^3 \left(\frac{\Gamma(n+C)}{n!\Gamma(C)}\right)^2 \\
&\sim t^2 \sum_{n=0}^{\infty} t^{4n}(n+1)^{1+2C} \\
&\sim t^2 \sum_{n=0}^{\infty} t^{4n} \frac{\Gamma(n+2+2C)}{n!} \\
&\sim \frac{t^2}{(1-t^2)^{2+2C}}.
\end{aligned}$$

Thus,

$$\|T_{\varphi d\sigma_t}\|_{\mathcal{L}(\mathcal{A}^2)} \geq \frac{At}{(1-t)^a}$$

for all $a < 2$, where A depends on a.

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