**C**ommunications in **M**athematical **A**nalysis
Volume 20, Number 2, pp. 62–67 (2017) www.math-res-pub.org/cma
ISSN 1938-9787

# On the Oscillation of Solutions of First-Order Difference Equations with Delay

**Yutaka Shoukaku***
Faculty of Enginnering
Kanazawa University
Kanazawa 920-1192, JAPAN

(Communicated by Michal Feckan)

**Abstract**

Consider the first order delay difference equation

$$\Delta x_n + p_n x_{\sigma(n)} = 0, \quad n \in \mathbb{N}_0,$$

where $\{p_n\}_{n\in\mathbb{N}_0}$ is a sequence of nonnegative real numbers, and $\{\sigma(n)\}_{n\in\mathbb{N}_0}$ is a sequence of integers such that $\sigma(n) \le n-1$, and $\lim_{n\to\infty} \sigma(n) = +\infty$. We obtain similar oscillation criteria of delay differential equations. This criterion is used by more simple method until now.

**AMS Subject Classification:** 39A21.

**Keywords**: difference equation, oscillation, delay.

## 1 Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions of the delay difference equation

$$\Delta x_n + p_n x_{\sigma(n)} = 0, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \tag{E}$$

where $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}_{n\in\mathbb{N}_0}$ is a sequence of nonnegative real numbers, and $\{\sigma(n)\}_{n\in\mathbb{N}_0}$ is a sequence of integers such that

$$\sigma(n) \le n-1 \quad \text{for } n \ge 0 \qquad \text{and} \qquad \lim_{n\to\infty} \sigma(n) = +\infty.$$

**Definition 1.** By a *solution* of the delay difference equation (E), we mean a sequence of real numbers $\{x_n\}_{n\ge -k}$ which is defined for $k = -\min_{n\in\mathbb{N}_0} \sigma(n)$, and satisfies (E) for all $n \in \mathbb{N}_0$. It is

*E-mail address: shoukaku@se.kanazawa-u.ac.jp

clear that, for each choice of real numbers $c_{-k}, c_{-k+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $\{x_n\}_{n \geq -k}$ of (E) which satisfies the initial conditions $x_{-k} = c_{-k}, x_{-k+1} = c_{-k+1}, \ldots, x_{-1} = c_{-1}, x_0 = c_0$.

**Definition 2.** A solution $\{x_n\}_{n \geq -k}$ of (E) is called *oscillatory* if the terms $x_n$ of the sequence are neither eventually positive nor eventually negative. Otherwise, it is called *nonoscillatory*.

In 1989 Ladas, Philos and Sficas [6] first proved that every solutions of difference equation

$$\Delta x_n + \sum_{j=1}^{m} p_j x_{n-k_j} = 0, \quad n = 0, 1, 2, \ldots$$

oscillates if and only if the characteristic equation

$$\lambda - 1 + \sum_{j=1}^{m} p_j \lambda^{-k_j} = 0$$

has no positive roots. The case of the difference equation

$$\Delta x_n + p_n x_{n-k} = 0 \tag{e}$$

was examined in [4, 7] where the following result was found.

**Theorem A.** *If*

$$\liminf_{n \to \infty} \left( \frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right) > \frac{k^k}{(k+1)^{k+1}}, \tag{c1}$$

*then every solution of Eq.*(e) *oscillates.*

Theorem A should be looked upon as a discrete analogue of the well known theorem [4, 5] about the oscillation of the first order delay differential equation

$$x'(t) + p(t)x(\sigma(t)) = 0, \ t \geq t_0,$$

where $p(t) \in C([t_0, \infty), \mathbb{R}_+)$, $\sigma(t) \leq t$, and $\lim_{t \to \infty} \sigma(t) = \infty$, which oscillates provided that

$$\liminf_{t \to \infty} \int_{\sigma(t)}^{t} p(s)ds > \frac{1}{e}. \tag{c2}$$

One should notice that condition (c1) can be written as

$$\liminf_{n \to \infty} \sum_{i=n-k}^{n-1} p_i > \left( \frac{k}{k+1} \right)^{k+1} \tag{c3}$$

and that

$$\lim_{k \to \infty} \left( \frac{k}{k+1} \right)^{k+1} = \lim_{k \to \infty} \left[ \frac{1}{\left(1 + \frac{1}{k}\right)^k} \cdot \frac{1}{1 + \frac{1}{k}} \right] = \frac{1}{e}.$$

In 1991, Philos [8] investigated the special case of (E) when $\sigma(n) = n - \sigma_n$, and established the following result.

**Theorem B.** *Suppose that the sequence $(n-\sigma_n)_{n\in\mathbb{N}_0}$ is increasing. If*

$$\liminf_{n\to\infty}\left(\frac{1}{\sigma_n}\sum_{i=n-\sigma_n}^{n-1} p_i\right) > \limsup_{n\to\infty}\frac{\sigma_n^{\sigma_n}}{(\sigma_n+1)^{\sigma_n+1}},$$

*then every solution of Eq.*(E) *with $\sigma(n)=n-\sigma_n$ oscillates.*

Then, it is interesting to establish sufficient conditions for first order difference equations with general delay. The problem of establishing sufficient conditions for the oscillation of every solutions of (E) have been the subject of many investigation. See [1, 3] and the references cited therein. In 1998, Zhang and Tian [9] studied the Eq. (E) and established:

**Theorem C.** *If*

$$\limsup_{n\to\infty} p_n > 0 \quad \textit{and} \quad \liminf_{n\to\infty}\sum_{i=\sigma(n)}^{n-1} p_i > \frac{1}{e},$$

*then every solutions of Eq.*(E) *oscillates.*

In 2006, Chatzarakis, Koplatadze and Stavroulakis [2] studied the Eq. (E) and established:

**Theorem D.** *If*

$$\limsup_{n\to\infty}\sum_{i=\sigma(n)}^{n-1} p_i < \infty \quad \textit{and} \quad \liminf_{n\to\infty}\sum_{i=\sigma(n)}^{n-1} p_i > \frac{1}{e},$$

*then every solutions of Eq.*(E) *oscillates.*

Then we arise interesting question whether there exists discrete analogue condition (c2) for (E) by using the simple method. Therefore, the main purpose of this paper is to establish the sharper condition than Theorems B–D.

## 2 Main Results

**Theorem 2.1.** *If*

$$\liminf_{n\to\infty}\sum_{i=\sigma(n)}^{n-1} p_i > \frac{1}{e}, \tag{C}$$

*then every solution of Eq.*(E) *oscillates.*

*Proof.* Assume, for the sake of contradiction, that there exists a eventually positive solution $\{x_n\}_{n\geq -k}$ of (E). Then there exists $n_1 > -k$ such that $\{x_n\} > 0$ and $\{x_{\sigma(n)}\} > 0$. Hence, from (E) we have

$$\Delta x_n = -p_n x_{\sigma(n)} \leq 0,\ n \geq n_1,$$

and so $\{x_n\}$ is nonincreasing sequence. (E) can be rewritten as follows

$$x_{n+1} - x_n + p_n x_n \leq 0$$

or

$$p_n \leq 1 - \frac{x_{n+1}}{x_n}.$$

From this, it is obvious that

$$\sum_{i=\sigma(n)}^{n-1} p_i \le \sum_{i=\sigma(n)}^{n-1} \left(1 - \frac{x_{i+1}}{x_i}\right). \tag{2.1}$$

Thus it follows from the condition (C) that we can choose a positive constant $\beta$ such that

$$\sum_{i=\sigma(n)}^{n-1} p_i \ge \beta > \frac{1}{e},\ n \ge n_2$$

for some $n_2 \ge n_1$. In view of (2.1) we obtain

$$\beta \le \sum_{i=\sigma(n)}^{n-1} \left(1 - \frac{x_{i+1}}{x_i}\right). \tag{2.2}$$

The right hand side of (2.2) means that

$$\sum_{i=\sigma(n)}^{n-1} \left(1 - \frac{x_{i+1}}{x_i}\right) = \sum_{i=\sigma(n)}^{n-1} \left\{1 - \exp\left(\ln\left(\frac{x_{i+1}}{x_i}\right)\right)\right\}.$$

Since

$$\frac{x_{i+1}}{x_i} < 1 \qquad \text{and} \qquad e^{-z} \ge 1 - z \quad (z > 0)$$

holds, it is obvious that

$$\begin{aligned} \sum_{i=\sigma(n)}^{n-1} \left(1 - \frac{x_{i+1}}{x_i}\right) &\le \sum_{i=\sigma(n)}^{n-1} \left\{1 - \left\{\ln\left(\frac{x_{i+1}}{x_i}\right) + 1\right\}\right\} \\ &= \sum_{i=\sigma(n)}^{n-1} \ln\left(\frac{x_{i+1}}{x_i}\right) = -\ln\left(\frac{x_n}{x_{\sigma(n)}}\right). \end{aligned} \tag{2.3}$$

Combining (2.2) with (2.3) yields

$$\beta \le \ln\left(\frac{x_{\sigma(n)}}{x_n}\right),$$

which implies that

$$e^{\beta} \le \frac{x_{\sigma(n)}}{x_n}.$$

Therefore we can see that

$$x_{\sigma(n)} \ge (e\beta)x_n. \tag{2.4}$$

Substituting (2.4) into (E) yields

$$\Delta x_n + p_n(e\beta)x_n \le 0.$$

By repeating the above arguments $m$ times, there exists an integer $n_{m+1}$ such that

$$\left(\frac{x_{\sigma(n)}}{x_n}\right) \ge (e\beta)^m \tag{2.5}$$

for $n \geq n_{m+1}$ and $m = 1, 2, \ldots$. On the other hand, it follows from the condition (C) that

$$\sum_{\sigma(n)}^{n} p_i \geq \sum_{\sigma(n)}^{n-1} p_i \geq \beta.$$

There exists an integer $n^* \in (\sigma(n), n)$ such that

$$\sum_{i=\sigma(n)}^{n^*} p_i \geq \frac{\beta}{2} \qquad \text{and} \qquad \sum_{i=n^*}^{n} p_i \geq \frac{\beta}{2}.$$

Summing up (E) from $\sigma(n)$ to $n^*$, and using the fact that $\{x_n\}$ is nonincreasing, we have

$$x_{n^*+1} - x_{\sigma(n)} \leq -\frac{\beta}{2} x_{\sigma(n^*)},$$

and so

$$\frac{\beta}{2} x_{\sigma(n^*)} \leq x_{\sigma(n)}. \tag{2.6}$$

Summing up (E) from $n^*$ to $n$ yields

$$x_{n+1} - x_{n^*} \leq -\frac{\beta}{2} x_{\sigma(n)},$$

and then

$$\frac{\beta}{2} x_{\sigma(n)} \leq x_{n^*}. \tag{2.7}$$

Combining (2.6) with (2.7), we obtain

$$\left(\frac{\beta}{2}\right)^2 x_{\sigma(n^*)} \leq x_{n^*}.$$

This and (2.5) can lead to the following contradiction

$$(e\beta)^m \leq \left(\frac{x_{\sigma(n^*)}}{x_{n^*}}\right) \leq \left(\frac{2}{\beta}\right)^2,$$

and completes the proof of the theorem. □

**Acknowledgments**

The author thanks the referees for their careful reading of the manuscript and insightful comments.

## References

[1] G. E. Chatzarakis, R. Koplatadze and I. P. Stavroulakis, Oscillation criteria of first order linear difference equations with delay arguments. *Nonlinear Anal.* **68** (2008), pp 994–1005.

[2] G. E. Chatzarakis, R. Koplatadze and I. P. Stavroulakis, Optimal oscillation criteria for first order difference equations with delay argument. *Pacific J. Math.* **235** (2008), pp 15–33.

[3] G. E. Chatzarakis and Özkan Öcalan, Oscillations of difference equations with non-monotone retarded arguments. *Appl. Math. Comput.* **258** (2015), pp 60–66.

[4] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, (1991).

[5] R. G. Koplatadze and T. A. Canturija, On the oscillatory and monotone solutions of the first order differential equations with deviating arguments. *Differential'nye Uravnenija*, **18** (1982), pp 1463–1465.

[6] G. Ladas, Ch. G. Philos and Y. G. Sficas, Necessary and sufficient conditions for the oscillation of difference equations. *Libertas Math.* **9** (1989), pp 121–125.

[7] G. Ladas, Ch. G. Philos, and Y. G. Sficas, Sharp conditions for the oscillation of delay difference equations. *J. Appl. Math. Simulation* **2** (1989), pp 101–111.

[8] Ch. G. Philos, On oscillations of some difference equations. *Funkcialoj Ekvacioj* **34** (1991), pp 157–172.

[9] B. G. Zhang and C. J. Tian, Nonexistence and existence of positive solutions for difference equations with unbounded delay. *Comput. Math. Appl.* **36** (1998), pp 1–8.