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D F S G B
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T I *
Mathematical Institute
Tohoku University
Sendai 980-8578 JAPAN USA

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Abstract

We investigate the relation between Besov spaces generated by the Dirichlet Laplacian and the Neumann Laplacian in one space dimension from the view point of the boundary value of functions. Derivatives on spaces with such boundary conditions are defined, and it is proved that the derivative operator is isomorphic from one to the other.

AMS Subject Classification: 42B35, 42B37

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1 Introduction

There are several methods to define function spaces on general domains such as Sobolev spaces, Besov spaces, etc. If the boundary of the domain is bounded and smooth, the basic notion is available; the restriction method of the function on $\mathbb{R}^d$ to the domain, the zero extension to the outside of the domain, and certain intrinsic characterization (see [6–8, 10–15]). In the book by Triebel [10], it is indicated that function spaces becomes different depending on the boundary value of functions by comparing the space defined by the restriction of functions on $\mathbb{R}^d$ with the completion of the set of smooth functions with compact support. Recently, the study of function spaces generated by the operators are also known, and we refer the papers by Bui-Duong-Yan [1], Kerkyacharian-Petrushev [5] (see also [4]) and references therein, there, one can understand that function spaces would be different from each space depending on operators. In this paper, we focus on the Besov

*E-mail address: t-iwabuchi@m.tohoku.ac.jp

spaces generated by the Dirichlet Laplacian and the Neumann Laplacian in one space dimension to make the relation of the spaces clear by showing that the derivative operators define isomorphic mappings from one to the other. This is an aspect of functions approximated under the Dirichlet and the Neumann boundary conditions. The motivation comes from the application to the nonlinear partial differential equations with the Dirichlet and Neumann boundary conditions.

Let A_D, A_N be the Dirichlet Laplacian $-\partial_x^2|_D$, the Neumann Laplacian $-\partial_x^2|_N$, respectively. We note that A_D, A_N are operators on $L^2(\mathbb{R})$ initially, while they are regarded as ones on Besov spaces, some spaces of distributions based on the uniform boundedness of scaled multipliers $\varphi(\theta A_D)$, $\varphi(\theta A_N)$ in $L^1(\Omega)$. (see e.g. [4]).

Let us recall the deinition of Besov spaces along the paper [4] (see also [9]). Let $\phi_0(\cdot) \in C_0^\infty(\mathbb{R})$ be a non-negative function on $\mathbb{R}$ such that

$$\operatorname{supp}\phi_0 \subset \{\lambda \in \mathbb{R} | 2^{-1} \le \lambda \le 2\}, \quad \sum_{j\in\mathbb{Z}} \phi_0(2^{-j}\lambda) = 1 \quad \text{for } \lambda > 0,$$

and $\{\phi_j\}_{j\in\mathbb{Z}}$ is defined by

$$\phi_j(\lambda) = \phi_0(2^{-j}\lambda) \quad \text{for } \lambda \in \mathbb{R}.$$

The following is the definition of the test function spaces and their duals generated by the opeartors $A = A_D, A_N$.

Definition (Test function spaces and distributions).

(i) (Linear topological spaces $\mathcal{X}(\Omega, A)$ and $\mathcal{X}'(\Omega, A)$). A linear topological space $\mathcal{X}_V(\Omega, A)$ is defined by

$$\mathcal{X}(\Omega, A) := \{f \in L^1(\Omega) \cap \mathcal{D}(A) \,\big|\, A^M f \in L^1(\Omega) \cap \mathcal{D}(A) \text{ for all } M \in \mathbb{N}\}$$

equipped with the family of semi-norms $\{p_{A,M}(\cdot)\}_{M=1}^\infty$ given by

$$p_{A,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{N}} 2^{Mj} \|\phi_j(\sqrt{A})f\|_{L^1(\Omega)}.$$

$\mathcal{X}'(\Omega, A)$ denotes the topological dual of $\mathcal{X}(\Omega, A)$.

(ii) (Linear topological spaces $\mathcal{Z}(\Omega, A)$ and $\mathcal{Z}'(\Omega, A)$). A linear topological space $\mathcal{Z}(\Omega, A)$ is defined by

$$\mathcal{Z}(\Omega, A) := \Big\{f \in \mathcal{X}(\Omega, A) \,\Big|\, \sup_{j\le 0} 2^{M|j|} \big\|\phi_j(\sqrt{A})f\big\|_{L^1(\Omega)} < \infty \text{ for all } M \in \mathbb{N}\Big\}$$

equipped with the family of semi-norms $\{q_{A,M}(\cdot)\}_{M=1}^\infty$ given by

$$q_{A,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{Z}} 2^{M|j|} \|\phi_j(\sqrt{A})f\|_{L^1(\Omega)}.$$

$\mathcal{Z}'(\Omega, A)$ denotes the topological dual of $\mathcal{Z}(\Omega, A)$.

Now, we define the homogeneous Besov spaces as follows.

Definition. For $s \in \mathbb{R}$ and $1 \le p,q \le \infty$, $\dot{B}^s_{p,q}(A)$ is defined by

$$\dot{B}^s_{p,q}(A) := \{f \in \mathcal{Z}'(\Omega,A) \,|\, \|f\|_{\dot{B}^s_{p,q}(A)} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(A)} := \Big\| \{2^{sj}\|\phi_j(\sqrt{A})f\|_{L^p(\Omega)}\}_{j\in\mathbb{Z}} \Big\|_{\ell^q(\mathbb{Z})}.$$

Here we note that the fundamental properties of spaces $\mathcal{X}(\Omega,A)$, $\mathcal{Z}(\Omega,A)$, their duals and Besov spaces are established in the paper [4], such as completeness, duality, embedding, lifting properties as well as the case when $\Omega = \mathbb{R}^d$.

We will prove that the derivative operator ∂_x maps from test function spaces $\mathcal{Z}(\Omega,A_D)$, $\mathcal{Z}(\Omega,A_N)$ to $\mathcal{Z}(\Omega,A_N)$, $\mathcal{Z}(\Omega,A_D)$, respectively. So we define ∂_x on spaces of distributions as follows.

Definition (Derivatives in the sense of distributions).

(i) For any $f \in \mathcal{Z}'(\Omega,A_D)$, we define $\partial_x f$ as an element of $\mathcal{Z}'(\Omega,A_N)$ by

$$_{\mathcal{Z}'(\Omega,A_N)}\langle \partial_x f, g\rangle_{\mathcal{Z}(\Omega,A_N)} := -_{\mathcal{Z}'(\Omega,A_D)}\langle f, \partial_x g\rangle_{\mathcal{Z}(\Omega,A_D)} \quad \text{for any } g \in \mathcal{Z}(\Omega,A_N). \tag{1.1}$$

(ii) For any $f \in \mathcal{Z}'(\Omega,A_N)$, we define $\partial_x f$ as an element of $\mathcal{Z}'(\Omega,A_D)$ by

$$_{\mathcal{Z}'(\Omega,A_D)}\langle \partial_x f, g\rangle_{\mathcal{Z}(\Omega,A_D)} := -_{\mathcal{Z}'(\Omega,A_N)}\langle f, \partial_x g\rangle_{\mathcal{Z}(\Omega,A_N)} \quad \text{for any } g \in \mathcal{Z}(\Omega,A_D). \tag{1.2}$$

Based on the above definition, the derivatives of higher order are also introduced. Namely, for $f \in \mathcal{Z}'(\Omega,A_D)$ and odd numbers $\alpha = 1,3,\cdots$, $\partial_x^\alpha f$ is defined by

$$_{\mathcal{Z}'(\Omega,A_N)}\langle \partial_x^\alpha f, g\rangle_{\mathcal{Z}(\Omega,A_N)} := (-1)^\alpha\, _{\mathcal{Z}'(\Omega,A_D)}\langle f, \partial_x^\alpha g\rangle_{\mathcal{Z}(\Omega,A_D)} \quad \text{for any } g \in \mathcal{Z}(\Omega,A_N).$$

For even numvers $\alpha = 2,4,\cdots$, it is

$$_{\mathcal{Z}'(\Omega,A_D)}\langle \partial_x^\alpha f, g\rangle_{\mathcal{Z}(\Omega,A_D)} := (-1)^\alpha\, _{\mathcal{Z}'(\Omega,A_D)}\langle f, \partial_x^\alpha g\rangle_{\mathcal{Z}(\Omega,A_D)} \quad \text{for any } g \in \mathcal{Z}(\Omega,A_D).$$

Similar definition of derivatives for $f \in \mathcal{Z}'(\Omega,A_N)$ is also introduced by replacing the role of A_D and A_N with each other.

The following is the main result of this paper.

Theorem 1.1. *Let $\Omega = (0,1)$ or $(0,\infty)$. Then the following hold.*

(i) *The operator ∂_x is a continuous linear operator from $\mathcal{Z}(\Omega,A_D)$ to $\mathcal{Z}(\Omega,A_N)$ and from $\mathcal{Z}'(\Omega,A_D)$ to $\mathcal{Z}'(\Omega,A_N)$ defined by* (1.1).

(ii) *For any $s \in \mathbb{R}$ and $1 \le p,q \le \infty$, the operator ∂_x defines a continuous linear operator from $\dot{B}^s_{p,q}(A_D)$ to $\dot{B}^{s-1}_{p,q}(A_N)$ and*

$$C^{-1}\|f\|_{\dot{B}^s_{p,q}(A_D)} \le \|\partial_x f\|_{\dot{B}^{s-1}_{p,q}(A_N)} \le C\|f\|_{\dot{B}^s_{p,q}(A_D)} \quad \textit{for any } f \in \dot{B}^s_{p,q}(A_D).$$

(iii) *The above assertions* (i) *and* (ii) *also hold by replacing* A_D *and* A_N *with each other.*

Let us give two comments. When we consider derivatives in the whole space $\mathbb{R}^d$ case, they are well-defined as maps from the space of tempered distributions to itself, where we do not have to change the test function space. This also applies to derivatives in the sense of distribution on general domains, where the test function space consists of smooth functions with the compact support. On the other hand, if we consider to analize in function spaces with several boundary conditions such as Dirichlet or Neumann types, we need to choose suitable boundary condition for derivatives from Theorem 1.1, which would be the main novelty of this paper due to the definition of derivatives above. In higher dimensional case, there seems to appear a difficulty to extend the above theorem. In fact, let us consider the half space $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d \,|\, x_d > 0\}$ for instance. The derivative ∂_{x_d} plays the same roll as the derivative ∂_x in 1D from the view point of boundary conditions of the Dirichlet Laplacian and the Neumann Laplacian, but ∂_{x_d} concerns with only the smoothness in the x_n direction only. Hence one can not consider ∂_{x_n} from $\dot{B}^s_{p,q}$ to $\dot{B}^{s-1}_{p,q}$ similarly to (ii) in Theorem 1.1, while only the corresponding second inequality would be proved.

This paper is organized as follows. In section 2, we prove that the derivative ∂_x is considered as maps from one test function space to the other. In section 3, we prove Theorem 1.1.

2 Preliminary

Let $G_D(t,x,y)$ and $G_N(t,x,y)$ be the kernels of the Dirichlet Laplacian and the Neumann Laplacian on the half line $(0,\infty)$, respectively, namely,

$$
\begin{aligned}
G_D(t,x,y) &:= (4\pi t)^{-\frac{1}{2}} \frac{e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}}{2}, \qquad t > 0, x \in (0,\infty),\\
G_N(t,x,y) &:= (4\pi t)^{-\frac{1}{2}} \frac{e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}}{2}, \qquad t > 0, x \in (0,\infty).
\end{aligned}
$$

Since $G_D(t,x,y)$ and $G_N(t,x,y)$ satisfy the Gaussian upper bounds and the gradient estimates in $L^\infty(\mathbb{R}^d)$, we can apply the argument in [2,3] (see also [9]) to veryfy the spectral multiplier theorem together with gradient estimates. As for the case when $\Omega = (0,1)$, we also see that the kernels in the case when $\Omega = (0,1)$ satisfy the Gaussian upper bounds and the gradient estimates thanks to the series given by sine or cosine functions. It should be noted here that the kernel for the Neumann Laplacian does not satisfy the Gaussian upper bounds for large t, but it is satisfied if we consider functions without the spectrum at zero. So we can utilize the Gaussian upper bounds also for the heat kernel for the Neumann Laplacian, since we consider the homogeneous spaces.

Lemma 2.1. *Let* $\Omega = (0,1)$ *or* $(0,\infty)$, $1 \le p \le \infty$. *Then*

$$\sup_{j\in\mathbb{Z}} \|\phi_j(\sqrt{A_D})\|_{L^p\to L^p} < \infty, \quad \sup_{j\in\mathbb{Z}} \|\phi_j(\sqrt{A_N})\|_{L^p\to L^p} < \infty, \tag{2.1}$$

$$\sup_{j\in\mathbb{Z}} 2^{-j}\|\partial_x\phi_j(\sqrt{A_D})\|_{L^p\to L^p} < \infty, \quad \sup_{j\in\mathbb{Z}} 2^{-j}\|\partial_x\phi_j(\sqrt{A_N})\|_{L^p\to L^p} < \infty, \tag{2.2}$$

$$\sup_{j\in\mathbb{Z}} 2^{-j}\|\phi_j(\sqrt{A_D})\partial_x\|_{L^p\to L^p} < \infty, \quad \sup_{j\in\mathbb{Z}} 2^{-j}\|\phi_j(\sqrt{A_N})\partial_x\|_{L^p\to L^p} < \infty. \tag{2.3}$$

The following lemma is essential in this paper, which is on the derivativite operator as a mapping on spaces of test functions.

Lemma 2.2. *Let* $\Omega = (0,1)$ *or* $(0,\infty)$. *The operator* ∂_x *is a continuous linear operator from* $\mathcal{Z}(\Omega, A_D)$ *to* $\mathcal{Z}(\Omega, A_N)$. *Also,* ∂_x *is a continuous linear operator from* $\mathcal{Z}(\Omega, A_N)$ *to* $\mathcal{Z}(\Omega, A_D)$.

Proof. If $\Omega = (0,1)$, it is easy to prove, since any $f \in \mathcal{Z}(\Omega, A_D)$ satisfies

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \text{and} \quad \partial_x f(x) = \sum_{n=1}^{\infty} n a_n \cos(n\pi x) \in \mathcal{Z}(\Omega, A_N),$$

with $a_n = 2\int_0^1 f(y)\sin(n\pi y)\,dy$, and any $f \in \mathcal{Z}(\Omega, A_D)$ satisfies

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(n\pi x) \quad \text{and} \quad \partial_x f(x) = \sum_{n=1}^{\infty} (-n a_n) \sin(n\pi x) \in \mathcal{Z}(\Omega, A_D),$$

with $a_n = 2\int_0^1 f(y)\cos(n\pi y)\,dy$, which are assured by $\sup_{n\in\mathbb{N}} |n|^M |a_n| < \infty$ for any $M \in \mathbb{N}$ thanks to $f \in \mathcal{Z}(\Omega, A_D)$, $\mathcal{Z}(\Omega, A_D)$, respectively.

We turn to prove the case when $\Omega = (0,\infty)$. Let us prove the former part of the assertion. For any $f \in \mathcal{Z}(\Omega, A_D)$, we have from $\partial_x^{2M} f \in H_0^1(\Omega)$ $(M \in \mathbb{N})$ that

$$\partial_x \partial_x^{2(M-1)} f \in D(A_N), \quad A_N^M \partial_x f = (-\partial_x^2)^M \partial_x f = \partial_x (-\partial_x^2)^M f = \partial_x A_D^M f, \quad M = 1, 2, \cdots.$$

On each norm of $\partial_x f$ in the definition of $q_{A_D,M}(f)$, the above identity and the gradient estimates (2.2) yield that

$$\|\partial_x f\|_{L^1} \le \sum_{k\in\mathbb{Z}} \|\partial_x \phi_k(\sqrt{A_D}) f\|_{L^1} \le C \sum_{k\in\mathbb{Z}} 2^k \|\phi_k(\sqrt{A_D}) f\|_{L^1} \le C \sum_{k\in\mathbb{Z}} 2^k 2^{-2|k|} q_{A_D,2}(f),$$

for $j \ge 0$

$$\begin{aligned}
2^{Mj}\Big\|\phi_j(\sqrt{A_N})\partial_x \sum_{k\in\mathbb{Z}} \phi_k(\sqrt{A_D}) f\Big\|_{L^1} \le& 2^{Mj} \sum_{k\in\mathbb{Z}} \|\phi_j(\sqrt{A_N}) A_N^{-M} A_N^M \partial_x \phi_k(\sqrt{A_D}) f\|_{L^1} \\
\le& C 2^{Mj} 2^{-2Mj} \sum_{k\in\mathbb{Z}} \|\partial_x A_D^M \phi_k(\sqrt{A_D}) f\|_{L^1} \\
\le& C \sum_{k\in\mathbb{Z}} 2^{(1+2M)k} \|\phi_k(\sqrt{A_D}) f\|_{L^1} \\
\le& C q_{A_D, 2+2M}(f),
\end{aligned} \tag{2.4}$$

and for $j \leq 0$

$$\begin{aligned}
2^{M|j|}\Big\|\phi_j(\sqrt{A_N})\partial_x \sum_{k\in\mathbb{Z}}\phi_k(\sqrt{A_D})f\Big\|_{L^1} =&2^{M|j|}\sum_{k\in\mathbb{Z}}\|\phi_j(\sqrt{A_N})\partial_x A_D^M A_D^{-M}\phi_k(\sqrt{A_D})f\|_{L^1}\\
=&2^{M|j|}\sum_{k\in\mathbb{Z}}\|\phi_j(\sqrt{A_N})A_N^M\partial_x A_D^{-M}\phi_k(\sqrt{A_D})f\|_{L^1}\\
\leq& C2^{M|j|}2^{2Mj}\sum_{k\in\mathbb{Z}}\|\partial_x A_D^{-M}\phi_k(\sqrt{A_D})f\|_{L^1}\\
\leq& C\sum_{k\in\mathbb{Z}}2^{(1-2M)k}\|\phi_k(\sqrt{A_D})f\|_{L^1}\\
\leq& Cq_{A_D,2M}(f).
\end{aligned} \tag{2.5}$$

We deduce from the above three estimates that $q_{A_N,M}(\partial_x f) \leq Cq_{A_D,2+2M}(f)$, which proves the former part of Lemma 2.2 for $\Omega = (0,\infty)$.

As for the latter part of Lemma 2.2 for $\Omega = (0,\infty)$, for any $f \in \mathcal{Z}(\Omega, A_N)$, it follows from $\partial_x^{2M-1} f \in H_0^1(\Omega)$ $(M \in \mathbb{N})$ that

$$\partial_x^{2M-1} f \in D(A_D), \quad A_D^M \partial_x f = (-\Delta)^M \partial_x f = \partial_x(-\Delta)^M f = \partial_x A_N^M f, \quad M = 1,2,\cdots.$$

Hence it is possible to apply the previous argument by replacing A_D and A_N with each other. We complete the proof of Lemma 2.2. □

3 Proof of Theorem 1.1

We prove Theorem 1.1 in this section. We omit the proof of (iii), since it is proved in the analogous way to those for (i) and (ii).

Proof of (i) in Theorem 1.1. It is possible to understand that Lemma 2.2 yields that the definition (1.1) makes sense. We finish the proof. □

Proof of (ii) in Theorem 1.1. Let $f \in \dot{B}^s_{p,q}(A_D)$. We first prove that

$$\partial_x f \in \dot{B}^{s-1}_{p,q}(A_N) \quad \text{and} \quad \|\partial_x f\|_{\dot{B}^{s-1}_{p,q}(A_N)} \leq C\|f\|_{\dot{B}^s_{p,q}(A_D)}. \tag{3.1}$$

Since $f \in \mathcal{Z}'(\Omega, A_D)$ and $\partial_x f \in \mathcal{Z}'(\Omega, A_N)$, it suffices to prove the above inequality. It follows from the partition of the identity of f in $\mathcal{Z}'(\Omega, A_D)$ that

$$\|\partial_x f\|_{\dot{B}^{s-1}_{p,q}(A_N)} \leq \Big\{\sum_{j\in\mathbb{Z}}\Big(2^{(s-1)j}\sum_{k\in\mathbb{Z}}\|\phi_j(\sqrt{A_N})\partial_x\phi_k(\sqrt{A_D})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}.$$

We divide into two cases of $k \leq j$ and $k \geq j$. If $k \leq j$, we have from the similar argument to (2.4) that

$$\begin{aligned}
2^{(s-1)j}\|\phi_j(\sqrt{A_N})\partial_x\phi_k(\sqrt{A_D})f\|_{L^p} \leq& C2^{(s-1)j}2^{-2Mj}\|\partial_x A_D^M\phi_k(\sqrt{A_D})f\|_{L^p}\\
\leq& C2^{(s-1-2M)j}2^{(-s+1+2M)k}2^{sk}\|\phi_k(\sqrt{A_D})f\|_{L^p}.
\end{aligned}$$

For any $M \in \mathbb{N}$ with $1+2M > s$, the above inequality and the transformation $k = j-k'$ yield that

$$
\begin{aligned}
&\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{(s-1)j}\sum_{k\le j}\|\phi_j(\sqrt{A_N})\partial_x\phi_k(\sqrt{A_D})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k'\ge 0}2^{(s-1-2M)j}2^{(-s+1+2M)(j-k')}2^{s(j-k')}\|\phi_{j-k'}(\sqrt{A_D})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
=& C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k'\ge 0}2^{(s-1-2M)k'}2^{s(j-k')}\|\phi_{j-k'}(\sqrt{A_D})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\Big(\sum_{k'\ge 0}2^{(s-1-2M)k'}\Big)\|f\|_{\dot{B}^s_{p,q}}.
\end{aligned}
$$

If $k \ge j$, we also have the similar argument to (2.5) that

$$
\begin{aligned}
2^{(s-1)j}\|\phi_j(\sqrt{A_N})\partial_x\phi_k(\sqrt{A_D})f\|_{L^p} \le& C2^{(s-1)j}2^{2Mj}\|\partial_x A_D^{-M}\phi_k(\sqrt{A_D})f\|_{L^p}\\
\le& C2^{(s-1+2M)j}2^{(-s+1-2M)k}2^{sk}\|\phi_k(\sqrt{A_D})f\|_{L^p}.
\end{aligned}
$$

For any $M \in \mathbb{N}$ with $2M > 1-s$, the above estimate and the transformation $k = j+k'$ enables us to get that

$$
\begin{aligned}
&\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{(s-1)j}\sum_{k\ge j}\|\phi_j(\sqrt{A_N})\partial_x\phi_j(\sqrt{A_D})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k'\ge 0}2^{(s-1+2M)j}2^{(-s+1-2M)(j+k')}2^{s(j+k')}\|\phi_{j+k'}(\sqrt{A_D})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
=& C\Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{k'\ge 0}2^{(-s+1-2M)k'}2^{s(j+k')}\|\phi_{j+k'}(\sqrt{A_D})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\
\le& C\Big(\sum_{k'\ge 0}2^{(-s+1-2M)k'}\Big)\|f\|_{\dot{B}^s_{p,q}}.
\end{aligned}
$$

Therefore, we obtain the inequality in (3.1), which also prove the latter inequality in (ii).

We turn to prove the former inequality in (ii). For this purpose, we use a part of the assertion (iii) which corresponds to (3.1), namely,

$$
\partial_x g \in \dot{B}^{s-2}_{p,q}(A_D) \quad \text{and} \quad \|\partial_x g\|_{\dot{B}^{s-2}_{p,q}(A_D)} \le C\|g\|_{\dot{B}^{s-1}_{p,q}(A_N)} \quad \text{for any } g \in \dot{B}^{s-1}_{p,q}(A_N). \tag{3.2}
$$

By following the proof of (3.1), we get the above assertion (3.2) although we omit the detail. We already know $\partial_x f \in \dot{B}^{s-1}_{p,q}(A_N)$ by (3.1) and it follows from $f = A_D^{-1}A_D f$, the lifting property $A_D^{-1} : \dot{B}^{s-2}_{p,q}(A_D) \to \dot{B}^s_{p,q}(A_D)$ (see Proposition 3.2 (ii) in [4]) and the above (3.2) for $g = \partial_x f$ that

$$
\|f\|_{\dot{B}^s_{p,q}(A_D)} \le C\|A_D f\|_{\dot{B}^{s-2}_{p,q}(A_D)} = C\|\partial_x(\partial_x f)\|_{\dot{B}^{s-2}_{p,q}(A_D)} \le C\|\partial_x f\|_{\dot{B}^{s-1}_{p,q}(A_N)}.
$$

Therefore the former inequality in (ii) is proved, and we complete the proof. □

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