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Weak Solutions for Implicit Hilfer Fractional Differential Equations With Not Instantaneous Impulses

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Abstract

In this paper, we present results concerning the existence of weak solutions for some functional implicit Hilfer fractional differential equations with not instantaneous impulses in Banach spaces. The main results are proved by applying Mönch's fixed point theorem associated with the technique of measure of weak noncompactness, and we present an illustrative example.

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1 Introduction

Fractional differential and integral equations have recently been applied in various areas of engineering, mathematics, physics, bio-engineering and other applied sciences [18, 29]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas *et al.* [3, 4], Kilbas *et al.* [22] and Zhou [33]. Implicit differential equations have been considered by many authors [5, 9, 31].

In pharmacotherapy, instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of the drugs into the bloodstream and the consequential absorption for the body are a gradual and continuous process. In [1, 2, 21, 27] the authors initially studied some new classes of abstract impulsive differential equations with not instantaneous impulses.

The measure of weak noncompactness was introduced by De Blasi [14]. The strong measure of noncompactness was developed first by Banaŝ and Goebel [8] and subsequently developed and used in many papers; see for example, Akhmerov *et al.* [6], Alvàrez [7], Benchohra *et al.* [12], Guo *et al.* [17], and the references therein. In [12, 25] the authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers obtained other results by application of the technique of measure of weak noncompactness; see [4, 10, 11], and the references therein.

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; see [15, 16, 18, 19, 20, 30, 32]. In this paper, we discuss the existence of weak solutions for the following problem of implicit Hilfer fractional differential equation with not instantaneous impulses

$$\begin{cases} (D^{\alpha,\beta}_{s_k}u)(t) = f(t,u(t),D^{\alpha,\beta}_{s_k}u(t)); \text{ if } t \in I_k,\ k = 0,\dots,m, \\ u(t) = g_k(t,u(t_k^-)); \text{ if } t \in J_k,\ k = 1,\dots,m, \\ (I_1^{1-\gamma}u)(t)|_{t=0} = \phi_0, \end{cases} \tag{1.1}$$

where $I_0 := [0,t_1],\ J_k := (t_k,s_k],\ I_k := (s_k,t_{k+1}];\ k = 1,\dots,m,\ \alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha+\beta-\alpha\beta,\ \phi_0 \in E,\ f : I_k \times E \times E \to E,\ g_k : J_k \times E \to E$ are given continuous functions such that $(I^{1-\gamma}_{s_k}g_k)(t,u(t_k^-))|_{t=s_k} = \phi_k \in E;\ k = 1,\dots,m,$ E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, $I^{1-\gamma}_{s_k}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma \in (0,1]$, and $D^{\alpha,\beta}_{s_k}$ is the generalized Riemann-Liouville derivative operator of order α and type β, introduced by Hilfer in [18], $0 = s_0 < t_1 \le s_1 < t_2 \le s_2 < \cdots \le s_{m-1} < t_m \le s_m < t_{m+1} = T$.
In this work, we give some existence results for implicit Hilfer fractional differential equations with not instantaneous impulses in Banach spaces. We initiate the application of measure of weak noncompactness for such a class of problems.

2 Preliminaries

Let $I := [0,T]$ and let $C := C(I)$ be the Banach space of all continuous functions v from I into E with the supremum (uniform) norm

$$\|v\|_C := \sup_{t\in I}\|v(t)\|_E.$$

Denote by

$$\mathcal{P}C = \{u : I \to E : u \in C(I_0 \Cup \Cup_{k=1}^{m}(t_k,t_{k+1})),\ u(t_k^-) = u(t_k)\},$$

which is a Banach space equipped with the standard supremum norm

$$\|u\|_\infty := \sup_{t\in I}\|u(t)\|_E.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into E. We denote by $AC^1(I)$ the space defined by

$$AC^1(I) := \{w : I \to E : \frac{d}{dt}w(t) \in AC(I)\}.$$

By $C_\gamma(I)$, $C_\gamma^1(I)$, $\mathcal{P}C_\gamma(I)$ and $\mathcal{P}C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{w : (0,T] \to E : t^{1-\gamma}w(t) \in C\},\ C_\gamma^1(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\},$$

$$\mathcal{P}C_\gamma(I) = \{w : (0,T] \to E : t^{1-\gamma}w(t) \in \mathcal{P}C\},$$

with the norm

$$\|w\|_{\mathcal{P}C_\gamma} := \sup_{t\in I}\|t^{1-\gamma}w(t)\|_E,$$

and

$$\mathcal{P}C_\gamma^1(I) = \{w \in \mathcal{P}C : \frac{dw}{dt} \in \mathcal{P}C_\gamma\},$$

with the norm

$$\|w\|_{\mathcal{P}C_\gamma^1} := \|w\|_\infty + \|w'\|_{\mathcal{P}C_\gamma}.$$

In the following we denote $\|w\|_{\mathcal{P}C_\gamma}$ by $\|w\|_{\mathcal{P}C}$, and let $(E,w) = (E,\sigma(E,E^*))$ be the Banach space E with its weak topology.

Definition 2.1. A Banach space X is called weakly compactly generated (WCG, for short) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.2. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E,w) then $h(u_n) \to h(u)$ in (E,w)).

Definition 2.3. [26] The function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\varphi(u_J) = \int_J \varphi(u(s))ds$ for all $\varphi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_J = \int_J u(s)ds$).

Let $P(I,E)$ be the space of all $E-$valued Pettis integrable functions on I, and $L^1(I,E)$, be the Banach space of Lebesgue measurable functions $u : I \to E$. Define the class $P_1(I,E)$ by

$$P_1(I,E) = \{u \in P(I,E) : \varphi(u) \in L^1(I,E);\ for\ every\ \varphi \in E^*\}.$$

The space $P_1(I,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi\in E^*,\ \|\varphi\|\le 1} \int_1^T |\varphi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on I.

The following result is due to Pettis (see [[26], Theorem 3.4 and Corollary 3.41]).

Proposition 2.4. *[26] If $u \in P_1(I,E)$ and h is a measurable and essentially bounded $E-$valued function, then $uh \in P_1(J,E)$.*

Definition 2.5. The function $f : I\times E\times E \to E$ is said to be weakly-Carathéodory if

(i) for a.e. $v,w \in E$, the function $t \to f(t,v,w)$ is Pettis integrable a.e. on I,

(ii) for a.e. $t \in I$, the functions $v \to f(t,v,\cdot)$ and $w \to f(t,\cdot,w)$ are weakly sequentially continuous.

For all that follows, the symbol "$\int$" denotes the Pettis integral.

Now, we give some results and properties of fractional calculus.

Definition 2.6. [3, 22, 28] The left-sided mixed Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$(I_\theta^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s)ds;\ for\ a.e.\ t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt;\ \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t);\ for\ a.e.\ t \in I.$$

Definition 2.7. [3, 22, 28] The Riemann-Liouville fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned}(D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s)ds;\ for\ a.e.\ t \in I.\end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$ and $w \in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows.

$$(D_0^r I_0^r w)(t) = w(t);\ for\ all\ t \in (0,T].$$

Moreover, if $I_0^{1-r} w \in C_{1-\gamma}^1(I)$, then the following composition is proved in [28]

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r} w)(0^+)}{\Gamma(r)} t^{r-1};\ for\ all\ t \in (0,T].$$

Definition 2.8. [3, 22, 28] The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned} ({}^c D_0^r w)(t) &= \left(I_0^{1-r} \frac{d}{dt} w \right)(t) \\ &= \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} \frac{d}{ds} w(s) ds;\ for\ a.e.\ t \in I. \end{aligned}$$

In [18], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as specific cases (see also [19, 20, 30]).

Definition 2.9. (Hilfer derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, $I_0^{(1-\alpha)(1-\beta)} \in AC^1(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{(1-\alpha)(1-\beta)} w \right)(t);\ for\ a.e.\ t \in I. \tag{2.1}$$

Properties. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.
1. The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} w \right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} w \right)(t);\ for\ a.e.\ t \in I.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1],\ \gamma \geq \alpha,\ \gamma > \beta,\ 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (2.1) for $\beta = 0$, coincides with the Riemann-Liouville derivative, and for $\beta = 1$, concides with the Caputo derivative,

$$D_0^{\alpha,0} = D_0^{\alpha},\ and\ D_0^{\alpha,1} = {}^c D_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)} w$ exists and belongs to $L^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t);\ for\ a.e.\ t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)} w \in C_\gamma^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = w(t);\ for\ a.e.\ t \in I.$$

4. If $D_0^{\gamma} w$ exists and belongs to $L^1(I)$, then

$$(I_0^{\alpha} D_0^{\alpha,\beta} w)(t) = (I_0^{\gamma} D_0^{\gamma} w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1};\ for\ a.e.\ t \in I.$$

Corollary 2.10. *Let $h \in C_\gamma(I)$. A function $u \in L^1(I,E)$ is said to be a solution of the problem*

$$\begin{cases}(D_0^{\alpha,\beta}u)(t) = h(t); \; t \in I := [0,T],\\ \\ (I_0^{1-\gamma}u)(t)|_{t=0} = \phi \in E,\end{cases}$$

if and only if u satisfies the following Volterra integral equation

$$w(t) = \frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + (I_0^\alpha h)(t).$$

From the above corollary and [5, Lemma 5.1], we have the following lemma.

Lemma 2.11. *Let $f : I_k \times E \times E \to E$ be such that $f(\cdot,u(\cdot),v(\cdot)) \in C_\gamma(I_k);\; k = 0,\dots,m$, and $g_k(t,u) : J_k \times E \to E;\; k = 1,\dots,m$, be continuous functions for any $u \in \mathcal{P}C_\gamma(I)$. Then problem (1.1) is equivalent to obtaining the solutions of the equations*

$$\begin{cases}u(t) = \frac{\phi_k}{\Gamma(\gamma)}t^{\gamma-1} + (I_{s_k}^\alpha h)(t); \; if\; t \in I_k,\; k = 0,\dots,m,\\ u(t) = g_k(t,u(t_k^-)); \; if\; t \in J_k,\; k = 1,\dots,m,\end{cases} \tag{2.2}$$

where $h \in C_\gamma(I_k);\; k = 0,\dots,m$, such that

$$h(t) = f\left(t, \frac{\phi_k}{\Gamma(\gamma)}t^{\gamma-1} + (I_{s_k}^\alpha h)(t), h(t)\right); \; k = 0,\dots,m.$$

Remark 2.12. Let $h \in P_1(I,E)$. For every $\varphi \in E^*$, we have

$$\varphi(I_0^\alpha h)(t) = (I_0^\alpha \varphi h)(t); \; for\; a.e.\; t \in I.$$

Definition 2.13. [14] Let E be a Banach space, Ω_E be the bounded subsets of E, and B_1 be the unit ball of E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0,\infty)$ defined by

$$\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact } \Omega \subset E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \le \beta(B)$,

(b) $\beta(A) = 0 \Leftrightarrow$ A is weakly relatively compact,

(c) $\beta(A \cup B) = \max\{\beta(A),\beta(B)\}$,

(d) $\beta(\overline{A}^\omega) = \beta(A)$, ($\overline{A}^\omega$ denotes the weak closure of A),

(e) $\beta(A+B) \le \beta(A)+\beta(B)$,

(f) $\beta(\lambda A) = |\lambda|\beta(A)$,

(g) $\beta(conv(A)) = \beta(A)$,

(h) $\beta(\cup_{|\lambda|\leq h}\lambda A) = h\beta(A)$.

The next result follows directly from the Hahn-Banach theorem.

Proposition 2.14. *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$ let us denote by

$$V(t) = \{v(t) : v \in V\};\ t \in I,$$

and

$$V(I) = \{v(t) : v \in V,\ t \in I\}.$$

Lemma 2.15. *[17] Let $H \subset C$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on I, and*

$$\beta_C(H) = \max_{t\in I} \beta(H(t)),$$

and

$$\beta\left(\int_I u(s)ds\right) \leq \int_I \beta(H(s))ds,$$

where $H(s) = \{u(s) : u \in H\},\ s \in I$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

For our purpose we will need the following fixed point theorem:

Theorem 2.16. *[24] Let Q be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space C such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \text{ is relatively weakly compact}, \tag{2.3}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence of Weak Solutions

Let us start by defining what we mean by a weak solution of the problem (1.1).

Definition 3.1. By a weak solution of the problem (1.1) we mean a measurable function $u \in \mathcal{P}C_\gamma$ that satisfies the condition $(I_0^{1-\gamma}u)(t)|_{t=0} = \phi_0$, and the equations $(D_{s_k}^{\alpha,\beta}u)(t) = f(t,u(t),(D_{s_k}^{\alpha,\beta}u)(t))$ on $I_k;\ k = 0,\dots,m$, and $u(t) = g_k(t,u(t_k^-))$ on $J_k;\ k = 1,\dots,m$.

The following hypotheses will be used in the sequel.

(H_1) The function $f : I_k \times E \times E \to E;\ k = 0,\dots,m$, is weakly-Carathéodory.

(H_2) There exists $p_k \in C(I_k, [0,\infty))$; $k = 0, \ldots, m$, such that for all $\varphi \in E^*$, we have

$$|\varphi(f(t,u,v)| \le \frac{p_k(t)\|\varphi\|}{1+\|\varphi\|+\|u\|_E+\|v\|_E};\ for\ a.e.\ t \in I_k,\ and\ each\ u,v \in E,$$

and for each bounded and measurable set $B \subset E$, we have

$$\beta(f(t,B,D^{\alpha,\beta}_{s_k}B)) \le t^{1-\gamma}p_k(t)\beta(B);\ for\ each\ t \in I_k;\ k = 0,\ldots,m,$$

where $D^{\alpha,\beta}_{s_k}B = \{D^{\alpha,\beta}_{s_k}w : w \in B\}$,

(H_3) There exists $q_k \in C(J_k, [0,\infty))$; $k = 1, \ldots, m$, such that for all $\varphi \in E^*$, we have

$$|\varphi(g_k(t,u_{t_k^-}))| \le \frac{q_k(t)\|\varphi\|}{1+\|\varphi\|};\ for\ a.e.\ t \in J_k;\ k = 1,\ldots,m.$$

Set

$$p^* = \max_{k=0,\ldots,m}\sup_{t\in I_k} p_k(t),\ q^* = \max_{k=1,\ldots,m}\sup_{t\in J_k} q_k(t).$$

Theorem 3.2. *Assume that the hypotheses $(H_1)-(H_3)$ hold. If*

$$L := \frac{p^*T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} < 1, \tag{3.1}$$

then the problem (1.1) has at least one weak solution defined on I.

Proof. Transform the problem (1.1) into a fixed point equation. Consider the operator $N : \mathcal{P}C_\gamma \to \mathcal{P}C_\gamma$ defined by:

$$\begin{cases}(Nu)(t) = \frac{\phi_k}{\Gamma(\gamma)}t^{\gamma-1} + \int_{s_k}^t (t-s)^{\alpha-1}\frac{h(s)}{\Gamma(\alpha)}ds;\ \text{if } t \in I_k,\ k = 0,\ldots,m,\\ (Nu)(t) = g_k(t,u(t_k^-));\ \text{if } t \in J_k,\ k = 1,\ldots,m,\end{cases} \tag{3.2}$$

where $h \in C_\gamma(I_k, E)$; $k = 0, \ldots, m$, with

$$h(t) = f\left(t, \frac{\phi_k}{\Gamma(\gamma)}t^{\gamma-1} + (I^\alpha_{s_k}h)(t), h(t)\right).$$

First notice that, the hypotheses imply that $s \mapsto (t-s)^{\alpha-1}\frac{h(s)}{s}$, for all $t \in I_k$, is Pettis integrable, and for each $u \in \mathcal{P}C_\gamma$, the function

$$t \mapsto f\left(t, \frac{\phi_k}{\Gamma(\gamma)}t^{\gamma-1} + \int_{s_k}^t (t-s)^{\alpha-1}\frac{h(s)}{\Gamma(\alpha)}ds, h(t)\right)$$

is Pettis integrable over I_k. Thus, the operator N is well defined.
Let $R > 0$ be such that

$$R \ge \max\left\{\frac{p^*T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}, q^*T^{1-\gamma}\right\},$$

and consider the set

$$Q=\Big\{u\in\mathcal{P}C_\gamma:\|u\|_{\mathcal{P}C}\le R \text{ and } \|t_2^{1-\gamma}u(t_2)-t_1^{1-\gamma}u(t_1)\|_E\le \\ \frac{p^*}{\Gamma(1+\alpha)}T^{1-\gamma}|t_2-t_1|^\alpha+\frac{p^*}{\Gamma(\alpha)}\int_1^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1}-t_1^{1-\gamma}(t_1-s)^{\alpha-1}|ds, \\ \text{and } \|t_2^{1-\gamma}u(t_2)-t_1^{1-\gamma}u(t_1)\|_E\le \\ \|t_2^{1-\gamma}g_k(t_2,u(t_k^-))-t_1^{1-\gamma}g_k(t_1,u(t_k^-))\|_E \text{ on } J_k,\ k=1,\dots,m\Big\}.$$

Clearly, the subset Q is closed, convex end equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 2.16. The proof will be given in several steps.

Step 1. *N maps Q into itself.*
Let $u\in Q$, $t\in I$ and assume that $(Nu)(t)\neq 0$. Then there exists $\varphi\in E^*$ such that $\|t^{1-\gamma}(Nu)(t)\|_E = \varphi(|t^{1-\gamma}(Nu)(t))$. Thus, for each $t\in I_k,\ k=0,\dots,m$, we have

$$\|t^{1-\gamma}(Nu)(t)\|_E=\varphi\left(\frac{\phi_k}{\Gamma(\gamma)}+\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds\right),$$

where $h\in C_\gamma(I_k),\ k=0,\dots,m$, with

$$h(t)=f\left(t,\frac{\phi_k}{\Gamma(\gamma)}t^{\gamma-1}+(I_0^\alpha h)(t),h(t)\right).$$

Then

$$\begin{aligned}\|t^{1-\gamma}(Nu)(t)\|_E &\le \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|\varphi(h(s))|ds\\ &\le \frac{p^*T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\\ &\le \frac{p^*T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\\ &\le R.\end{aligned}$$

Also, for each $t\in J_k,\ k=1,\dots,m$, it is clear that

$$\|t^{1-\gamma}(Nu)(t)\|_E\le q^*T^{1-\gamma}\le R.$$

Hence,

$$\|N(u)\|_{\mathcal{P}C}\le R.$$

Next, let $t_1,t_2\in I_k,\ k=0,\dots,m$, be such that $t_1<t_2$ and let $u\in Q$, with

$$(\ln t_2)^{1-r}(Nu)(t_2)-(\ln t_1)^{1-r}(Nu)(t_1)\neq 0.$$

Then there exists $\varphi\in E^*$ such that

$$\|t_2^{1-\gamma}(Nu)(t_2)-t_1^{1-\gamma}(Nu)(t_1)\|_E=\varphi(t_2^{1-\gamma}(Nu)(t_2)-t_1^{1-\gamma}(Nu)(t_1)),$$

and $\|\varphi\| = 1$. Then

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E = \varphi(t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)),$$
$$\leq \varphi\left(t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\alpha-1}\frac{h(s)}{\Gamma(\alpha)}ds - t_1^{1-\gamma}\int_0^{t_1}(t_1-)^{\alpha-1}\frac{h(s)}{\Gamma(\alpha)}ds\right),$$

where $h \in C_\gamma(I_k)$ with

$$h(t) = f\left(t, \frac{\phi_k}{\Gamma(\gamma)}t^{\gamma-1} + (I_0^\alpha h)(t), h(t)\right).$$

Then

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E \leq t_2^{1-\gamma}\int_{t_1}^{t_2}|t_2-s|^{\alpha-1}\frac{|\varphi(h(s))|}{\Gamma(\alpha)}ds$$
$$+\int_1^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{|\varphi(h(s))|}{\Gamma(\alpha)}ds$$
$$\leq t_2^{1-\gamma}\int_{t_1}^{t_2}|t_2-s|^{\alpha-1}\frac{p(s)}{\Gamma(\alpha)}ds$$
$$+\int_1^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{p(s)}{\Gamma(\alpha)}ds.$$

Thus, we get

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E \leq \frac{p^*}{\Gamma(1+\alpha)}T^{1-\gamma}|t_2-t_1|^\alpha$$
$$+\frac{p^*}{\Gamma(\alpha)}\int_1^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|ds.$$

Also, for $t_1, t_2 \in J_k,\ k = 1,\ldots,m$, such that $t_1 < t_2$, let $u \in Q$, with

$$t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1) \neq 0,$$

then there exists $\varphi \in E^*$ such that

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E \leq \|t_2^{1-\gamma}g_k(t_2, u(t_k^-)) - t_1^{1-\gamma}g_k(t_1, u(t_k^-))\|_E.$$

Hence $N(Q) \subset Q$.

Step 2. *N is weakly-sequentially continuous.*
Let (u_n) be a sequence in Q and let $(u_n(t)) \to u(t)$ in (E,ω) for each $t \in I$. Fix $t \in I$, since f satisfies the assumption (H_1), we have $f(t, u_n(t), D_{s_k}^{\alpha,\beta}u_n(t))$ converges weakly uniformly to $f(t, u(t), D_{s_k}^{\alpha,\beta}u(t))$ on $I_k,\ k = 0,\ldots,m$. Hence the Lebesgue dominated convergence theorem for Pettis integrals implies $(Nu_n)(t)$ converges weakly uniformly to $(Nu)(t)$ in (E,ω), for each $t \in I$. Thus, $N(u_n) \to N(u)$. Hence, $N : Q \to Q$ is weakly-sequentially continuous.

Step 3. *The implication (2.3) holds.*
Let V be a subset of Q such that $\overline{V} = \overline{conv}(N(V) \cup \{0\})$. Obviously

$$V(t) \subset \overline{conv}(NV)(t)) \cup \{0\}),\ t \in I.$$

Further, as V is bounded and equicontinuous, by [13, Lemma 3], the function $t \to v(t) = \beta(V(t))$ is continuous on I. From (H_2), Lemma 2.15 and the properties of the measure β, for any $t \in I_k;\ k = 0, \ldots, m$, we have

$$\begin{aligned} t^{1-\gamma}v(t) &\le \beta(t^{1-\gamma}(NV)(t) \cup \{0\}) \\ &\le \beta(t^{1-\gamma}(NV)(t)) \\ &\le \frac{T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} p(s)\beta(V(s))ds \\ &\le \frac{T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} s^{1-\gamma} p(s)v(s)ds \\ &\le \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} \|v\|_{\mathcal{P}C}. \end{aligned}$$

Thus

$$\|v\|_{\mathcal{P}C} \le L\|v\|_{\mathcal{P}C}.$$

From (3.1), we get $\|v\|_{\mathcal{P}C} = 0$, that is $v(t) = \beta(V(t)) = 0$, for each $t \in I$, and then by [23, Theorem 2], V is weakly relatively compact in $\mathcal{P}C_\gamma$. Applying now Theorem 2.16, we conclude that N has a fixed point which is a solution of the problem (1.1). □

4 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

As an application of our results we consider the following problem of implicit Hilfer fractional differential equation of the form

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}} u_n)(t) = f_n(t, u(t), (D_0^{\frac{1}{2},\frac{1}{2}} u)(t)),\ t \in [0,1], \\ u(t) = g(t, e^-),\ t \in (1,2], \\ (D_2^{\frac{1}{2},\frac{1}{2}} u_n)(t) = f_n(t, u(t), (D_2^{\frac{1}{2},\frac{1}{2}} u)(t)),\ t \in (2,3], \\ (I_0^{\frac{1}{4}} u)(t)|_{t=0} = 0, \end{cases} \tag{4.1}$$

where

$$f_n(t, u(t), (D_k^{\frac{1}{2},\frac{1}{2}} u)(t)) = \frac{ct^2(e^{-7} + e^{-t-5})}{1 + \|u\|_E + \|D_k^{\frac{1}{2},\frac{1}{2}} u\|_E} u_n(t),\ t \in [0,1] \cup (2,3],\ k \in \{0,2\},$$

$$g(t, e^-) = \frac{e^{-2t}}{1+e},\ t \in (1,2],$$

with

$$u = (u_1, u_2, \ldots, u_n, \ldots), \text{ and } c := \frac{e^5}{4 \times 3^{\frac{13}{4}}} \Gamma\left(\frac{1}{2}\right).$$

Set $f = (f_1, f_2, \ldots, f_n, \ldots)$. Clearly, the function f is continuous. For each $u \in E$ and $t \in [0,1] \cup (2,3]$, we have

$$\|f(t, u(t), (D_k^{\frac{1}{2},\frac{1}{2}})(t))\|_E \leq ct^2 \left(e^{-7} + \frac{1}{e^{t+5}}\right), \; k \in \{0,2\}.$$

Hence, the hypothesis (H_2) is satisfied with $p^* = 18ce^{-5}$.

We shall show that condition (3.1) holds with $T = 3$. Indeed,

$$\frac{p^* T^{\frac{5}{4}}}{\Gamma(\frac{3}{2})} = \frac{36c3^{\frac{5}{4}} e^{-5}}{\Gamma(\frac{1}{2})} = \frac{1}{2} < 1.$$

Simple computations show that all conditions of Theorem 3.2 are satisfied. It follows that the problem (4.1) has at least one solution on $[0,3]$.

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