Communications in **M**athematical **A**nalysis
Volume 20, Number 2, pp. 1–7 (2017) www.math-res-pub.org/cma
ISSN 1938-9787

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(Communicated by Simeon Reich)

Abstract

Let A and B be bounded linear operators in a Banach space. We consider the following problem: if $\sum_{k=0}^{\infty}\|A^k\|\|B^k\| < \infty$, under what conditions $\sum_{k=0}^{\infty}\|(AB)^k\| < \infty$?

AMS Subject Classification: 47A10, 47B10, 15A42

Keywords: Banach space, powers of linear operators, norm estimates

1 Introduction and statement of the main result

Let $\mathcal{X}$ be a Banach space with a norm $\|.\|$ and $\mathcal{B}(\mathcal{X})$ the algebra of bounded linear operators in $\mathcal{X}$. $\|A\|$, $\sigma(A)$ and $r_s(A)$ denote the operator norm, spectrum and spectral radius of $A \in \mathcal{B}(\mathcal{X})$, respectively.

We consider the following problem: let $A, B \in \mathcal{B}(\mathcal{X})$ and $\sum_{k=0}^{\infty}\|A^k\|\|B^k\| < \infty$. What conditions provide the inequality $\sum_{k=0}^{\infty}\|(AB)^k\| < \infty$?

The theory of powers of bounded operators is a significant part of the operator theory, cf. [1, 2, 8, 10], and references given therein. In particular, below we derive conditions that provide the power boundedness of AB. The power bounded operators have remarkable spectral properties and attract the attention of many mathematicians, cf. [3, 4, 5, 9, 11].

To the best of our knowledge the above stated problem was not considered in the available literature. Put

$$\zeta_m := \sum_{k=0}^{m-1}\|A^k\|\|B^k\| \ \ (m > 1) \text{ and } K = AB - BA.$$

Now we are in a position to formulate our main result.

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Theorem 1.1. *Let $A, B \in \mathcal{B}(\mathcal{X})$ and for some integer $m \geq 2$ the condition*

$$\zeta_{m-1}(\zeta_m - 1)\|K\| < 1 \tag{1.1}$$

hold. Then

$$\max_{k=0,\dots,m} \|(AB)^k\| \leq \frac{\max_{k=0,\dots,m}\|A^k B^k\|}{1 - \|K\|\zeta_{m-1}(\zeta_m - 1)} \quad \textit{and}$$

$$\max_{k=2,\dots,m} \|(AB)^k - A^k B^k\| \leq \frac{\|K\|\zeta_{m-1}(\zeta_m - 1)\max_{k=0,\dots,m}\|A^k B^k\|}{1 - \|K\|\zeta_{m-1}(\zeta_m - 1)}.$$

In addition,

$$\sum_{k=0}^{m} \|(AB)^k\| \leq \frac{\zeta_{m+1}}{1 - \|K\|\zeta_{m-1}(\zeta_m - 1)} \quad \textit{and} \quad \sum_{k=0}^{m} \|(AB)^k - A^k B^k\| \leq \frac{\|K\|\zeta_{m-1}(\zeta_m - 1)\zeta_{m+1}}{1 - \|K\|\zeta_{m-1}(\zeta_m - 1)}.$$

The proof of this theorem is presented in the next section. The theorem is sharp: if $K = 0$, then $(AB)^k = A^k B^k$ for all $k \geq 0$.

Let

$$\zeta_\infty := \sum_{k=0}^{\infty} \|A^k\|\|B^k\| < \infty$$

and

$$\zeta_\infty(\zeta_\infty - 1)\|K\| < 1. \tag{1.2}$$

Then due to Theorem 1.1 we have

$$\max_{k=0,1,\dots} \|(AB)^k\| \leq \frac{\max_{k=0,1,\dots}\|A^k B^k\|}{1 - \|K\|\zeta_\infty(\zeta_\infty - 1)},$$

$$\max_{k=0,1,\dots} \|(AB)^k - A^k B^k\| \leq \frac{\|K\|(\zeta_\infty - 1)\zeta_\infty}{1 - \|K\|\zeta_\infty(\zeta_\infty - 1)} \max_{k=0,1,\dots} \|A^k B^k\|,$$

$$\sum_{k=0}^{\infty} \|(AB)^k\| \leq \frac{\zeta_\infty}{1 - \|K\|\zeta_\infty(\zeta_\infty - 1)} \quad \text{and} \quad \sum_{k=0}^{\infty} \|(AB)^k - A^k B^k\| \leq \frac{\|K\|(\zeta_\infty - 1)\zeta_\infty^2}{1 - \|K\|\zeta_\infty(\zeta_\infty - 1)}. \tag{1.3}$$

Corollary 1.2. *Let condition (1.2) hold. Then $r_s(AB) < 1$ and therefore the difference equation*

$$x_{k+1} = ABx_k \ \ (k = 1, 2, \dots)$$

is exponentially stable, i.e. $\|x_k\| \leq const\, \rho^k$ $(0 < \rho < 1)$ for any its solution x_k $(k = 1, 2, \dots)$.

Indeed, from (1.3) it follows that $\|(AB)^k\| \to 0$ as $k \to \infty$, provided condition (1.2) holds. Hence due to the spectral mapping theorem $r_s^k(AB) \leq \|(AB)^k\| \to 0$. So really $r_s(AB) < 1$.

Furthermore, from the well-known representation

$$A^k = \frac{1}{2\pi i}\int_{|z|=r_A} z^k (zI - A)^{-1} dz \ \ (k = 1, 2, \dots),$$

for any $r_A > r_s(A)$ it follows that $\|A^k\| \le c_A r_A^k$ ($c_A = const \ge 1$). Similarly,$\|B^k\| \le c_B r_B^k$ ($r_B > r_s(B); c_B = const \ge 1$). Assuming that $r_s(A) r_s(B) < 1$ we can take $r_A r_B < 1$. Besides,

$$\zeta_\infty \le c_A c_B \sum_{k=0}^{\infty} (r_A r_B)^k = \frac{c_A c_B}{1 - r_B r_A}.$$

So, if

$$\frac{\|K\| c_A c_B}{(1 - r_B r_A)^2} (c_A c_B - 1 + r_A) < 1, \tag{1.4}$$

then $r_s(AB) < 1$ due to Corollary 1.2.

2 Proof of Theorem 1.1

Put $X_m = (AB)^m$ and $Y_m = A^m B^m$ for $m = 1, 2, \dots$, $X_0 = Y_0 = I$, and

$$J_m = \sum_{j=1}^{m-1} \sum_{k=0}^{j-1} \|A^k\| \|A^{j-k}\| \|B^j\| \ \ (m = 2, 3, \dots).$$

Lemma 2.1. *If*

$$\|K\| J_m < 1 \tag{2.1}$$

for some integer $m \ge 2$, then

$$\max_{0 \le k \le m} \|X_k\| \le \frac{\max_{0 \le k \le m} \|Y_k\|}{1 - \|K\| J_m}$$

and

$$\max_{0 \le k \le m} \|X_k - Y_k\| \le \frac{\max_{k \le m} \|Y_k\| \|K\| J_m}{1 - \|K\| J_m}.$$

Proof. We have

$$X_{m+1} = A B X_m \ \ (m = 0, 1, \dots) \tag{2.2}$$

and

$$Y_{m+1} = A^{m+1} B^{m+1} = A A^m B B^m = A B A^m B^m + A[A^m, B] B^m,$$

where $[A^m, B] = A^m B - B A^m$. Hence,

$$Y_{m+1} = A B Y_m + F_m \ \ (m = 0, 1, \dots), \tag{2.3}$$

with

$$F_m = A[A^m, B] B^m \ \ (m \ge 1), F_0 = 0.$$

Subtracting (2.2) from (2.3), we get

$$Y_{m+1} - X_{m+1} = A B (Y_m - X_m) + F_m \ \ (m = 2, 3, \dots)$$

with $Y_1 - X_1 = 0$. By induction we can write

$$Y_m - X_m = \sum_{j=1}^{m-1} (AB)^{m-j-1} F_j = \sum_{j=1}^{m-1} X_{m-j-1} F_j \ \ (m \geq 2), \tag{2.4}$$

and therefore,

$$\|X_m - Y_m\| \leq \sum_{j=1}^{m-1} \|X_{m-1-j}\| \|F_j\| \quad (m \geq 2). \tag{2.5}$$

As is checked in [7, formula (2.4)],

$$[A^j, B] := A^j B - BA^j = \sum_{k=0}^{j-1} A^{j-k-1} [A, B] A^k \ \ (j = 1, 2, ...). \tag{2.6}$$

Consequently,

$$F_j = \sum_{k=0}^{j-1} A^{j-k} K A^k B^j, j \geq 1,$$

and

$$\sum_{j=1}^{m-1} \|F_j\| \leq \|K\| J_m. \tag{2.7}$$

Put

$$x_\nu := \max_{0 \leq m \leq \nu} \|X_m\|, y_\nu := \max_{0 \leq m \leq \nu} \|Y_m\|.$$

Since $X_0 = Y_0 = I, X_1 = Y_1 = AB$, due to (2.5) and (2.7),

$$\max_{0 \leq m \leq \nu} \|X_m - Y_m\| = \max_{2 \leq m \leq \nu} \|X_m - Y_m\| \leq x_\nu \|K\| J_\nu \ \ (\nu = 2, 3, ...), \tag{2.8}$$

Consequently, $x_\nu \leq y_\nu + \|K\| x_\nu J_\nu \ \ (\nu = 2, 3, ...)$. According to (2.1)

$$x_\nu \leq \frac{y_\nu}{1 - \|K\| J_\nu}.$$

Hence, by (2.8) we finish the proof. □

Lemma 2.2. *If condition (2.1) holds for some $m \geq 2$, then*

$$\sum_{k=0}^{m} \|X_k\| \leq \frac{1}{1 - \|K\| J_m} \sum_{k=0}^{m} \|Y_k\| \tag{2.8}$$

and

$$\sum_{k=0}^{m} \|X_k - Y_k\| \leq \frac{\|K\| J_m}{1 - \|K\| J_m} \sum_{k=1}^{m} \|Y_k\|. \tag{2.9}$$

Proof. Since $X_1 = Y_1$ and $X_0 = Y_0$, from (2.5) we have

$$\sum_{m=2}^{\nu} \|X_m - Y_m\| \le \sum_{m=2}^{\nu} \sum_{j=1}^{m-1} \|X_{m-1-j}\| \, \|F_j\| = \sum_{m=2}^{\nu} \sum_{i=2}^{m} \|X_{m-i}\| \, \|F_{i-1}\| =$$

$$\sum_{i=2}^{\nu} \sum_{m=i}^{\nu} \|X_{m-i}\| \|F_{i-1}\| = \sum_{i=2}^{\nu} \|F_{i-1}\| \sum_{k=0}^{\nu-i} \|X_k\| \le \sum_{t=1}^{\nu-1} \|F_t\| \sum_{k=0}^{\nu} \|X_k\| \;\; (\nu \ge 2).$$

Hence, due to (2.7)

$$\sum_{m=0}^{\nu} \|X_m - Y_m\| \le J_\nu \|K\| \sum_{k=0}^{\nu} \|X_k\|.$$

Thus,

$$\sum_{m=0}^{\nu} \|X_m\| \le \sum_{m=0}^{\nu} \|Y_m\| + J_\nu \|K\| \sum_{m=0}^{\nu} \|X_m\|.$$

Now (2.1) implies

$$\sum_{m=0}^{\nu} \|X_m\| \le \frac{1}{1 - J_\nu \|K\|} \sum_{m=0}^{\nu} \|Y_m\|$$

and

$$\sum_{m=0}^{\nu} \|X_m - Y_m\| \le \frac{J_\nu \|K\|}{1 - J_\nu \|K\|} \sum_{m=0}^{\nu} \|Y_m\|,$$

as claimed. □

Furthermore,

$$J_m = \sum_{j=1}^{m-1} \sum_{k=0}^{j-1} \|A^k\| \|A^{j-k}\| \|B^j\| = \sum_{t=0}^{m-2} \sum_{k=0}^{t} \|A^k\| \|A^{t+1-k}\| \|B^{t+1}\| =$$

$$\sum_{k=0}^{m-2} \|A^k\| \sum_{t=k}^{m-2} \|A^{t+1-k}\| \|B^{t+1}\| = \sum_{k=0}^{m-2} \|A^k\| \sum_{s=0}^{m-2-k} \|A^{s+1}\| \|B^{s+k+1}\|$$

$$\le \sum_{k=0}^{m-2} \|A^k\| \|B^k\| \sum_{s=0}^{m-2} \|A^{s+1}\| \|B^{s+1}\|.$$

Thus $J_m \le \zeta_{m-1}(\zeta_m - 1)$. Now the assertion of Theorem 1.1 follows from Lemmas 2.1 and 2.2, and the obvious inequality

$$\sum_{k=0}^{m} \|Y_k\| \le \zeta_{m+1}.$$

□

3 Particular cases

3.1 Operators in a Euclidean space

In this subsection A and B are $n \times n$-matrices. Introduce the quantity (the departure from normality of A)

$$g(A) = [N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2]^{1/2},$$

where $\lambda_k(A)$ $(k = 1, ..., n)$ are the eigenvalues of A taking with their multiplicities, and $N_2(A) = (\text{trace } (AA^*))^{1/2}$ is the Hilbert-Schmidt (Frobenius) norm of A. The following relations are checked in [6, Section 2.1]:

$$g^2(A) \le N_2^2(A) - |\text{trace } (A^2)| \text{ and } g^2(A) \le \frac{N_2^2(A - A^*)}{2}.$$

If A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$. By Corollary 2.7.2 from [6] we have

$$\|A^m\| \le \sum_{k=0}^{n-1} \frac{m! g^k(A) r_s^{m-k}(A)}{(m-k)!(k!)^{3/2}} \quad (m = 1, 2, ...).$$

Note that $1/(k!) = 0$ if $k < 0$. Thus $\zeta_\infty \le \hat{\zeta}_{\infty,n}$, where

$$\hat{\zeta}_{\infty,n} := \sum_{m=0}^{\infty} \sum_{j,k=0}^{n-1} \frac{g^j(A) g^k(B) (m!)^2 r_s^{m-j}(A) r_s^{m-k}(B)}{(j!k!)^{3/2}(m-j)!(m-k)!}.$$

Now we can directly apply Corollary 1.2, provided $\hat{\zeta}_{\infty,n}(\hat{\zeta}_{\infty,n} - 1)\|K\| < 1$. If A is normal, then

$$\hat{\zeta}_{\infty,n} := \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \frac{g^k(B) m! r_s^m(A) r_s^{m-k}(B)}{(k!)^{3/2}(m-k)!}.$$

If both A and B are normal, then

$$\hat{\zeta}_{\infty,n} = \sum_{m=0}^{\infty} r_s^m(A) r_s^m(B) = \frac{1}{1 - r_s(A) r_s(B)}.$$

3.2 Operators in a Hilbert space

In this subsection, $\mathcal{X}$ is a Hilbert space, $A, B \in \mathcal{B}(\mathcal{X})$ and, in addition, $\Im A = (A - A^*)/2i, \Im B$ are Hilbert-Schmidt operators, i.e. $N_2(\Im A) = (\text{trace } (\Im A)^2)^{1/2} < \infty$, $N_2(\Im B) < \infty$. As is shown in [6, Example 7.15.5],

$$\|A^m\| \le \sum_{k=0}^{m} \frac{m! u^k(A) r_s^{m-k}(A)}{(m-k)!(k!)^{3/2}} \quad (m = 1, 2, ...),$$

where $u(A) = \sqrt{2} N_2(\Im A)$. Thus $\zeta_\infty \le \hat{\zeta}$, where

$$\hat{\zeta} := \sum_{m=0}^{\infty} \sum_{j,k=0}^{m} \frac{(m!)^2 u^j(A) r_s^{m-j}(A) u^k(B) r_s^{m-k}(B)}{(k!j!)^{3/2}(m-k)!(m-j)!}.$$

Now we can directly apply Corollary 1.2, provided $\hat{\zeta}(\hat{\zeta}-1)\|K\| < 1$. If A is selfadjoint, then

$$\hat{\zeta} = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{m! r_s^m(A) u^k(B) r_s^{m-k}(B)}{(k!)^{3/2}(m-k)!}.$$

If both A and B are selfadjoint, then $\hat{\zeta} = 1/(1 - r_s(A) r_s(B))$.

I am very grateful to the referee of this paper for his (her) really helpful remarks

References

[1] A. Atzmon, Power regular operators, *Transactions of AMS*, **347**, no. 8, (1995) , pp 3101-3109.

[2] Aymen Ettaieb, Hammam Sousse and Habib Ouerdiane, Higher powers of analytical operators and associated *-Lie algebras, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, **19**, no. 2 (2016), article 1650013 (20 pages).

[3] A.G. Baskakov, Harmonic and spectral analysis of power bounded operators and bounded semigroups of operators on a Banach space. *Mat. Zametki* **97** (2015), no. 2, pp 174-190 (Russian); translation in Math. Notes 97 (2015), no. 1-2, pp 164-178.

[4] C. Cuny, Almost everywhere convergence of generalized ergodic transforms for invertible power-bounded operators in Lp. *Colloq. Math.* **124** (2011), no. 1, pp 61–77.

[5] M.F. Gamal', A class of power bounded operators which are similar to contractions *Acta Sci. Math. (Szeged)* **80** (2014), pp 625-637.

[6] M.I. Gil', *Operator Functions and Localization of Spectra*, Lecture Notes In Mathematics vol. 1830, Springer-Verlag, Berlin, 2003.

[7] M.I. Gil, A bound for the Hilbert-Schmidt norm of generalized commutators of nonself-adjoint operators, *Operators and Matrices*, **11**, no. 1 (2017), pp 115-123.

[8] K. Hedayatian and S. Yarmahmoodi, Power regularity of isometric N-Jordan operators. *Honam Math. J.* **38** (2016), no. 2, pp 317–323.

[9] Y. Katznelson and L. Tzafriri, On power bounded operators, *J. Funct. Anal.* **68 (3)**, (1986), pp 313–328 .

[10] Masoumeh Faghih-Ahmadi, Powers of A-m-isometric operators and their supercyclicity, *Bull. Malays. Math. Sci. Soc.* **39** (2016) pp 901-911.

[11] A. Ulger, and O. Yavuz, A Fredholm alternative-like result on power bounded operators. *Turkish J. Math.* **35** (2011), no. 3, pp 473–478.