GENERAL ADJOINT ON A BANACH SPACE

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Abstract

In this paper, we show that the continuous dense embedding of a separable Banach space $\mathcal{B}$ into a Hilbert space $\mathcal{H}$ offers a new tool for studying the structure of operators on a Banach space. We use this embedding to demonstrate that the dual of a Banach space is not unique. As a application, we consider this non-uniqueness within the $\mathbb{C}[0,1] \subset L^2[0,1]$ setting. We then extend our theory every separable Banach space $\mathcal{B}$. In particular, we show that every closed densely defined linear operator A on $\mathcal{B}$ has a unique adjoint A^* defined on $\mathcal{B}$ and that $\mathcal{L}[\mathcal{B}]$, the bounded linear operators on $\mathcal{B}$, are continuously embedded in $\mathcal{L}[\mathcal{H}]$. This allows us to define the Schatten classes for $\mathcal{L}[\mathcal{B}]$ as the restriction of a subset of $\mathcal{L}[\mathcal{H}]$. Thus, the structure of $\mathcal{L}[\mathcal{B}]$, particularly the structure of the compact operators $\mathbb{K}[\mathcal{B}]$, is unrelated to the basis or approximation problems for compact operators. We conclude that for the Enflo space $\mathcal{B}_e$, we can provide a representation for compact operators that is very close to the same representation for a Hilbert space, but the norm limit of the partial sums may not converge, which is the only missing property.

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1   Introduction

A Banach space with a Schauder basis (S-basis) $\{e_n\}_{n=1}^{\infty}$ has two important properties. The first property is uniqueness; for each $u \in \mathcal{B}$, there is a unique set of scalers $\{u_n\}_{n=1}^{\infty}$ such that $u = \sum_{n=1}^{\infty} u_n e_n$. This property is important for studying the geometry of Banach spaces and for studying Lebesgue measure on Banach spaces (see Section 2.2 in [GZ]). The second property is approximation for compact operators (approximation property); every compact operator on $\mathcal{B}$ can be approximated by a sequence of operators of finite rank. A separable Banach space $\mathcal{B}$ has the bounded approximation property if and only if there is a sequence of finite rank operators $\{T_n\} \subset \mathcal{L}[\mathcal{B}]$ such that $lim_n T_n u = u$ for each $u \in \mathcal{B}$.

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Historically, in 1932, Banach [BA] asked if every separable Banach space possessed an S-basis. Grothendieck [GR] introduced the bounded approximation property and demonstrated that this implied the approximation property for separable reflexive Banach spaces. In 1973, Enflo showed that a separable reflexive Banach space $\mathcal{B}_e$ exists without the approximation property and hence without an S-basis (see [EN]). In the same year, Figiel and Johnson demonstrated that the approximation property does not imply the bounded approximation property (see [FJ]). Finally, in 1987, Szarek demonstrated that the bounded approximation property does not imply the existence of an S-basis (see [SZ]).

In 1965, Gross [G] proved that every separable Banach space contains a separable Hilbert space as a continuous dense embedding. This work was a generalization of Wiener's theory, which used the (densely embedded Hilbert) Sobolev space $\mathbb{H}_0^1[0,1] \subset \mathbb{C}_0[0,1]$.

In 1970, Kuelbs [K], generalized Gross' theorem to include the Hilbert space rigging $\mathbb{H}_0^1[0,1] \subset \mathbb{C}_0[0,1] \subset L^2[0,1]$. A general version of this theorem can be stated as follows:

Theorem 1.1. (Gross-Kuelbs) *Let $\mathcal{B}$ be a separable Banach space. Then, separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and a positive trace class operator T_{12} defined on $\mathcal{H}_2$ exist such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ all as continuous dense embeddings, with $\left(T_{12}^{1/2}u,\ T_{12}^{1/2}v\right)_1 = (u,\ v)_2$ and $\left(T_{12}^{-1/2}u,\ T_{12}^{-1/2}v\right)_2 = (u,\ v)_1$.*

(Since we use $\mathcal{H}_2$, we prove that $\mathcal{B} \subset \mathcal{H}_2$ in the appendix.)

Purpose

The purpose of this paper is to show how $\mathcal{H} = \mathcal{H}_2$ provides a completely different approach to the study of operator theory on separable Banach spaces.

Summary

After establishing a few background results, we investigate the basis problem from a new perspective. We show how to obtain the uniqueness property for the Enflo space $\mathcal{B}_e$ using $\mathcal{H}$. In the second section, we show that the dual of every separable Banach space $\mathcal{B}$ has two different representations for a class of linear functionals that are bijectively related to $\mathcal{B}$. In the next section, we use one representation to investigate the case when $\mathcal{B} = \mathbb{C}[0,1]$, the space of continuous functions on $[0,1]$ and $\mathcal{H} = L^2[0,1]$. In the last section, we apply our approach to obtain a new structure theory for the bounded linear operators on a separable Banach space, which is close to that of a Hilbert space. We show that the structure of the compact operators on $\mathcal{B}$ is almost identical to that of $\mathcal{H}$. The important difference is that the bounded linear operators on $\mathcal{B}$ do not form a C^*-algebra.

1.1 Preliminaries

Let $\mathcal{B}$ be a separable Banach space with dual space $\mathcal{B}^*$, let $C[\mathcal{B}]$ be the closed densely defined linear operators, and let $\mathcal{L}[\mathcal{B}]$ be the bounded linear operators on $\mathcal{B}$.

Definition 1.2. *A duality map $\mathcal{J} : \mathcal{B} \mapsto \mathcal{B}^*$, is a set*

$$\mathcal{J}(u) = \left\{ u^* \in \mathcal{B}^* \left| \langle u, u^* \rangle = \|u\|_{\mathcal{B}}^2 = \left\|u^*\right\|_{\mathcal{B}'}^2 \right. \right\}, \ \forall u \in \mathcal{B}.$$

If T is an operator, we let $\sigma(T)$ denote the spectrum of T and $\sigma_p(T) \subset \sigma(T)$ denote the point spectrum of T. The following theorem is due to Lax [L]. (A proof is provided in the appendix.)

Theorem 1.3. (Lax's Theorem) *Let $\mathcal{B}$ be a separable Banach space that is continuously and densely embedded in a Hilbert space $\mathcal{H}$, and let T be a bounded linear operator on $\mathcal{B}$ that is symmetric with respect to the inner product of $\mathcal{H}$ (i.e., $(Tu,v)_{\mathcal{H}} = (u,Tv)_{\mathcal{H}}$ for all $u,v \in \mathcal{B}$). Then,*

1. *T is bounded with respect to the $\mathcal{H}$ norm, and*

$$\left\|T^*T\right\|_{\mathcal{H}} = \|T\|_{\mathcal{H}}^2 \leqslant k\|T\|_{\mathcal{B}}^2,$$

   where k is a positive constant.

2. *$\sigma(T)$ relative to $\mathcal{H}$ is a subset of $\sigma(T)$ relative to $\mathcal{B}$.*

3. *$\sigma_p(T)$ relative to $\mathcal{H}$ is equal to $\sigma_p(T)$ relative to $\mathcal{B}$.*

Let $\mathcal{B}_e$ be a separable Banach space without an S-basis, and construct $\mathcal{H}$ such that $\mathcal{B}_e \subset \mathcal{H}$ as a continuous dense embedding. Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis for $\mathcal{H}$ such that for each $u \in \mathcal{H}$, a unique set of scalers $\{\bar{u}_n\}_{n=1}^{\infty}$ exists with

$$u = \sum_{n=1}^{\infty} \bar{u}_n e_n, \quad \text{with,} \quad \|u\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |\bar{u}_n|^2.$$

This is also true for $u \in \mathcal{B}_e$; then, we can rewrite the above representation as follows:

$$u = \sum_{n=1}^{\infty} u_n h_n, \quad \text{where,} \quad h_n = \frac{e_n}{\|e_n\|_{\mathcal{B}_e}} \text{ and } u_n = \|e_n\|_{\mathcal{B}_e} \bar{u}_n.$$

Now, $\|h_n\|_{\mathcal{B}_e} = 1$, and the scalers $\{u_n\}_{n=1}^{\infty}$ are still unique. Thus, the mapping from $\mathcal{B}_e \to \mathbb{R}^{\infty}$, taking $u = \sum_{n=1}^{\infty} u_n h_n \to (u_1, u_2, \cdots)$, is a unique, so the uniqueness property of an S-basis is actually true for all separable Banach spaces. (However, since we do not require that the family $\{e_n\}_{n=1}^{\infty} \subset \mathcal{B}_e$, the family $\{h_n\}_{n=1}^{\infty}$ is not a Schauder basis.)

1.2 Representation of Linear Functionals

Historically, it has been implicitly assumed that $\mathcal{B}^*$, the dual space of $\mathcal{B}$, has a unique representation. In this section, we use $\mathcal{H}$ to construct two special families of linear functionals that are bijectively related to $\mathcal{B}$.

Let $\mathbf{J}$ be the natural linear mapping from $\mathcal{H} \to \mathcal{H}^*$, and let $\mathbf{J}_{\mathcal{B}}$ be the restriction of $\mathbf{J}$ to $\mathcal{B}$. Since $\mathcal{B}$ is a continuous dense embedding in $\mathcal{H}$, $\mathbf{J}_{\mathcal{B}}$ is an isometric isomorphism of $\mathcal{B}$ onto $\mathbf{J}_{\mathcal{B}}(\mathcal{B}) \subset \mathcal{H}^*$ as a continuous dense embedding.

1.2.1 The Hilbert Representation

For our first representation let $u \in \mathcal{B}$.

Definition 1.4. *We define $u_h = \mathbf{J}_{\mathcal{B}}(u)$ and $\mathcal{B}_h^* = \{u_h \in \mathcal{B}^* : u \in \mathcal{B}\}$, so that $\langle u, u_h \rangle = (u,u)_{\mathcal{H}} = \|u\|_{\mathcal{H}}^2$. It is clear from our construction that the mapping $u \to u_h$ is a isometric (conjugate) isomorphism from $\mathcal{B}$ onto $\mathcal{B}_h^* \subset \mathcal{B}^*$. We call $\mathcal{B}_h^*$ the Hilbert representation for $\mathcal{B}$ in $\mathcal{B}^*$.*

1.2.2   The Steadman Representation

For each $u \in \mathcal{B}$, let $\hat{u}_s = \frac{\|u\|_{\mathcal{B}}^2}{\|u\|_{\mathcal{H}}^2} u_h$. Let the seminorm $p_u(\cdot)$ be the Minkowski functional defined on the span of $\{u\}$ in $\mathcal{B}$ by $p_u(v) = \|u\|_{\mathcal{B}} \|v\|_{\mathcal{B}}$. By the Hahn-Banach Theorem, $\hat{u}_s(\cdot)$ has an extension $u_s(\cdot)$ to all of $\mathcal{B}$ such that $|u_s(v)| \leqslant p_u(v) = \|u\|_{\mathcal{B}} \|v\|_{\mathcal{B}}$ for all $v \in \mathcal{B}$. From the definition of $p_u(\cdot)$, we find that $\|u_s\|_{\mathcal{B}^*} \leq \|u_s\|_{\mathcal{B}}$. Meanwhile, $u_s(u) = \|u\|_{\mathcal{B}}^2 \leqslant \|u\|_{\mathcal{B}} \|u_s\|_{\mathcal{B}^*}$ so that $u_s(\cdot)$ is a duality mapping for u (i.e., $\|u_s\|_{\mathcal{B}^*} = \|u\|_{\mathcal{B}}$). We call $\mathcal{B}_s^*$ the Steadman representation of $\mathcal{B}$ in $\mathcal{B}^*$.

Remark 1.5. *In general, the embedding of $\mathcal{B}_h^*$ (respectively $\mathcal{B}_s^*$) is a proper subspace of $\mathcal{B}^*$.*

2   Special Case

In this section, we investigate the Hilbert representation $\mathcal{B}_h^*$, in a familiar setting with $\mathcal{B} = \mathbb{C}[0,1] \subset L^2[0,1]$, where $\mathbb{C}[0,1]$ is the set of continuous functions on $[0,1]$ and $L^2[0,1]$ is the set of functions f, such that $\int_0^1 |f(x)|^2 dx < \infty$, with the Lebesgue measure on $[0,1]$. In this case, $\mathbb{C}[0,1] \subset L^2[0,1]$ as a continuous dense embedding (i.e., $\|f\|_2 \leqslant \|f\|_{\mathbb{C}}$, $f \in \mathbb{C}[0,1]$). It is well known that every bounded linear functional $l(\cdot)$ on $\mathbb{C}[0,1]$ has a representation of the form

$$l(f) = \int_0^1 f(x) d\alpha(x), \text{ where } \alpha(x) \in \mathbb{C}^*[0,1] = NBV[0,1], \tag{2.1}$$

the functions of normalized bounded variation on $[0,1]$ (i.e., $\alpha(0) = 0$). However, every bounded linear functional on $L^2[0,1]$, when restricted to $\mathbb{C}[0,1]$, is a bounded linear functional on $\mathbb{C}[0,1]$. We conclude that the set of functionals $\mathbb{C}_2^* = \left\{ (\cdot, u)_2 \mid u \in L^2[0,1] \right\}$ represent a subset of the functionals in $NBV[0,1]$. It follows that, for each $u \in L^2[0,1]$, there is a function $\alpha_u \in NBV[0,1]$ and a constant $c_u > 0$, depending on u, such that

$$(f, u)_2 = c_u^{-1} \int_0^1 f(x) d\alpha_u(x), \text{ for all } f \in \mathbb{C}[0,1].$$

Thus, $\mathbb{C}_2^*$ is a new representational subspace of $NBV[0,1]$. This inner product representation offers a new and unexpected perspective on the adjoint problem. Let $\mathbf{J}_2 : L^2[0,1] \to \{L^2[0,1]\}^*$ be the standard conjugate isomorphism, and let $\mathbf{J}_{\mathbb{C}}$ be the restriction of $\mathbf{J}_2$ to $\mathbb{C}[0,1]$ such that $\mathbf{J}_{\mathbb{C}} : \mathbb{C}[0,1] \to \{L^2[0,1]\}^*$.

Define $\mathbb{C}_h^* = \{u_h = (\cdot, u)_2 \mid u \in \mathbb{C}[0,1]\}$ such that $\mathbf{J}_{\mathbb{C}}(u) = u_h$. Let $C[\mathbb{C}[0,1]]$ be the set of closed densely defined linear operators on $\mathbb{C}[0,1]$.

Theorem 2.1. *If $A \in C[\mathbb{C}[0,1]]$, then there is a unique operator $A^* \in C[\mathbb{C}[0,1]]$ that satisfies the following:*

1. *$(aA)^* = \bar{a} A^*$,*

2. *$A^{**} = A$,*

3. *$(A^* + B^*) = A^* + B^*$,*

4. $(AB)^* = B^*A^*$ on $D(A^*) \bigcap D(B^*)$, and

5. if A is bounded ($A \in \mathcal{L}[\mathbb{C}[0,1]]$), then $\|A^*A\|_{\mathcal{B}} \leq M\|A\|_{\mathcal{B}}^2$ (for some constant M) and A has a bounded extension to $L^2[0,1]$.

Proof. If $A \in C[\mathbb{C}[0,1]]$, then the dual operator $A' : NBV[0,1] \to NBV[0,1]$. As a mapping on $NBV[0,1]$, A' is closed and weak* dense. However, since $\mathbb{C}[0,1]$ is dense in $L^2[0,1]$, $\mathbb{C}_h^*$ is strongly dense in $\{L^2[0,1]\}^*$. It follows that $A'\mathbf{J}_\mathbb{C}$, mapping $\mathbb{C}_h^* \subset \{L^2[0,1]\}^* \to \{L^2[0,1]\}^*$ is a closed (strongly) dense linear operator. Thus, $\mathbf{J}_\mathbb{C}^{-1}A'\mathbf{J}_\mathbb{C} : \mathbb{C}[0,1] \to \mathbb{C}[0,1]$ is a closed and densely defined linear operator. We define $A^* = [\mathbf{J}_\mathbb{C}^{-1}A'\mathbf{J}_\mathbb{C}] \in C[\mathbb{C}[0,1]]$. If A is bounded, A^* is defined on all of $\mathbb{C}[0,1]$. According to the closed graph theorem, A^* is bounded. The proofs of (1)-(3) are straightforward. To prove (4), let $u \in D(A^*) \bigcap D(B^*)$; then,

$$\begin{aligned}
(BA)^*u &= [\mathbf{J}_\mathbb{C}^{-1}(BA)'\mathbf{J}_\mathbb{C}]u = [\mathbf{J}_\mathbb{C}^{-1}A'B'\mathbf{J}_\mathbb{C}]u \\
&= \left[\mathbf{J}_\mathbb{C}^{-1}A'\mathbf{J}_\mathbb{C}\right]\left[\mathbf{J}_\mathbb{C}^{-1}B'\mathbf{J}_\mathbb{C}\right]u = A^*B^*u.
\end{aligned} \tag{2.2}$$

If we replace B by A^* in equation (2.2), noting that $A^{**} = A$, we also find that $(A^*A)^* = A^*A$.

The proof of the first part of (5) follows from

$$\left\|A^*A\right\|_\mathbb{C} \leqslant \left\|A^*\right\|_\mathbb{C}\|A\|_\mathbb{C} \leqslant \|\mathbf{J}_\mathbb{C}\|_{\mathbb{C}^*}\left\|\mathbf{J}_\mathbb{C}^{-1}\right\|_\mathbb{C}\left\|A'\right\|_{\mathbb{C}^*}\|A\|_\mathbb{C} = M\|A\|_\mathbb{C}^2,$$

for some constant M. A proof of the second part is a special case of Theorem 1.3. From (4), $S = A^*A$ is self-adjoint; thus, from Theorem 1.3, S has a bounded extension to $L^2[0,1]$. Therefore, A has a bounded extension $\bar{A}$ to $L^2[0,1]$ such that $\mathcal{L}[\mathbb{C}[0,1]]$ is continuously embedded into $\mathcal{L}[L^2[0,1]]$, the bounded linear operators on $L^2[0,1]$. $\qquad\square$

The last result also implies that $\mathcal{L}[\mathbb{C}[0,1]]$ is a *algebra.

Theorem 2.2. *(Polar Representation) If $A \in C[\mathbb{C}[0,1]]$, then there exists a partial isometry U and a self-adjoint operator T, $T = T^*$, with $D(T) = D(A)$ and $A = UT$.*

Proof. Let $\bar{A}$ be the (closed densely defined) extension of A to $L^2[0,1]$. On $L^2[0,1]$, $\bar{T}^2 = \bar{A}^*\bar{A}$ is self-adjoint, and there exists a unique partial isometry $\bar{U}$, with $\bar{A} = \bar{U}\bar{T}$. Thus, the restriction to $\mathbb{C}[0,1]$ provides us $A = UT$, and U is a partial isometry on $\mathbb{C}[0,1]$. (It is easy to check that $A^*A = T^2$.) $\qquad\square$

Theorem 2.3. (Spectral Representation) *Let $A \in C[\mathbb{C}[0,1]]$ be a self-adjoint linear operator. There exists a operator-valued spectral measure E_x defined for each $x \in \mathbb{R}$, and for each $u \in D(A)$,*

$$Au = \int_{-\infty}^{\infty} x dE_x(u).$$

The next result easily follows from examination of the previous proofs.

Theorem 2.4. *Let $\mathcal{B}$ be any Banach space that is a continuous dense embedding in $L^2[0,1]$; then, all the results of this section hold for $\mathcal{B}$.*

Since $L^p[0,1] \subset L^2[0,1]$, $p > 2$ as a continuous dense embedding, we conclude that Theorems 1.1, 1.2, and 1.3 hold for all $L^p[0,1]$, $p > 2$.

If $u \in L^p[0,1]$, $2 < p < \infty$, then the standard duality mapping is

$$u^* = \|u\|_p^{2-p} |u(x)|^{p-2} u(x) \in L^q[0,1], \ \tfrac{1}{p} + \tfrac{1}{q} = 1. \tag{2.3}$$

Furthermore,

$$\langle u, u^* \rangle = \|u\|_p^{2-p} \int_0^1 |u(x)|^p \, d\lambda_n(x) = \|u\|_p^2 = \left\| u^* \right\|_q^2$$

Applying our earlier observation to $L^p[0,1]$, $p > 2$, we find that for each $u \in L^p[0,1]$, there is a $\alpha_u \in NBV[0,1]$, a constant $c_u > 0$ and a unique $u^* \in L^q[0,1]$, $\tfrac{1}{p} + \tfrac{1}{q} = 1$ such that (see (2.3))

$$\langle f, u^* \rangle_p = \|u\|_p^{2-p} \int_0^1 f(x)|u(x)|^{p-2} u(x) dx = c_u^{-1} \int_0^1 f(x) d\alpha_u(x),$$

for all $f \in \mathbb{C}[0,1]$. It follows that there are an infinite number of possible of representations for the linear functionals on $\mathbb{C}[0,1]$.

One of the most important implications of this section is the possibility of a new representation theory for compact operators. Let A be a compact operator on $\mathbb{C}[0,1]$, and let $\{e_n\}_{n=1}^\infty \subset \mathbb{C}[0,1]$ be an orthonormal basis for $L^2[0,1]$. From Theorem 1.1(5), A has a bounded extension $\bar{A}$ to $L^2[0,1]$. On $L^2[0,1]$, $\bar{A}$ has the following representation:

$$\bar{A}f = \sum_{n=1}^\infty \mu_n(\bar{A})(f, e_n)_2 \bar{U} e_n,$$

for all $f \in L^2[0,1]$, where the $\mu_n(\bar{A})$ are the eigenvalues of $[\bar{A}^* \bar{A}]^{1/2}$ counted by multiplicity in decreasing order. Since $(\cdot, e_n)_2$ is a linear functional on $\mathbb{C}[0,1]$, for each n, there is a $\alpha_n \in NVB[0,1]$ and constant $c_n > 0$ such that $(f, e_n)_2 = \langle f, \alpha_n^* \rangle = c_n^{-1} \int_0^1 f(x) d\alpha_n(x)$. From Theorem 1.3 (3), $\mu_n(\bar{A}) = \mu_n(A)$. It follows that we can also represent A on $\mathbb{C}[0,1]$ as

$$Af = \sum_{n=1}^\infty \mu_n(A)(f, e_n)_2 U e_n = \sum_{n=1}^\infty \mu_n(A) c_n^{-1} \langle f, \alpha_n^* \rangle U e_n.$$

for all $f \in \mathbb{C}[0,1]$. The partial sums always converge in $\mathcal{H}$, but convergence in $\mathbb{C}[0,1]$ requires proof.

3 The Adjoint Problem

The important properties of operators on a Hilbert space do not directly depend on the inner product; however, they do depend on the existence of an adjoint operator A^* for each closed densely defined linear operator A. In this section, we extend the special case of $\mathbb{C}[0,1]$ to all separable Banach spaces.

Theorem 3.1. *Let $\mathcal{B}$ be a separable Banach space. If $A \in C[\mathcal{B}]$, then there is a unique operator $A^* \in C[\mathcal{B}]$ that satisfies the following:*

    1. $(aA)^* = \bar{a}A^*$;

2. $A^{**} = A$;

3. $(A + B)^* = A^* + B^*$;

4. $(AB)^* = B^* A^*$ on $D(A^*) \bigcap D(B^*)$;

5. if $A \in \mathcal{L}[\mathcal{B}]$, then $\|A^* A\|_{\mathcal{B}} \le M \|A\|_{\mathcal{B}}^2$, for some constant M; and

6. if $A \in \mathcal{L}[\mathcal{B}]$, it has a bounded extension to $\mathcal{L}[\mathcal{H}]$.

Proof. Let $\mathbf{J}_{\mathcal{B}}$ be the restriction of $\mathbf{J}$ to $\mathcal{B}$, and let $\mathbf{J}_{\mathcal{B}}(\mathcal{B}) = \mathcal{B}_{\mathbf{h}}^* \subset \mathcal{H}^*$. Since $\mathcal{B}$ is dense in $\mathcal{H}$, $\mathcal{B}_h^*$ is dense in $\mathcal{H}^*$. It follows that if $A \in C[\mathcal{B}]$, $A' : \mathcal{B}_h^* \to \mathcal{H}^*$ is closed and densely defined. Thus, $\mathbf{J}_{\mathcal{B}}^{-1} A' \mathbf{J}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is a closed and densely defined linear operator. We define $A^* = [\mathbf{J}_{\mathcal{B}}^{-1} A' \mathbf{J}_{\mathcal{B}}] \in C[\mathcal{B}]$. If $A \in L[\mathcal{B}]$, $A^* = \mathbf{J}_{\mathcal{B}}^{-1} A' \mathbf{J}_{\mathcal{B}}$ is defined on all of $\mathcal{B}$ such that $A^* \in L[\mathcal{B}]$ by the closed graph theorem. The proofs of (1)-(3) are straightforward. To prove (4), let $u \in D(A^*) \bigcap D(B^*)$; then,

$$\begin{aligned}
(BA)^* u &= [\mathbf{J}_{\mathcal{B}}^{-1} (BA)' \mathbf{J}_{\mathcal{B}}] u = [\mathbf{J}_{\mathcal{B}}^{-1} A' B' \mathbf{J}_{\mathcal{B}}] u \\
&= \left[\mathbf{J}_{\mathcal{B}}^{-1} A' \mathbf{J}_{\mathcal{B}} \right] \left[\mathbf{J}_{\mathcal{B}}^{-1} B' \mathbf{J}_{\mathcal{B}} \right] u = A^* B^* u.
\end{aligned} \tag{3.1}$$

If we replace B by A^* in equation (3.1), noting that $A^{**} = A$, we also find that $(A^* A)^* = A^* A$ (self-adjoint). To prove (5), we observe that

$$\left\| A^* A \right\|_{\mathcal{B}} \le \left\| A^* \right\|_{\mathcal{B}} \|A\|_{\mathcal{B}} \le \|\mathbf{J}_{\mathcal{B}}\|_{\mathcal{B}^*} \left\| \mathbf{J}_{\mathcal{B}}^{-1} \right\|_{\mathcal{B}} \left\| A' \right\|_{\mathcal{B}^*} \|A\|_{\mathcal{B}} = M \|A\|_{\mathcal{B}}^2 .$$

It follows that

$$\left\| A^* A \right\|_{\mathcal{B}} \le M \|A\|_{\mathcal{B}}^2 . \tag{3.2}$$

To prove (6),

$$\langle A^* A v, \mathbf{J}_{\mathcal{B}}(u) \rangle = \langle A^* A v, u_h \rangle = (A^* A v, u)_{\mathcal{H}} = (v, A^* A u)_{\mathcal{H}},$$

so that $A^* A$ is symmetric. Thus, from Theorem 1.3 (Lax), $A^* A$ has a bounded extension to $\mathcal{H}$ and $\|A^* A\|_{\mathcal{H}} \le k \|A^* A\|_{\mathcal{B}}$, where k is a positive constant. It follows that A has a bounded extension to $\mathcal{L}[\mathcal{H}]$ (i.e., $\mathcal{L}[\mathcal{B}] \hookrightarrow \mathcal{L}[\mathcal{H}]$ is a continuous embedding). $\quad\square$

Remark 3.2. *The continuous embedding of $\mathcal{B}$ in $\mathcal{H}$ does not imply the dense embedding of $\mathcal{L}[\mathcal{B}]$ in $\mathcal{L}[\mathcal{H}]$. We conjecture that the embedding is dense.*

Definition 3.3. *Let U be bounded, $A \in C[\mathcal{B}]$, and let $\mathcal{U}, \mathcal{V}$ be subspaces of $\mathcal{B}$. Then,*

1. *A is said to be self-adjoint if $D(A) = D(A^*)$ and $A = A^*$.*

2. *A is said to be normal if $D(A) = D(A^*)$ and $AA^* = A^* A$.*

3. *U is unitary if $UU^* = U^* U = I$.*

4. *The subspace $\mathcal{U}$ is $\perp$ to $\mathcal{V}$ if and only, for each $v \in \mathcal{V}$ and $\forall u \in \mathcal{U}$, $(v, u)_{\mathcal{H}} = 0$ and, for each $u \in \mathcal{U}$ and $\forall v \in \mathcal{V}$, $(u, v)_{\mathcal{H}} = 0$.*

The last definition is transparent since orthogonal subspaces in $\mathcal{H}$ induce orthogonal subspaces in $\mathcal{B}$.

3.0.3 Self-adjointness

With respect to our definition of self-adjointness, the following related definition is due to Palmer [P], where the operator is called symmetric. This operator is essentially the same as a Hermitian operator as defined by Lumer [LU]. (An operator A is dissipative if $-A$ is accretive.)

Definition 3.4. *A closed densely defined linear operator A on $\mathcal{B}$ is called self-conjugate if both iA and $-iA$ are dissipative.*

Theorem 3.5. *(Vidav-Palmer) A linear operator A, defined on $\mathcal{B}$, is self-conjugate if and only if iA and $-iA$ are generators of isometric semigroups.*

Theorem 3.6. *The operator A, defined on $\mathcal{B}$, is self-conjugate if and only if it is self-adjoint.*

Proof. Let $\bar{A}$ be the closed densely defined extension of A to $\mathcal{H}$. On $\mathcal{H}$, $\bar{A}$ is self-adjoint if and only if $i\bar{A}$ generates a unitary group and if and only if it is self-conjugate. Thus, both definitions coincide on $\mathcal{H}$. It follows that the restrictions coincide on $\mathcal{B}$.   □

The proof of the last theorem represents a general approach for proving new results for $\mathcal{B}$. The following two are representative.

Theorem 3.7. (Polar Representation) *Let $\mathcal{B}$ be a separable Banach space. If $A \in \mathbb{C}[\mathcal{B}]$, then there exists a partial isometry U and a self-adjoint operator T, with $D(T) = D(A)$ and $A = UT$. Furthermore, $T = [A^*A]^{1/2}$ in a well-defined sense.*

Theorem 3.8. (Spectral Representation) *Let $\mathcal{B}$ be a separable Banach space, and let $A \in \mathbb{C}[\mathcal{B}]$ be a self-adjoint linear operator. Then, there exists a operator-valued spectral measure $E_x, x \in \mathbb{R}$, and for each $u \in D(A)$,*

$$Au = \int_{\mathbb{R}} x \, dE_x(u).$$

3.1 Example

The following (separable) Hilbert space is a concrete construction of $\mathcal{H}$. It is related to one that was first found by Steadman [ST]. This particular $\mathcal{H}$ was used in [GZ] to provide a rigorous foundation for the path integral formulation of quantum mechanics in the manner originally suggested by Feynman. It is also important because it contains the L^p spaces and the test functions $\mathcal{D}(\mathbb{R}^n)$ as continuous dense embeddings.

3.2 The space $KS^2[\mathbb{R}^n]$

On $\mathbb{R}^n$, let $\mathbb{Q}^n$ be the set $\{\mathbf{x} = (x_1, x_2 \cdots, x_n) \in \mathbb{R}^n\}$ such that x_i is rational for each i. Since this is a countable dense set in $\mathbb{R}^n$, we can arrange it as $\mathbb{Q}^n = \left\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \cdots\right\}$. For each l and i, let $\mathbf{B}_l(\mathbf{x}^i)$ be the closed cube centered at $\mathbf{x}^i$, with sides parallel to the coordinate axes and edge $e_l = \frac{1}{2^l\sqrt{n}}, l \in \mathbb{N}$. Now choose the natural order that maps $\mathbb{N} \times \mathbb{N}$ bijectively to $\mathbb{N}$:

$$\{(1,1), (2,1), (1,2), (1,3), (2,2), (3,1), (3,2), (2,3), \cdots\}.$$

Let $\{\mathbf{B}_k, \ k \in \mathbb{N}\}$ be the resulting set of (all) closed cubes $\{\mathbf{B}_l(\mathbf{x}^i) \ |(l,i) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$, and let $\mathcal{E}_k(\mathbf{x})$ be the characteristic function of $\mathbf{B}_k$ so that $\mathcal{E}_k(\mathbf{x})$ is in $L^p[\mathbb{R}^n]$ for $1 \le p \le \infty$. Define $F_k(\,\cdot\,)$ on $L^1[\mathbb{R}^n]$ by

$$F_k(f) = \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\lambda_n(\mathbf{x}). \tag{3.3}$$

It is clear that $F_k(\,\cdot\,)$ is a bounded linear functional on $L^p[\mathbb{R}^n]$ for each k, $\|F_k\| \le 1$ and, if $F_k(f) = 0$ for all k, $f = 0$ so that $\{F_k\}$ is fundamental on $L^p[\mathbb{R}^n]$ for $1 \le p \le \infty$. Set $t_k = 2^{-k}$ such that $\sum_{k=1}^{\infty} t_k = 1$, and define a measure $d\mu$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$d\mu = \left[\sum_{k=1}^{\infty} t_k \mathcal{E}_k(\mathbf{x})\mathcal{E}_k(\mathbf{y}) \right] d\lambda_n(\mathbf{x})d\lambda_n(\mathbf{y}).$$

To construct our Hilbert space, define an inner product $(\,\cdot\,)$ on $L^1[\mathbb{R}^n]$ by

$$
\begin{aligned}
(f,g) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(\mathbf{x})g(\mathbf{y})^* d\mu \\
&= \sum_{k=1}^{\infty} t_k \left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\lambda_n(\mathbf{x}) \right] \left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y})g(\mathbf{y})d\lambda_n(\mathbf{y}) \right]^*.
\end{aligned}
\tag{3.4}
$$

We call the completion of $L^1[\mathbb{R}^n]$, with the above inner product, the Kuelbs-Steadman space, $KS^2[\mathbb{R}^n]$. Steadman [ST] constructed a version of this space by modifying a method developed by Kuelbs [K] for other purposes. Her interest was in demonstrating that $L^1[\mathbb{R}]$ can be densely and continuously embedded in a Hilbert space that contains the non-absolutely integrable functions. To show that this is the case, suppose that f is a non-absolutely integrable function, say a Henstock-Kurzweil integral; then,

$$
\begin{aligned}
\|f\|_{KS^2}^2 &= \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\lambda_n(\mathbf{x}) \right|^2 \\
&\le \sup_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\lambda_n(\mathbf{x}) \right|^2 < \infty.
\end{aligned}
$$

Since the absolute value is outside the integral, we find that $f \in KS^2[\mathbb{R}^n]$ for any of the classical definitions of a non-absolute integral (see [GO]). A discussion of this space, its relationship to the Feynman path integral formulation of quantum mechanics and proofs of the following can be found in [GZ].

Theorem 3.9. *The space $KS^2[\mathbb{R}^n]$ contains $L^p[\mathbb{R}^n]$ (for each p, $1 \le p \le \infty$) as a continuous dense compact embedding.*

Remark 3.10. *The fact that $L^\infty[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ as a continuous dense and compact embedding, while $KS^2[\mathbb{R}^n]$ is separable clearly indicates in a very forceful manner that separability is not an inherited property. We note that since $L^1[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ and $KS^2[\mathbb{R}^n]$ is reflexive, the second dual $L^1[\mathbb{R}^n]^{**} = \mathfrak{M}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$. Recall that $\mathfrak{M}[\mathbb{R}^n]$ is the space of bounded finitely additive set functions defined on the Borel sets $\mathfrak{B}[\mathbb{R}^n]$. This space contains the Dirac delta measure and free-particle Green's function for the Feynman path integral.*

The space $KS^2[\mathbb{R}^n]$ uses a base composed of characteristic functions of cubes, and the volume of the largest cube is $\mathrm{vol}(\mathbf{B}_k) \le \left[\frac{1}{2\sqrt{n}} \right]^n$. This observation leads to the following interesting theorem.

Theorem 3.11. *The space of test functions $\mathcal{D}(\mathbb{R}^n) \subset KS^2[\mathbb{R}^n]$ as a continuous dense embedding.*

Proof. Suppose that $\phi_j \to \phi$ in $\mathcal{D}(\mathbb{R}^n)$. By definition, a compact set $K \subset \mathbb{R}^n$ exists that is the support of $\phi_j - \phi$, and $D^\alpha \phi_j$ converges to $D^\alpha \phi$ uniformly on K for every multi-index α. Let $\{\mathcal{E}_{K_l}\}$ be the set of all $\mathcal{E}_l$, with support $K_l \subset K$. If α is a multi-index, then we have the following:

$$\lim_{k \to \infty} \left\| D^\alpha \phi_k - D^\alpha \phi \right\|_{KS^2}^2$$

$$= \lim_{k \to \infty} \sum_{l=1}^{\infty} t_l \left| \int_{\mathbb{R}^n} \mathcal{E}_l(\mathbf{x}) \left[D^\alpha \phi_k(\mathbf{x}) - D^\alpha \phi(\mathbf{x}) \right] d\lambda_n(\mathbf{x}) \right|^2$$

$$\leqslant \lim_{k \to \infty} \sup_l \left| \int_{\mathbb{R}^n} \mathcal{E}_l(\mathbf{x}) \left[D^\alpha \phi_k(\mathbf{x}) - D^\alpha \phi(\mathbf{x}) \right] d\lambda_n(\mathbf{x}) \right|^2$$

$$\leqslant \left[\frac{1}{2\sqrt{n}} \right]^{2n} \lim_{k \to \infty} \sup_{x \in K} \left| D^\alpha \phi_k(\mathbf{x}) - D^\alpha \phi(\mathbf{x}) \right|^2 = 0.$$

Since α is arbitrary, we find that $\mathcal{D}(\mathbb{R}^n) \subset KS^2[\mathbb{R}^n]$ as a continuous embedding. Since $\mathcal{D}(\mathbb{R}^n)$ is a dense topological vector subspace of $KS^2[\mathbb{R}^n]$, according to the Hahn-Banach theorem, each continuous linear functional T on $\mathcal{D}(\mathbb{R}^n)$ has a continuous extension to $KS^2[\mathbb{R}^n]$. However, from the Riesz representation theorem, every continuous linear functional on $KS^2[\mathbb{R}^n]$ is of the form $T(f) = (f,g)_{KS^2}$ for some unique $g \in KS^2[\mathbb{R}^n]$. $\square$

Example 3.12. *Let A be a second-order differential operator on $L^p[\mathbb{R}^n]$ of the form*

$$A = \sum_{i,j=1}^n a_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^n x_i b_{ij}(\mathbf{x}) \frac{\partial}{\partial x_j},$$

where $\mathbf{a}(\mathbf{x}) = \left\| a_{ij}(\mathbf{x}) \right\|$ and $\mathbf{b}(\mathbf{x}) = \left\| b_{ij}(\mathbf{x}) \right\|$ are matrix-valued functions in $\mathbb{C}_c^\infty[\mathbb{R}^n \times \mathbb{R}^n]$ (infinitely differentiable functions with compact support). We also assume that, for all $\mathbf{x} \in \mathbb{R}^n$ $\det \left\| a_{ij}(\mathbf{x}) \right\| > \varepsilon$ and the imaginary part of the eigenvalues of $\mathbf{b}(\mathbf{x})$ are bounded above by $-\varepsilon$ for some $\varepsilon > 0$. Note that since we do not require $\mathbf{a}$ or $\mathbf{b}$ to be symmetric, $A \neq A'$.

It is well known that $\mathbb{C}_c^\infty[\mathbb{R}^n] \subset L^p[\mathbb{R}^n] \cap L^q[\mathbb{R}^n]$ is dense for all $1 < p \leq q < \infty$. Furthermore, since A' is invariant on $\mathbb{C}_c^\infty[\mathbb{R}^n]$,

$$A' : \mathbb{C}_c^\infty[\mathbb{R}^n] \subset L^p[\mathbb{R}^n] \to \mathbb{C}_c^\infty[\mathbb{R}^n] \subset L^p[\mathbb{R}^n].$$

It follows that A' has a closed extension to $L^q[\mathbb{R}^n]$. (In this case, we do not need $\mathcal{H}$ directly, and we can identify $\mathbf{J}$ with the identity on $\mathcal{H}$ and A^* with A'.)

A number of other examples are given in Chapter 3 of [GZ].

4   Schatten Classes on Banach Spaces

Let $\mathbb{K}(\mathcal{B})$ be the class of compact operators on $\mathcal{B}$, and let $\mathbb{F}(\mathcal{B})$ represent the class of operators of finite rank. For separable Banach spaces, $\mathbb{K}(\mathcal{B})$ is an ideal that need not be the maximal ideal in $\mathcal{L}[\mathcal{B}]$. If $\mathbb{M}(\mathcal{B})$ is the set of weakly compact operators and $\mathbb{N}(\mathcal{B})$ is the

set of operators that map weakly convergent sequences into strongly convergent sequences, it is known that both are closed two-sided ideals in the operator norm and, in general, $\mathbb{F}(\mathcal{B}) \subset \mathbb{K}(\mathcal{B}) \subset \mathbb{M}(\mathcal{B})$ and $\mathbb{F}(\mathcal{B}) \subset \mathbb{K}(\mathcal{B}) \subset \mathbb{N}(\mathcal{B})$ (see part I of Dunford and Schwartz [DS], pg. 553). For reflexive Banach spaces, $\mathbb{K}(\mathcal{B}) = \mathbb{N}(\mathcal{B})$ and $\mathbb{M}(\mathcal{B}) = \mathcal{L}[\mathcal{B}]$. For the space of continuous functions $\mathbb{C}[\Omega]$ on a compact Hausdorff space Ω, Grothendieck [GR] has shown that $\mathbb{M}(\mathcal{B}) = \mathbb{N}(\mathcal{B})$. However, it is shown in part I of Dunford and Schwartz [DS] that, if (Ω, Σ, μ) is a positive measure space, then for $L^1(\Omega, \Sigma, \mu)$, we have $\mathbb{M}(\mathcal{B}) \subset \mathbb{N}(\mathcal{B})$. In this section, we present a natural definition of the Schatten class of operators on $\mathcal{B}$ and show that the structure of $\mathcal{L}[\mathcal{B}]$ is almost identical to that of $\mathcal{L}[\mathcal{H}]$ (see [SC]).

4.1   Background: Compact Operators on Banach Spaces

In the appendix of Chapter 5 in [GZ], we assumed that $\mathcal{B}$ was uniformly convex with an S-basis. In this section, we weaken that assumption to a separable Banach. Here, we show that the structure of $\mathcal{L}[\mathcal{B}]$ is almost identical to $\mathcal{L}[\mathcal{H}]$, except it is not a C^*-algebra. This result follows from the following theorem.

Theorem 4.1. *For every $\phi \in \mathcal{H}$, there exists a $\varphi^* \in \mathcal{B}^*$ and a constant $c_\phi > 0$ depending on ϕ such that $(f, \phi)_{\mathcal{H}} = c_\phi^{-1} \langle f, \varphi^* \rangle_{\mathcal{B}^*}$ for all $f \in \mathcal{B}$.*

Proof. The proof is easy. For each $f \in \mathcal{B}$ and all $\phi \in \mathcal{H}$, $(f, \phi)_{\mathcal{H}}$ is well defined. It follows that $(\cdot, \phi)_{\mathcal{H}}$ is a bounded linear functional on $\mathcal{B}$ for all $\phi \in \mathcal{H}$. Thus, a $\varphi^* \in \mathcal{B}^*$ and a constant $c_\phi > 0$ depending on ϕ exist such that $(f, \phi)_{\mathcal{H}} = c_\phi^{-1} \langle f, \varphi^* \rangle_{\mathcal{B}^*}$ for all $f \in \mathcal{B}$. $\square$

Let $A = U[A^*A]^{1/2}$ be a compact operator on $\mathcal{B}$, and let $\bar{A} = \bar{U}[\bar{A}^*\bar{A}]^{1/2}$ be its extension to $\mathcal{H}$. For each compact operator $\bar{A}$, an orthonormal basis $\{\phi_n \,|\, n \geqslant 1\}$ for $\mathcal{H}$ exists such that

$$\bar{A} = \sum_{n=1}^{\infty} \mu_n(\bar{A})(\cdot, \phi_n)_{\mathcal{H}} \, \bar{U}\phi_n. \tag{4.1}$$

Here, the $\mu_n(\bar{A})$ are the eigenvalues of $[\bar{A}^*\bar{A}]^{1/2} = |\bar{A}|$, counted by multiplicity and in decreasing order. Without loss of generality, we can assume that $\{\phi_n \,|\, n \geqslant 1\} \subset \mathcal{B}$. From Theorem 4.1 and the fact that $\mu_n(\bar{A}) = \mu_n(A)$ by Lax's theorem, we can write A as follows:

$$A = \sum_{n=1}^{\infty} \mu_n(A) c_n^{-1} \langle \cdot, \varphi_n^* \rangle_{\mathcal{B}^*} U\phi_n. \tag{4.2}$$

If $\bar{A} \in \mathbb{S}_p[\mathcal{H}]$, the Schatten class of order p in $\mathcal{L}[\mathcal{H}]$, its norm can be represented as follows:

$$\left\| \bar{A} \right\|_p^{\mathcal{H}} = \left\{ Tr[\bar{A}^*\bar{A}]^{p/2} \right\}^{1/p} = \left\{ \sum_{n=1}^{\infty} \left(\bar{A}^*\bar{A}\phi_n, \phi_n \right)_{\mathcal{H}}^{p/2} \right\}^{1/p}$$

$$= \left\{ \sum_{n=1}^{\infty} \left| \mu_n(\bar{A}) \right|^p \right\}^{1/p}.$$

Definition 4.2. *We define $\mathbb{S}_p[\mathcal{B}]$, the Schatten class of order p in $\mathcal{L}[\mathcal{B}]$, as follows:*

$$\mathbb{S}_p[\mathcal{B}] = \left\{ A \in \mathbb{K}[\mathcal{B}] : \|A\|_p^{\mathcal{B}} = \left\{ \sum_{n=1}^{\infty} |\mu_n(A)|^p \right\}^{1/p} < \infty \right\}.$$

Since $\mu_n(A) = \mu_n(\bar{A})$, we have the following:

Corollary 4.3. *Let $A \in \mathbb{S}_p[\mathcal{B}]$; then, $\bar{A} \in \mathbb{S}_p[\mathcal{H}]$ and $\|A\|_p^{\mathcal{B}} = \|\bar{A}\|_p^{\mathcal{H}}$.*

It is clear that all of the theory of operator ideals on Hilbert spaces extends to separable Banach spaces. As in [GZ], we state a few of the more important results to provide a sense of the power provided by the existence of adjoints. The first result extends the theorems due to Weyl [W], Horn [HO], Lalesco [LA] and Lidskii [LI]. The proofs are all straightforward: for a given A, extend it to $\mathcal{H}$, use the Hilbert space result, and then restrict back to $\mathcal{B}$.

Theorem 4.4. *Let $A \in \mathbb{K}(\mathcal{B})$, and let $\{\lambda_n\}$ be the eigenvalues of A counted up to algebraic multiplicity. If Φ is a mapping on $[0, \infty]$ that is nonnegative and monotone increasing, then we have the following:*

1. *(Weyl)*

$$\sum_{n=1}^{\infty} \Phi(|\lambda_n(A)|) \leqslant \sum_{n=1}^{\infty} \Phi(\mu_n(A))$$

   and

2. *(Horn) If A_1, $A_2 \in \mathbb{K}(\mathcal{B})$*

$$\sum_{n=1}^{\infty} \Phi(|\lambda_n(A_1 A_2)|) \leqslant \sum_{n=1}^{\infty} \Phi(\mu_n(A_1)\mu_n(A_2)).$$

   In the case where $A \in \mathbb{S}_1(\mathcal{B})$, we have the following:

3. *(Lalesco)*

$$\sum_{n=1}^{\infty} |\lambda_n(A)| \leqslant \sum_{n=1}^{\infty} \mu_n(A) < \infty$$

   and

4. *(Lidskii)*

$$\sum_{n=1}^{\infty} \lambda_n(A) = tr(A).$$

Remark 4.5. *According to Pisier [PS], when $\mathcal{B}$ is a Hilbert space, equation (4) was first discovered by Grothendieck [GR1].*

4.2   Discussion

On a Hilbert space $\mathcal{H}$, the Schatten classes $\mathbb{S}_p(\mathcal{H})$ are the only ideals in $\mathbb{K}(\mathcal{H})$, and $\mathbb{S}_1(\mathcal{H})$ is minimal. In a Banach space, this is far from true. A complete history of the subject can be found in the recent book by Pietsch [PI1] (see also Retherford [RE], for a nice review). We limit our discussion to a few major topics. Grothendieck [GR] defined an important class of nuclear operators as follows:

Definition 4.6. *If $A \in \mathbb{F}(\mathcal{B})$ (the operators of finite rank), define the ideal $\mathbf{N}_1(\mathcal{B})$ by:*

$$\mathbf{N}_1(\mathcal{B}) = \{A \in \mathbb{F}(\mathcal{B}) \mid \mathbf{N}_1(A) < \infty\},$$

where

$$\mathbf{N}_1(A) = \mathrm{glb}\left\{\sum_{n=1}^{m} \|u_n^*\| \|u_n\| \,\Big|\, u_n^* \in \mathcal{B}^*, \, u_n \in \mathcal{B}, \, A = \sum_{n=1}^{m} u_n \langle \cdot, u_n^* \rangle \right\}$$

and the greatest lower bound is over all possible representations for A.

Grothendieck has shown that $\mathbf{N}_1(\mathcal{B})$ is the completion of the finite rank operators. Furthermore, $\mathbf{N}_1(\mathcal{B})$ is a Banach space with norm $\mathbf{N}_1(\cdot)$ and is a two-sided ideal in $\mathbb{K}(\mathcal{B})$. He also noted that if $\mathcal{B}$ has the approximation property, then for every $A \in N_1[\mathcal{B}]$, $tr[A] = \sum_{n=1}^{\infty} \langle u_n, u_n^* \rangle$ (trace) is well defined. Johnson and Szankowski [JS] have asked if the family eigenvalues of A, $\{\lambda_n[A]\}$ are absolutely convergent, can the above trace formula be replaced by

$$tr[A] = \sum_{n=1}^{\infty} \lambda_n(A).$$

Absolutely convergence is necessary as it is known that if $\mathcal{B}$ is not a Hilbert space, there is an $A \in N_1[\mathcal{B}]$, with $\sum_{n=1}^{\infty} |\lambda_n(A)| = \infty$ (see [JKMR] or Simon [SI]).

In order to compensate for the cloudy view of $\mathbb{K}(\mathcal{B})$ due to the lack of an adjoint for Banach spaces, Pietsch [PI2], [PI3] defined a number of classes of operator ideals for a given $\mathcal{B}$. Of particular importance for our discussion is the class $\mathbb{C}_p(\mathcal{B})$, defined by

$$\mathbb{C}_p(\mathcal{B}) = \left\{A \in \mathbb{K}(\mathcal{B}) \,\Big|\, \mathbb{C}_p(A) = \sum_{i=1}^{\infty} [s_i(A)]^p < \infty \right\},$$

where the singular numbers $s_n(A)$ are defined by:

$$s_n(A) = \inf\{\|A - K\|_{\mathcal{B}} \mid \text{rank of } K \leqslant n\}.$$

Pietsch has shown that, $\mathbb{C}_1(\mathcal{B}) \subset \mathbf{N}_1(\mathcal{B})$, while Johnson et al [JKMR] have shown that for each $A \in \mathbb{C}_1(\mathcal{B})$, $\sum_{n=1}^{\infty} |\lambda_n(A)| < \infty$. It is known that, if $\mathbb{C}_1(\mathcal{B}) = \mathbf{N}_1(\mathcal{B})$, then $\mathcal{B}$ is isomorphic to a Hilbert space (see [JKMR]). It is clear from the above discussion, that:

Corollary 4.7. $\mathbb{C}_p(\mathcal{B})$ *is a two-sided *ideal in $\mathbb{K}(\mathcal{B})$, and $\mathbb{S}_1(\mathcal{B}) \subset \mathbb{C}_1(\mathcal{B}) \subset \mathbf{N}_1(\mathcal{B})$.*

For a given separable Banach space, it is not clear how the spaces $\mathbb{C}_p(\mathcal{B})$ of Pietsch relate to our Schatten Classes $\mathbb{S}_p(\mathcal{B})$ (clearly $\mathbb{S}_p(\mathcal{B}) \subseteq \mathbb{C}_p(\mathcal{B})$). Thus, one question is that of the equality of $\mathbb{S}_p(\mathcal{B})$ and $\mathbb{C}_p(\mathcal{B})$. (We suspect that $\mathbb{S}_1(\mathcal{B}) = \mathbb{C}_1(\mathcal{B})$.)

4.3   Conclusion

In this paper, we have shown that the continuous dense embedding of a separable Banach space into a Hilbert space is a powerful tool for studying the structure of operators on Banach spaces. This approach also offers some new insights into the structure of Banach spaces themselves. We have first used this embedding to show that the lack of an S-basis does not invalidate the uniqueness property that a basis affords. We have also used this embedding to show that the representation of the dual of a Banach space is not unique. We

first investigated the non-uniqueness within the $\mathbb{C}[0, 1] \subset L^2[0, 1]$ setting. We then extended our study to all separable Banach spaces, showing that every closed densely defined linear operator A on $\mathcal{B}$ has a unique adjoint A^* defined on $\mathcal{B}$. We have further shown that $\mathcal{L}[\mathcal{B}]$, the bounded linear operators on $\mathcal{B}$, are continuously embedded in $\mathcal{L}[\mathcal{H}]$. This result allowed us to define the Schatten classes for $\mathcal{L}[\mathcal{B}]$ as the restriction of a subset of the ones in $\mathcal{L}[\mathcal{H}]$. Thus, the structure of $\mathcal{L}[\mathcal{B}]$, particularly that of the compact operators $\mathbb{K}[\mathcal{B}]$, is unrelated to the basis problem or the approximation problem for compact operators. We conclude that for the Enflo space $\mathcal{B}_e$, we can provide a representation for compact operators that is very close to the same representation for a Hilbert space, but the norm limit of the partial sums need not converge, which is the only missing property.

5 Appendix

5.1 Partial Proof of Gross-Kuelbs

We now construct $\mathcal{H}_2 \supset \mathcal{B}$ as a continuous dense embedding.

Proof. Let $\{e_n\}$ be a countable dense sequence on the unit ball of $\mathcal{B}$, and let $\{e_n^*\}$ be any fixed set of corresponding duality mappings (i.e., $e_n^* \in \mathcal{J}(e_n)$ for each n and $e_n^*(e_n) = \langle e_n, e_n^* \rangle = \|e_n\|_{\mathcal{B}}^2 = \|e_n^*\|_{\mathcal{B}^*}^2 = 1$). For each n, let $t_n = \frac{1}{2^n}$, and define (u, v) as follows:

$$(u, v) = \sum_{n=1}^{\infty} t_n e_n^*(u) \bar{e}_n^*(v) = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n^*(u) \bar{e}_n^*(v).$$

It is clear that (u, v) is an inner product on $\mathcal{B}$. Let $\mathcal{H}_2$ be the completion of $\mathcal{B}$ with respect to this inner product. It is clear that $\mathcal{B}$ is dense in $\mathcal{H}_2$, and

$$\|u\|_{\mathcal{H}_2}^2 = \sum_{n=1}^{\infty} t_n \left|e_n^*(u)\right|^2 \leq \sup_n \left|e_n^*(u)\right|^2 = \|u\|_{\mathcal{B}}^2,$$

so the embedding is continuous. $\square$

5.2 Proof of Lax's Theorem

Proof. To prove (1), let $u \in \mathcal{B}$, and without loss of generality, we can assume that $k = 1$ and $\|u\|_{\mathcal{H}} = 1$. Since T is self-adjoint,

$$\|Tu\|_{\mathcal{H}}^2 = (Tu, Tu) = \left(u, T^2 u\right) \leq \|u\|_{\mathcal{H}} \left\|T^2 u\right\|_{\mathcal{H}} = \left\|T^2 u\right\|_{\mathcal{H}}.$$

Thus, we have $\|Tu\|_{\mathcal{H}}^4 \leq \left\|T^4 u\right\|_{\mathcal{H}}$; consequently, it is clear that $\|Tu\|_{\mathcal{H}}^{2n} \leq \left\|T^{2n} u\right\|_{\mathcal{H}}$ for all n. It follows that

$$\|Tu\|_{\mathcal{H}} \leq (\left\|T^{2n} u\right\|_{\mathcal{H}})^{1/2n} \leq (\left\|T^{2n} u\right\|_{\mathcal{B}})^{1/2n}$$
$$\leq (\left\|T^{2n}\right\|_{\mathcal{B}})^{1/2n} (\|u\|_{\mathcal{B}})^{1/2n} \leq \|T\|_{\mathcal{B}} (\|u\|_{\mathcal{B}})^{1/2n}.$$

By allowing $n \to \infty$, we obtain that $\|Tu\|_{\mathcal{H}} \leq \|T\|_{\mathcal{B}}$ for u in a dense set of the unit ball of $\mathcal{H}$. It follows that

$$\|T\|_{\mathcal{H}} = \sup_{\|u\|_{\mathcal{H}} = 1} \|Tu\|_{\mathcal{H}} \leq \|T\|_{\mathcal{B}}.$$

To prove (2), suppose that $\lambda_0 \notin \sigma(T)$ over $\mathcal{B}$ such that $T - \lambda_0 I$ has a bounded inverse S on $\mathcal{B}$. Since T is symmetric on $\mathcal{H}$, S is also symmetric on $\mathcal{H}$. It follows that S extends to a bounded inverse on $\mathcal{H}$ such that $\lambda_0 \notin \sigma(T)$ over $\mathcal{H}$. It follows that the spectrum of T on the extension to $\mathcal{H}$ is a subset of the spectrum of T on $\mathcal{B}$.

To prove (3), suppose that $\lambda \in \sigma(T)_p$, the point spectrum of T, such that $T - \lambda I$ has a finite dimensional null space N and $dimN = dim\{\mathcal{B}/J\}$, where $J = (T - \lambda I)(\mathcal{B})$.

Since T is symmetric, every vector in J is orthogonal to N. Conversely, from $dimN = dim\{\mathcal{B}/J\}$, we find that J contains all vectors that are orthogonal to N. It follows that $(T - \lambda I)$ is a one-to-one onto mapping of $J \to J$ such that $T - \lambda I = S$ has an inverse on J, which is bounded (on J) by the closed graph theorem. It follows that the extension $\bar{S}$ of S to the closure of $J, \bar{J}$ in $\mathcal{H}$ is bounded on $\bar{J}$. This means that $(T - \lambda I)$ is the orthogonal compliment of N over $\mathcal{H}$; thus, λ belongs to the point spectrum of T on $\mathcal{H}$, and the null space of $(T - \lambda I)$ over $\mathcal{H}$ is N. It follows that $\sigma_p(T)$ is unchanged on extension to $\mathcal{H}$.   □

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