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# A Poincaré Inequality for Functions with Locally Bounded Variation in $\mathbb{R}^d$

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**Abstract**

We prove a weighted Poincaré inequality in a subspace of $BV_{\text{loc}}$ whose elements have variation measure in a Wiener amalgam space of Radon measures.

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## 1 Introduction

In the study of partial differential equations, a basic problem is to obtain controls on a function in terms of the norms of its distributional derivatives in some Banach space. Widely used examples of such controls are Poincaré inequalities. The global form of these inequalities are usually established in the Sobolev spaces (see [2], [8], [9]) and the space of functions of bounded variation (see [1], [4], [11]). In this paper we derive a global, weighted, weak-type Poincaré -Wirtinger inequality in the setting of a class of subspaces of the space of functions of locally bounded variation on $\mathbb{R}^d$, which contains properly Sobolev spaces.

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## 2 Results and notations

Let $d$ be a fixed positive integer. $\mathbb{R}^d$ is endowed with its usual Euclidean norm $x \mapsto |x|$ and Lebesgue measure $E \mapsto |E| = \int_E dx$.
If $\Omega$ is an open subset of $\mathbb{R}^d$ then

- we denote by $L^1_{\text{loc}}(\Omega)$ the standard Lebesgue space of (equivalence classes modulo equality almost everywhere in $\Omega$ of) locally integrable real valued functions on $\Omega$
- for any element $\alpha$ of $[1;\infty]$, $L^\alpha(\Omega)$ is the classical Lebesgue space on $\Omega$ equipped with its usual norm $\|\ \|_{\alpha,\Omega}$ ($\|\|_{\alpha,\mathbb{R}^d}$ is simply denoted $\|\|_\alpha$)
- for any element $f$ of $L^1_{\text{loc}}(\Omega)$ we set

$$\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_d})$$

  where $\frac{\partial f}{\partial x_j}$ stands for the partial derivative, in the sense of distribution, of $f$ with respect to the $j$-th coordinate ($1 \leq j \leq d$).

Let $M(\mathbb{R}^d)$ denotes the space of Radon measures on $\mathbb{R}^d$. We set

- $\Delta = \{Q(x,r) = \prod_{j=1}^{d} [x_j - \frac{r}{2}; x_j + \frac{r}{2}] \ / \ (x,r) \in \mathbb{R}^d \times (0;\infty)\}$
- $S = \{\{Q_i\}_{i\in I} \subset \Delta \ / \ I \text{ countable and } \quad Q_j \cap Q_i = \emptyset \quad \text{if} \quad j \neq i\}$
- for $1 \leq \alpha \leq p \leq \infty$,

$$T^{p,\alpha}(\mathbb{R}^d) = \{\mu \in M(\mathbb{R}^d) \ / \ \|\mu\|_{T^{p,\alpha}} < \infty\}$$

  with, for any element $\mu$ of $M(\mathbb{R}^d)$,

$$\|\mu\|_{T^{p,\alpha}} = \begin{cases} \sup\left\{ (\sum_{i\in I} (|Q_i|^{\frac{1}{\alpha}-1} |\mu|(Q_i))^p)^{\frac{1}{p}} \ / \ \{Q_i\}_{i\in I} \subset S \right\} & \text{if } p < \infty \\ \sup\left\{ |Q|^{\frac{1}{\alpha}-1} |\mu|(Q) \ / \ Q \in \Delta \right\} & \text{if } p = \infty \end{cases}$$

  where $|\mu|$ denotes the total variation of $\mu$.

Let us recall that:

- $BV_{\text{loc}}(\mathbb{R}^d) = \{f \in L^1_{loc}(\mathbb{R}^d) \ / \ \frac{\partial f}{\partial x_j} \in M(\mathbb{R}^d) \text{ for } j = 1, 2, \cdots, d\}$
  is the space of functions of locally bounded variation,
- the variation measure $|Df|$ of an element $f$ of $BV_{\text{loc}}(\mathbb{R}^d)$ is defined by

$$|Df|(\Omega) = \sup\left\{ \int_\Omega f(x) div\varphi(x) dx \ / \ \varphi = (\varphi_1, \varphi_2, \cdots, \varphi_d) \in \mathbf{C}^1_c(\Omega, \mathbb{R}^d), \ |\varphi| \leq 1 \right\}$$

  for any open subset $\Omega$ of $\mathbb{R}^d$, and there is a function $\sigma_f$ from $\mathbb{R}^d$ to $\mathbb{R}^d$, $|Df|$-measurable and such that

$$\begin{aligned} |\sigma_f(x)| &= 1, \quad \text{almost every } x \in \mathbb{R}^d, \\ \int_{\mathbb{R}^d} f(x) div\varphi(x) dx &= -\int_{\mathbb{R}^d} \varphi(x) \cdot \sigma_f(x) d|Df|(x), \quad \varphi \in \mathbf{C}^1_c(\mathbb{R}^d, \mathbb{R}^d). \end{aligned}$$

**Definition 2.1.** For $1 \le \alpha \le p \le \infty$, we set

$$BV^{p,\alpha}(\mathbb{R}^d) = \{f \in BV_{\text{loc}}(\mathbb{R}^d) \ / \ |Df| \in T^{p,\alpha}(\mathbb{R}^d)\}.$$

Our main result reads as follows.

**Theorem 2.2.** *Let us suppose that $1 \le \alpha < d$ and $f$ belongs to $BV^{\infty,\alpha}(\mathbb{R}^d)$.*

*1) There is a real number $f_{(\infty)}$ such that*

$$\lim_{r\to\infty} \frac{1}{|Q(x,r)|}\int_{Q(x,r)} f(y)dy = f_{(\infty)}, \quad x \in \mathbb{R}^d. \tag{2.1}$$

*2) If $0 \le \frac{1}{\theta} < \frac{\alpha}{d}$, $\frac{\frac{1}{\alpha}-\frac{1}{d}}{1-\frac{1}{\theta}} = \frac{1}{q} < \frac{1}{p} \le \frac{1}{\alpha}$, $f$ belongs to $BV^{p,\alpha}(\mathbb{R}^d)$ and $\nu$ is a non negative element of $T^{\infty,\theta}(\mathbb{R}^d)$ then*

$${}_{\nu}\|f - f_{(\infty)}\|^*_{q,\infty} := \sup_{\lambda>0} \lambda[\nu(\{x \in \mathbb{R}^d /|f(x) - f_{(\infty)}| > \lambda\})]^{\frac{1}{q}} \le A\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\|\,|Df|\,\|_{T^{p,\alpha}} \tag{2.2}$$

*where $A$ is a real number not depending on $f$ and $\nu$.*

Point 2) of Theorem 2.2 has the following non weighted form.

**Theorem 2.3.** *Let us suppose that: $1 < \alpha < d$, $\frac{1}{\alpha} - \frac{1}{d} = \frac{1}{p}$ and $f$ belongs to $BV^{p,\alpha}(\mathbb{R}^d)$. Then*

$$\|f - f_{(\infty)}\|^*_{p,\infty} := \sup_{\lambda>0} \lambda|\{x \in \mathbb{R}^d \ / \ |f(x) - f_{(\infty)}| > \lambda\}|^{\frac{1}{p}} \quad \le \quad A\|\,|Df|\,\|_{T^{p,\alpha}} \tag{2.3}$$

*where $A$ is a real number not depending on $f$.*

In the sequel, we shall identify each element $f$ of $L^1_{\text{loc}}(\mathbb{R}^d)$ to the Radon measure $\mu_f$ on $\mathbb{R}^d$ defined by $d\mu_f(x) = f(x)dx$. So $L^1_{\text{loc}}(\mathbb{R}^d)$ is viewed as the subspace of $M(\mathbb{R}^d)$ consisting in its absolutely continuous (with respect to the Lebesgue measure) elements.
Notice that

$$|\mu_f|(A) \quad = \quad \mu_{|f|}(A) = \int_A |f(x)|dx, \quad A \subset \mathbb{R}^d,\ f \in L^1_{\text{loc}}(\mathbb{R}^d).$$

Therefore, for $1 \le \alpha \le p \le \infty$,

$$F(1, p, \alpha)(\mathbb{R}^d) \quad = \quad \{f \in L^1_{\text{loc}}(\mathbb{R}^d) \ / \ \|f\|_{F(1,p,\alpha)} = \|\mu_f\|_{T^{p,\alpha}} < \infty\}$$

is a subspace of $T^{p,\alpha}(\mathbb{R}^d)$. These spaces have been introduced in [5]. Let us recall some of their properties and links with classical spaces.

**Proposition 2.4.** [5] *Let us assume that $1 \le \alpha \le p \le \infty$.*

*1) $(T^{p,\alpha}(\mathbb{R}^d), \|\ \|_{T^{p,\alpha}})$ and $(F(1, p, \alpha), \|\ \|_{F(1,p,\alpha)})$ are real Banach spaces.*

*2) If $p < q \le \infty$ then*

$$\|\mu\|_{T^{q,\alpha}} \le \|\mu\|_{T^{p,\alpha}}, \quad \mu \in M(\mathbb{R}^d)$$

*and therefore, $T^{p,\alpha}(\mathbb{R}^d)$ is continuously embedded in $T^{q,\alpha}(\mathbb{R}^d)$.*

*3) a)* $\|f\|_{F(1,p,\alpha)} \leq \|f\|_\alpha, \quad f \in L^1_{loc}(\mathbb{R}^d)$
*and so* $L^\alpha(\mathbb{R}^d)$ *is continuously embedded in* $F(1,p,\alpha)$.

*b)* $F(1,\infty,\alpha)(\mathbb{R}^d)$ *is the classical Morrey space* $L^{1,d(1-\frac{1}{\alpha})}(\mathbb{R}^d)$ *if* $\alpha < \infty$.

*c)* $F(1,p,\alpha)(\mathbb{R}^d) = L^\alpha(\mathbb{R}^d)$ *if* $\alpha$ *belongs to* $\{1,p\}$.

*d)* $L^\alpha(\mathbb{R}^d)$ *is a proper subspace of* $F(1,p,\alpha)(\mathbb{R}^d)$ *if* $1 < \alpha < p$.

*4) If* $p < \infty$ *then*

$$\lim_{r\to\infty} \|\chi_{\mathbb{R}^d\setminus Q(0,r)}\mu\|_{T^{p,\alpha}} = 0, \quad \mu \in T^{p,\alpha}(\mathbb{R}^d).$$

Let us recall that, if $1 \leq \alpha \leq \infty$, then

- for any open set $\Omega$ of $\mathbb{R}^d$ the Sobolev space $W^{1,\alpha}(\Omega)$ is defined by

$$W^{1,\alpha}(\Omega) = \left\{ f \in L^\alpha(\Omega) \;/\; \frac{\partial f}{\partial x_j} \in L^\alpha(\Omega) \text{ for } j = 1,2,\cdots,d \right\}$$

- $W^{1,\alpha}_{\text{loc}}(\mathbb{R}^d) = \{f \in L^1_{\text{loc}}(\mathbb{R}^d) \;/\; f \in W^{1,\alpha}(\Omega)$ for any open and bounded subset $\Omega$ of $\mathbb{R}^d\}$.

Notice that if $f$ belongs to $W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ then it is an element of $BV_{\text{loc}}(\mathbb{R}^d)$ and its variation measure satisfies

$$d|Df|(x) = |\nabla f(x)| dx$$

and therefore, by Proposition2.4

$$\| \, |Df| \, \|_{T^{p,\alpha}} = \| \, |\nabla f| \, \|_{F(1,p,\alpha)} \leq \| \, |\nabla f| \, \|_\alpha \, , \; 1 \leq \alpha \leq p \leq \infty$$

It is clear from what precedes that Theorem 2.2 and Theorem 2.3 are strongly related to the following result obtained by G.Lu and B.Ou.

**Proposition 2.5.** [9] *Let us assume that* $1 \leq \alpha < d$, $f \in W^{1,\alpha}_{loc}(\mathbb{R}^d)$ *and* $|\nabla f| \in L^\alpha(\mathbb{R}^d)$. *Then there exists:*

*1) a real number* $f_{(\infty)}$ *such that*

$$\lim_{r\to\infty} \frac{1}{|B(0,r)|} \int_{B(0,r)} f(y)dy = f_{(\infty)}; \tag{2.4}$$

*where* $B(0,r) = \{x \in \mathbb{R}^d \;/\; |x| < r\}$

*2) a real number* $A_{d,\alpha}$*, not depending on* $f$ *such that*

$$\|f - f_{(\infty)}\|_p \leq A_{d,\alpha} \| \, |\nabla f| \, \|_\alpha \tag{2.5}$$

*where* $\frac{1}{p} = \frac{1}{\alpha} - \frac{1}{d}$.

It is worth noticing that Theorem 2.2 and Theorem 2.3 deal with functions belonging to $BV_{\text{loc}}(\mathbb{R}^d)$, while Proposition 2.5 is concerned only in elements of $W^{1,1}_{\text{loc}}(\mathbb{R}^d)$. Even if we focus exclusively on elements of $W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ the following comments are relevant.

a) It is easy to see that the relations (2.1) and (2.4) are equivalent. But, if $1 < \alpha < \infty$ then $L^{\alpha}(\mathbb{R}^d)$ is properly included in $F(1, p, \alpha)$ [see point 3) d) of Proposition 2.4] and therefore the hypothesis on $f$ in Proposition 2.5 is stronger than that under which (2.1) is true in Theorem 2.2

b) Inequality (2.3) is a weak form of inequality (2.5). But again we notice that if $1 < \alpha < p$ then the hypothesis on $f$ in Theorem 2.3 is weaker than the one in Proposition 2.5.

The proof of Theorem 2.2 relies upon two results interesting for their own right.
The first one is a norm inequality for the fractional integral operator $I_{\gamma}(0 < \gamma < 1)$ defined by

$$I_{\gamma}\mu(x) = \int_{\mathbb{R}^d} |x - y|^{d(\gamma-1)} d\mu(y) \quad \text{and} \quad I_{\gamma} f(x) = \int_{\mathbb{R}^d} |x - y|^{d(\gamma-1)} f(y) dy$$

for $\mu$ in $M(\mathbb{R}^d)$, $f$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ and the points $x$ of $\mathbb{R}^d$ where the above integrals make sense.

**Proposition 2.6.** *Suppose that* $0 < \gamma < \frac{1}{\alpha} \leq 1$, $0 \leq \frac{1}{\theta} < \alpha\gamma$ *and* $\frac{\frac{1}{\alpha}-\gamma}{1-\frac{1}{\theta}} = \frac{1}{q} < \frac{1}{p} \leq \frac{1}{\alpha}$. *Then there exists a real number* $B > 0$ *such that for any non negative Radon measures* $\mu$ *and* $\nu$ *on* $\mathbb{R}^d$

$$_{\nu}\|I_{\gamma}\mu\|^{*}_{q,\infty} \quad \leq \quad B\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\|\mu\|_{T^{p,\alpha}}. \tag{2.6}$$

We have also the following result.

**Proposition 2.7.** *Let us assume that :* $0 < \gamma < \frac{1}{\alpha} \leq 1$, $0 \leq \frac{1}{\theta} \leq \alpha\gamma$, $\frac{\frac{1}{\alpha}-\gamma}{1-\frac{1}{\theta}} = \frac{1}{p} \leq \frac{1}{\alpha}$ *and* $\nu$ *is a non negative Radon measure on* $\mathbb{R}^d$ *satisfying the following condition* $(A_{\infty})$ *:*
*≪for any real number* $\delta > 0$ *there is a real number* $\rho > 0$ *such that, if* $Q$ *is a cube of* $\mathbb{R}^d$ *and* $E$ *a Borel subset of* $Q$ *then* $[|E| \leq \rho|Q| \Rightarrow \nu(E) \leq \delta\nu(Q)]$ *≫.*
*Then there is a real number* $C > 0$ *such that for any non negative Radon measure* $\mu$ *on* $\mathbb{R}^d$

$$_{\nu}\|I_{\gamma}\mu\|^{*}_{p,\infty} \quad \leq \quad C\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|\mu\|_{T^{p,\alpha}}. \tag{2.7}$$

Let $\rho$ be a fixed non negative element of $\mathbf{C}^{\infty}_{c}(\mathbb{R}^d, \mathbb{R})$, with support included in the unit ball $\overline{B}(0; 1) = \{x \in \mathbb{R}^d \ / \ |x| \leq 1\}$ of $\mathbb{R}^d$ and satisfying $\int_{\mathbb{R}^d} \rho(x)dx = 1$.
For any real number $\varepsilon > 0$, we set

$$\begin{aligned} \rho_{\varepsilon}(x) &= \varepsilon^{-d}\rho(\varepsilon^{-1}x), \quad x \in \mathbb{R}^d \\ f^{\varepsilon} &= \rho_{\varepsilon} * f, \quad f \in L^1_{\text{loc}}(\mathbb{R}^d). \end{aligned}$$

It is well known that, for any real number $\alpha \geq 1$ and any element $f$ of $L^{\alpha}_{\text{loc}}(\mathbb{R}^d)$, $\{f^{\varepsilon}\}_{\varepsilon>0}$ is a subset of $\mathbf{C}^{\infty}(\mathbb{R}^d, \mathbb{R})$ satisfying

$$\lim_{\varepsilon \to 0} \|(f - f^{\varepsilon})\chi_{\Omega}\|_{\alpha} \quad = \quad 0$$

for any bounded measurable subset $\Omega$ of $\mathbb{R}^d$.
The second result we shall use in the proof of Theorem 2.2 reads as follows.

**Proposition 2.8.** *Let us assume that* $1 \leq \alpha < d$*. We have*

1) $(\frac{1}{\alpha} - \frac{1}{d})\frac{d}{d-1} < \frac{1}{\alpha}$;

2) *if* $(\frac{1}{\alpha} - \frac{1}{d})\frac{d}{d-1} < \frac{1}{p} \leq \frac{1}{\alpha}$ *and* $f$ *belongs to* $BV^{p,\alpha}(\mathbb{R}^d)$ *then there is a subset* $N$ *of* $\mathbb{R}^d$ *such that*

   a) *for any element* $x$ *of* $\mathbb{R}^d \setminus N$,

$$\begin{aligned} f^*(x) &= \lim_{r \to 0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} f(y)dy \text{ exists in } \mathbb{R}, \\ f^*(x) &= \lim_{\varepsilon \to 0} f^{\varepsilon}(x) \quad \text{and} \quad \lim_{r \to 0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y) - f^*(x)|dy = 0; \end{aligned}$$

   b) $H^t(N) = 0, \quad t > dp(\frac{1}{\alpha} - \frac{1}{d})$;

   *where* $H^t$ *denotes the t-dimensional Hausdorff measure.*

Point 2) of Proposition 2.8 gives a measure of the thinness of the complementary set of the Lebesgue points of an element of $BV^{p,\alpha}(\mathbb{R}^d)$, in the spirit of the following classical result.

**Proposition 2.9.** [4] *Let us assume that* $f$ *belongs to* $BV_{loc}(\mathbb{R}^d)$*. Then there is a subset* $N$ *of* $\mathbb{R}^d$ *such that :*

1) *for any element* $x$ *of* $\mathbb{R}^d \setminus N$, $f^*(x) = \lim_{r \to 0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} f(y)dy$ *exists in* $\mathbb{R}$ *and satisfies* $f^*(x) = \lim_{\varepsilon \to 0} f^{\varepsilon}(x)$.

2) $H^{d-1}(N) = 0$.

The remainder of this paper is organized as follows. In Section 3 we study the asymptotical mean value of an element of the space $BV^{\infty,\alpha}(\mathbb{R}^d)$. Section 4 is devoted to the precise representative $f^*$ of an element $f$ of $BV^{p,\alpha}(\mathbb{R}^d)$ and the thinness of the complementary set of its Lebesgue points. In Section 5 we prove our main result, that is point 2) of Theorem 2.2. Finally in Section 6 we define on $BV^{p,\alpha}(\mathbb{R}^d)$ a structure of Banach space.

## 3 Existence of asymptotical mean value of elements of $BV^{\infty,\alpha}(\mathbb{R}^d)$

We consider a fixed element $f$ of $BV_{\text{loc}}(\mathbb{R}^d)$.
The variation measure $|Df|$ of $f$ dominates the gradients of its regularization $f^{\varepsilon}$ ($\varepsilon > 0$) as stated below.

**Lemma 3.1.** *Let* $\varepsilon$ *be a positive real number and* $\Omega$ *a bounded open subset of* $\mathbb{R}^d$*. We have*

$$\int_{\Omega} |\nabla f^{\varepsilon}(x)|dx \quad \leq \quad \int_{\mathbb{R}^d} |Df|(\Omega - y)\rho_{\varepsilon}(y)dy \leq |Df|(\Omega_{\varepsilon})$$

*where* $\Omega_{\varepsilon} = \{x \in \mathbb{R}^d \ / \ \inf\{|x-y| \ / \ y \in \Omega\} < \varepsilon\}$.

*Proof.* a) Let $\varphi$ be an element of $C^1_c(\Omega,\mathbb{R}^d)$ satisfying $|\varphi|\le 1$. We have :

$$\begin{aligned}\int_\Omega \nabla f^\varepsilon(x)\cdot\varphi(x)dx &= -\int_\Omega f^\varepsilon(x)div\varphi(x)dx = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_\varepsilon(y)f(x-y)div\varphi(x)dydx\\ &= -\int_{\mathbb{R}^d}[\int_{\mathbb{R}^d} f(x-y)div\varphi(x)dx]\rho_\varepsilon(y)dy\\ &= -\int_{\mathbb{R}^d}[\int_{\mathbb{R}^d} f(z)div\varphi(y+z)dz]\rho_\varepsilon(y)dy.\end{aligned}$$

Let us notice that, for any element $y$ of $\mathbb{R}^d$, $z\longmapsto\varphi(y+z)$ belongs to $C^1_c(\Omega-y,\mathbb{R}^d)$ and satisfies

$$|\varphi(y+z)|\le 1,\quad z\in\mathbb{R}^d.$$

Therefore one has

$$\int_\Omega \nabla f^\varepsilon(x)\cdot\varphi(x)dx \le \int_{\mathbb{R}^d}|Df|(\Omega-y)\rho_\varepsilon(y)dy.$$

Thus

$$\int_\Omega |\nabla f^\varepsilon(x)|dx \le \int_{\mathbb{R}^d}|Df|(\Omega-y)\rho_\varepsilon(y)dy.$$

b) Let us notice that, for any element $y$ of $\mathbb{R}^d$,

$$[|y|\ge\varepsilon\Rightarrow\rho_\varepsilon(y)=0]\quad\text{and}\quad[|y|\le\varepsilon\Rightarrow\Omega-y\subset\Omega_\varepsilon].$$

Therefore

$$\int_{\mathbb{R}^d}|Df|(\Omega-y)\rho_\varepsilon(y)dy \le |Df|(\Omega_\varepsilon)\int_{\mathbb{R}^d}\rho_\varepsilon(y)dy = |Df|(\Omega_\varepsilon).$$

□

**Notation 3.2.** For $g$ belonging to $L^1_{\text{loc}}(\mathbb{R}^d)$, $E$ a subset of $\mathbb{R}^d$ such that $|E|\neq 0$ and an element $(x,r)$ of $\mathbb{R}^d\times(0,\infty)$, we set

$$g_E = |E|^{-1}\int_E g(y)dy;\quad g_{(x,r)} = g_{Q(x,r)};\quad \Omega(x,r)=\prod_{j=1}^d (x_j-\frac{r}{2};x_j+\frac{r}{2}).$$

**Lemma 3.3.** *Let us assume that $(x,R)$ and $(y,r)$ are two elements of $\mathbb{R}^d\times(0;\infty)$ such that $Q(y,r)$ is included in $Q(x,R)$. Then*

$$\int_{Q(x,R)}|f(z)-f_{(y,r)}|dz \le C_d\frac{R^d}{r^{d-1}}|Df|(Q(x,R))$$

*where $C_d = |B(0,1)|^{1-\frac{1}{d}}d^{\frac{d}{2}}$.*

*Proof.* Let $\delta$ and $\varepsilon$ be two real numbers such that $0 < \delta < r$ and $0 < \varepsilon < \frac{1}{2}(r-\delta)$.
We notice that $\Omega(y,\delta) \subset \Omega(x, R-(r-\delta))$. Therefore, we have

$$\int_{\Omega(x,R-r+\delta)} |f^{\varepsilon}(z) - f^{\varepsilon}{}_{(y,\delta)}|dz \leq C_d (R-r+\delta)^d \delta^{1-d} \int_{Q(x,R-r+\delta)} |\nabla f^{\varepsilon}(z)|dz.$$

(see formula (7.45) in section 7.8 of [7]).
Thus, by an application of Lemma 3.1, we get

$$\int_{\Omega(x,R-r+\delta)} |f^{\varepsilon}(z) - f^{\varepsilon}{}_{(y,\delta)}|dz \leq C_d (R-r+\delta)^d \delta^{1-d} |Df|(Q(x, R-r+\delta+\varepsilon)).$$

Furthermore $\lim_{\varepsilon \to 0} \|(f^{\varepsilon} - f)\chi_{Q(x,R)}\|_1 = 0$. So,

$$\begin{aligned} \int_{\Omega(x,R-r+\delta)} |f(z) - f_{(y,\delta)}|dz &\leq \limsup_{\varepsilon \to 0} \int_{Q(x,R-r+\delta)} |f^{\varepsilon}(z) - f^{\varepsilon}{}_{(y,\delta)}|dz \\ &\leq C_d (R-r+\delta)^d \delta^{1-d} |Df|(Q(x, R-r+\delta)) \end{aligned}$$

and

$$\int_{Q(x,R)} |f(z) - f_{(y,r)}|dz = \lim_{\delta \to r} \int_{Q(x,R-r+\delta)} |f(z) - f_{(y,\delta)}|dz \leq C_d R^d r^{1-d} |Df|(Q(x,R)).$$

□

**Proposition 3.4.** *Let us assume that $f$ belongs to $BV^{\infty,\alpha}(\mathbb{R}^d)$ with $1 \leq \alpha \leq \infty$.*

*1) For any elements $(x,R)$ and $(y,r)$ of $\mathbb{R}^d \times (0;\infty)$ such that $Q(y,r)$ is included in $Q(x,R)$, we have*

$$|f_{(x,R)} - f_{(y,r)}| \leq C_d (\frac{R}{r})^{d(1-\frac{1}{\alpha})} r^{1-\frac{d}{\alpha}} \| \, |Df| \, \|_{T^{\infty,\alpha}}$$

*where $C_d = |B(0,1)|^{1-\frac{1}{d}} d^{\frac{d}{2}}$.*

*2) If $\alpha < d$ then there is a real number $f_{(\infty)}$ such that*

$$\lim_{R \to \infty} f_{(x,R)} = f_{(\infty)}, \quad x \in \mathbb{R}^d.$$

*Proof.* 1) Let $(x,R)$ and $(y,r)$ be elements of $\mathbb{R}^d \times (0;\infty)$ such that $Q(y,r)$ is included in $Q(x,R)$. We have

$$|f_{(x,R)} - f_{(y,r)}| = R^{-d} \left| \int_{Q(x,R)} [f(z) - f_{(y,r)}]dz \right| \leq R^{-d} \int_{Q(x,R)} |f(z) - f_{(y,r)}|dz$$

and therefore by Lemma 3.3

$$|f_{(x,R)} - f_{(y,r)}| \leq C_d \, r^{1-d} |Df|(Q(x,R)) \leq C_d \, r^{1-d} R^{d(1-\frac{1}{\alpha})} \| \, |Df| \, \|_{T^{\infty,\alpha}}.$$

2) Let us assume that $1 \leq \alpha < d$.

a) We consider an element $x$ of $\mathbb{R}^d$ and two real numbers $r$ and $R$ such that $0 < r < R$. Let $k$ be the unique non negative integer satisfying $2^k r < R \le 2^{k+1} r$. We have

$$|f_{(x,R)} - f_{(x,r)}| \le \sum_{j=0}^{k-1} |f_{(x,2^{j+1}r)} - f_{(x,2^j r)}| + |f_{(x,R)} - f_{(x,2^k r)}|$$

and therefore, by the result obtained in part 1),

$$|f_{(x,R)} - f_{(x,r)}| \le C_d[2^{d(1-\frac{1}{\alpha})}(\sum_{j=0}^{k-1} 2^{j(1-\frac{d}{\alpha})}) r^{1-\frac{d}{\alpha}} + (\frac{R}{2^k r})^{d(1-\frac{1}{\alpha})}(2^k r)^{1-\frac{d}{\alpha}}]\| \, |Df| \, \|_{T^{\infty,\alpha}}.$$

$$|f_{(x,R)} - f_{(x,r)}| \le C_d \, 2^{d(1-\frac{1}{\alpha})} \sum_{j=0}^{k} 2^{j(1-\frac{d}{\alpha})}) r^{1-\frac{d}{\alpha}} \| \, |Df| \, \|_{T^{\infty,\alpha}}.$$

Since $1 - \frac{d}{\alpha} < 0$ by hypothesis, we have $\sum_{j=0}^{k} 2^{j(1-\frac{d}{\alpha})} < \infty$ and $\lim_{r\to\infty} r^{1-\frac{d}{\alpha}} = 0$. Hence, for any real number $\varepsilon > 0$ there is a real number $\delta_\varepsilon > 0$ such that :

$$\delta_\varepsilon \le r < R \Rightarrow |f_{(x,R)} - f_{(x,r)}| < \varepsilon.$$

Therefore, there is a real number $f_{(x,\infty)}$ satisfying

$$\lim_{r\to\infty} f_{(x,r)} = f_{(x,\infty)}.$$

b) Let $x$ and $y$ be two elements of $\mathbb{R}^d$ such that $|x - y| = s > 0$. For any real number $r > 2s$, $Q(y, r-2s)$ is included in $Q(x, r)$ and therefore, by the result obtained in part 1)

$$|f_{(x,r)} - f_{(y,r-2s)}| \le C_d[r(r-2s)^{-1}]^{d(1-\frac{1}{\alpha})}(r-2s)^{1-\frac{d}{\alpha}} \| \, |Df| \, \|_{T^{\infty,\alpha}}.$$

So, because of the inequalities $d(1 - \frac{1}{\alpha}) \ge 0$ and $1 - \frac{d}{\alpha} < 0$, we have

$$\lim_{r\to\infty} |f_{(x,r)} - f_{(y,r-2s)}| = 0.$$

Therefore

$$f_{(x,\infty)} = \lim_{r\to\infty} f_{(x,r)} = \lim_{r\to\infty} f_{(y,r-2s)} = \lim_{r\to\infty} f_{(y,r)} = f_{(y,\infty)}.$$

This means that $f_{(x,\infty)}$ does not depend on $x$ and so, there is a real number $f_{(\infty)}$ such that

$$\lim_{r\to\infty} f_{(x,r)} = f_{(\infty)}, \quad x \in \mathbb{R}^d.$$

□

**Proposition 3.5.** *Let us suppose that $f$ belongs to $BV^{\infty,\alpha}(\mathbb{R}^d)$ with $1 \le \alpha < d$. Then*

$$f^{\varepsilon}{}_{(\infty)} = f_{(\infty)}, \quad \varepsilon \in (0, \infty).$$

*Proof.* Let $\varepsilon$ be a positive real number. For any element $(x,r)$ of $\mathbb{R}^d\times(0,\infty)$. We have

$$\begin{aligned} f^{\varepsilon}{}_{(x,r)} &= r^{-d}\int_{Q(x,r)}\int_{\mathbb{R}^d} f(y-z)\rho_\varepsilon(z)dzdy = r^{-d}\int_{\mathbb{R}^d}\int_{Q(x-z,r)} f(y)dy\rho_\varepsilon(z)dz \\ &= \int_{\mathbb{R}^d} f_{(x-z,r)}\rho_\varepsilon(z)dz = \int_{\mathbb{R}^d}(f_{(x-z,r)}-f_{(x,r+2\varepsilon)})\rho_\varepsilon(z)dz + f_{(x,r+2\varepsilon)}. \end{aligned}$$

It follows that

$$|f^{\varepsilon}_{(x,r)} - f_{(x,r+2\varepsilon)}| = |\int_{\mathbb{R}^d}(f_{(x-z,r)}-f_{(x,r+2\varepsilon)})\rho_\varepsilon(z)dz| \le \int_{\mathbb{R}^d}|f_{(x-z,r)}-f_{(x,r+2\varepsilon)}|\rho_\varepsilon(z)dz$$

and therefore, by point 1) of Proposition 3.4,

$$|f^{\varepsilon}{}_{(x,r)} - f_{(x,r+2\varepsilon)}| \le C_d[(r+2\varepsilon)r^{-1}]^{d(1-\frac{1}{\alpha})}r^{1-\frac{d}{\alpha}}\|\,|Df|\,\|_{T^{\infty,\alpha}}.$$

Thus, letting $r$ goes to infinity, we obtain, by point 2) of Proposition 3.4

$$|f^{\varepsilon}{}_{(\infty)} - f_{(\infty)}| = 0 \quad \text{that is} \quad f^{\varepsilon}{}_{(\infty)} = f_{(\infty)}.$$

□

# 4 Precise representative of an element of $BV^{p,\alpha}(\mathbb{R}^d)$

For any element $\beta$ of $[1;\infty]$ and non negative Radon measure $\mu$ on $\mathbb{R}^d$, we set

- $m_\beta\mu(x) = \sup\{|Q(x,r)|^{\frac{1}{\beta}-1}\mu(Q(x,r))\ /\ 0<r<\infty\}, \quad x\in\mathbb{R}^d$
- $m_{\beta,R}\mu(x) = \sup\{|Q(x,r)|^{\frac{1}{\beta}-1}\mu(Q(x,r))\ /\ 0<r\le R\}, \quad 0<R<\infty, \quad x\in\mathbb{R}^d.$

**Proposition 4.1.** *Assume that $f$ belongs to $BV_{loc}(\mathbb{R}^d)$, and $d<\beta\le\infty$.*

*1) There is a real number $C_{d,\beta}>0$, not depending on $f$ such that*

$$|f_{(x,R)} - f_{(x,r)}| \le R^{-d}\int_{Q(x,R)}|f(y)-f_{(x,r)}|dy \le C_{d,\beta}R^{1-\frac{d}{\beta}}m_{\beta,2R}|Df|(x),$$

*2) If $x$ is a point of $\mathbb{R}^d$ such that $m_{\beta,1}|Df|(x)<\infty$ then*

$$\begin{aligned} \lim_{r\to 0} f_{(x,r)} &= f^*(x) \text{ exists in } \mathbb{R}; \\ f^*(x) &= \lim_{\varepsilon\to 0} f^\varepsilon(x) \text{ and } \lim_{r\to 0} r^{-d}\int_{Q(x,r)}|f(y)-f^*(x)|dy = 0. \end{aligned}$$

*Proof.* 1) Let us consider a point $x$ of $\mathbb{R}^d$ and $0<r\le R<\infty$.

We denote by $k$ the unique positive integer satisfying $2^{-k}R<r\le 2^{-k+1}R$. We have

$$\begin{aligned} |f-f_{(x,r)}| &= |(f-f_{(x,R)}) + (f_{(x,R)}-f_{(x,2^{-1}R)}) + \ldots\ldots + (f_{(x,2^{-k+1}R)}-f_{(x,r)})| \\ &\le |f-f_{(x,R)}| + |f_{(x,R)}-f_{(x,2^{-1}R)}| + \ldots\ldots + |f_{(x,2^{-k+1}R)}-f_{(x,r)}| \end{aligned}$$

and therefore

$$R^{-d}\int_{Q(x,R)}|f(y)-f_{(x,r)}|dy \le R^{-d}\int_{Q(x,R)}|f(y)-f_{(x,R)}|dy + \sum_{j=1}^{k-1}(2^{-j+1}R)^{-d}\int_{Q(x,2^{-j+1}R)}|f(y)-f_{(x,2^{-j}R)}|dy + (2^{-k+1}R)^{-d}\int_{Q(x,2^{-k+1}R)}|f(y)-f_{(x,r)}|dy.$$

An application of Lemma 3.3 leads to

$$R^{-d}\int_{Q(x,R)}|f(y)-f_{(x,r)}|dy \le C_d[R^{1-d}|Df|(Q(x,R)) + \sum_{j=1}^{k-1}(2^{-j}R)^{1-d}|Df|(Q(x,2^{-j+1}R)) + r^{1-d}|Df|(Q(x,2^{-k+1}R))]$$

$$\begin{aligned} R^{-d}\int_{Q(x,R)}|f(y)-f_{(x,r)}|dy &\le C_d\sum_{j=0}^{k}(2^{-j}R)^{1-d}|Df|(Q(x,2^{-j+1}R)) \\ &\le C_d 2^{d(1-\frac{1}{\beta})}\sum_{j=0}^{k}(2^{-j}R)^{1-\frac{d}{\beta}}\sup_{0<\delta\le 2R}|Q(x,\delta)|^{\frac{1}{\beta}-1}|Df|(Q(x,\delta)) \\ &\le C_{d,\beta}R^{1-\frac{d}{\beta}}m_{\beta,2R}|Df|(x) \end{aligned}$$

where $C_{d,\beta}$ is a real number not depending on $f$, $R$ and $x$.
Furthermore, we have

$$|f_{(x,R)}-f_{(x,r)}| = |R^{-d}\int_{Q(x,R)}(f(y)-f_{(x,r)})dy| \le R^{-d}\int_{Q(x,R)}|f(y)-f_{(x,r)}|dy.$$

Therefore

$$|f_{(x,R)}-f_{(x,r)}| \le C_{d,\beta}R^{1-\frac{d}{\beta}}m_{\beta,2R}|Df|(x).$$

2) Let us suppose that $x$ is a point of $\mathbb{R}^d$ such that $m_{\beta,1}|Df|(x) < \infty$.

a) From the result obtained in point 1) we get

$$|f_{(x,R)}-f_{(x,r)}| < C_{d,\beta}\delta^{1-\frac{d}{\beta}}m_{\beta,1}|Df|(x), \quad 0<r\le R<\delta\le\frac{1}{2}.$$

Furthermore, we have

$$\lim_{\delta\to 0}C_{d,\beta}\delta^{1-\frac{d}{\beta}}m_{\beta,1}|Df|(x) = 0.$$

Thus there is a real number $f^*(x)$ such that

$$\lim_{r\to 0}f_{(x,r)} = f^*(x).$$

b) For any element $r$ of $(0;\frac{1}{2})$, we have

$$\begin{aligned} r^{-d}\int_{Q(x,r)}|f(y)-f^*(x)|dy &\le r^{-d}\int_{Q(x,r)}|f(y)-f_{(x,r)}|dy + |f_{(x,r)}-f^*(x)| \\ &\le C_{d,\beta}r^{1-\frac{d}{\beta}}m_{\beta,1}|Df|(x) + |f_{(x,r)}-f^*(x)|. \end{aligned}$$

Therefore

$$\lim_{r\to 0}r^{-d}\int_{Q(x,r)}|f(y)-f^*(x)|dy = 0.$$

c) For any real number $\varepsilon > 0$, we have also

$$|f^{\varepsilon}(x) - f^*(x)| = |\varepsilon^{-d} \int_{\mathbb{R}^d} [f(x-y) - f^*(x)]\rho(\varepsilon^{-1}y)dy| \leq \|\rho\|_\infty \varepsilon^{-d} \int_{Q(x,\varepsilon)} |f(x-y) - f^*(x)|dy$$

and therefore, by the result obtained in b)

$$\lim_{\varepsilon \to 0} |f^{\varepsilon}(x) - f^*(x)| = 0.$$

□

**Proposition 4.2.** *Let us assume that* $1 \leq \alpha \leq p < \infty$ *and* $f$ *belongs to* $BV^{p,\alpha}(\mathbb{R}^d)$*. Then*

$$H^{dp(\frac{1}{\alpha}-\frac{1}{\beta})}(\{x \in \mathbb{R}^d \ / \ m_\beta|Df|(x) = \infty\}) = 0, \quad \beta \in (\alpha, \infty].$$

*Proof.* Let us suppose that $\beta$ belongs to $(\alpha, \infty]$.

a) We consider a real number $\lambda > 0$ and set $E_\lambda = \{x \in \mathbb{R}^d \ / \ m_\beta|Df|(x) > \lambda\}$.
For any element $x$ of $E_\lambda$ there is a real number $r(x)$ such that :

$$\lambda < |Q(x, r(x))|^{\frac{1}{\beta}-1}|Df|(Q(x, r(x)))$$

and therefore
$$\lambda|Q(x, r(x))|^{\frac{1}{\alpha}-\frac{1}{\beta}} < |Q(x, r(x))|^{\frac{1}{\alpha}-1}|Df|(Q(x, r(x))) \leq \| \, |Df| \, \|_{T^{\infty,\alpha}} \leq \| \, |Df| \, \|_{T^{p,\alpha}}$$

$$r(x) < (\lambda^{-1}\||Df|\|_{T^{p,\alpha}})^{\frac{1}{d(\frac{1}{\alpha}-\frac{1}{\beta})}} < \infty.$$

By Vitali's covering Lemma, there is a subset $\{Q_i \ / \ i \in I\}$ of $\{Q(x, r(x)) \ / \ x \in E_\lambda\}$ such that

$$\begin{cases} E_\lambda \subset \bigcup_{i \in I} 5Q_i \\ Q_i \cap Q_j = \emptyset \quad \text{for } i, \ j \in I \text{ with } i \neq j \end{cases}$$

where $5Q_i$ is the cube of $\mathbb{R}^d$ having the same center as $Q_i$ and with side length five times that of $Q_i$.
We notice that

$$1 < \lambda^{-1}|Q_i|^{\frac{1}{\beta}-1}|Df|(Q_i), \quad i \in I$$

and therefore

$$\begin{aligned} \sum_{i \in I} |5Q_i|^{p(\frac{1}{\alpha}-\frac{1}{\beta})} &\leq 5^{pd(\frac{1}{\alpha}-\frac{1}{\beta})} \sum_{i \in I} |Q_i|^{p(\frac{1}{\alpha}-\frac{1}{\beta})}[\lambda^{-1}|Q_i|^{\frac{1}{\beta}-1}|Df|(Q_i)]^p \\ \sum_{i \in I} |5Q_i|^{p(\frac{1}{\alpha}-\frac{1}{\beta})} &\leq 5^{pd(\frac{1}{\alpha}-\frac{1}{\beta})}\lambda^{-p} \sum_{i \in I} |Q_i|^{p(\frac{1}{\alpha}-1)}[|Df|(Q_i)]^p \leq 5^{pd(\frac{1}{\alpha}-\frac{1}{\beta})}\lambda^{-p}\| \, |Df| \, \|^p_{T^{p,\alpha}}. \end{aligned}$$

Thus

$$H_\infty^{pd(\frac{1}{\alpha}-\frac{1}{\beta})}(E_\lambda) = \inf\left\{\sum_{i \in I} (\mathrm{diam} A_i)^{pd(\frac{1}{\alpha}-\frac{1}{\beta})} \ / \ E_\lambda \subset \bigcup_{i \in I} A_i\right\} \leq [C\lambda^{-1}\| \, |Df| \, \|_{T^{p,\alpha}}]^p,$$

where $C$ is a real number not depending on $f$ and $\lambda$.

b) We notice that

$$\{x \in \mathbb{R}^d \ / \ m_\beta |Df|(x) = +\infty\} \subset E_\lambda, \qquad \lambda \in (0,\infty).$$

Thus, by the result obtained in point a), we have

$$H^{pd(\frac{1}{\alpha}-\frac{1}{\beta})}(\{x \in \mathbb{R}^d \ / \ m_\beta |Df|(x) = +\infty\}) = 0.$$

(see Lemma 1 of Section 2.1 of [4]).

□

*Proof of Proposition 2.8.* It is easy to verify that the hypothesis $1 < \alpha < d$ implies that $(\frac{1}{\alpha} - \frac{1}{d})\frac{d}{d-1} < \frac{1}{\alpha}$.
Let us assume that $(\frac{1}{\alpha} - \frac{1}{d})\frac{d}{d-1} < \frac{1}{p} \leq \frac{1}{\alpha}$ and $f$ belongs to $BV^{p,\alpha}(\mathbb{R}^d)$.

a) It is easy to verify that: $\frac{1}{dp} \leq \frac{1}{\alpha} - \frac{d-1}{dp} < \frac{1}{d}$, $\beta \mapsto \varphi(\beta) = dp(\frac{1}{\alpha} - \frac{1}{\beta})$ is an increasing function on $\mathbb{R}$, $\varphi(d) = dp(\frac{1}{\alpha} - \frac{1}{d})$ and $d-1 = \varphi(\beta_0)$, with $\frac{1}{\beta_0} = \frac{1}{\alpha} - \frac{d-1}{dp}$. Thus $\varphi$ is a bijection of $(d,\beta_0]$ on $(dp(\frac{1}{\alpha} - \frac{1}{d}), d-1]$.

b) For any element $t$ of $(dp(\frac{1}{\alpha} - \frac{1}{d}), d-1]$ we set

$$N_t = \{x \in \mathbb{R}^d \ / \ m_{\beta,1}|Df|(x) = \infty\} \ \text{ with } \quad \beta = \varphi^{-1}(t).$$

We notice that for $0 < \beta_1 < \beta_2$ we have $m_{\beta_1,1}|Df| \leq m_{\beta_2,1}|Df|$.
Therefore

$$N_{t_1} \subset N_{t_2}, \qquad dp(\frac{1}{\alpha} - \frac{1}{d}) < t_1 < t_2 \leq d-1.$$

Let us set $N = \bigcap_{d-1 \geq t > dp(\frac{1}{\alpha}-\frac{1}{d})} N_t$.

c) Let us consider a point $x$ of $\mathbb{R}^d \setminus N$.
There is an element $t$ of $(dp(\frac{1}{\alpha} - \frac{1}{d}), d-1]$ such that $x \in \mathbb{R}^d \setminus N_t$. We notice that $\beta = \varphi^{-1}(t)$ satisfies $\quad 0 < \frac{1}{\beta} < \frac{1}{d} \quad$ and $\quad m_{\beta,1}|Df|(x) < \infty$.
Therefore, by Proposition 4.1,

- $f^*(x) = \lim_{r\to 0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} f(y)dy$ exists in $\mathbb{R}$
- $f^*(x) = \lim_{\varepsilon\to 0} f^\varepsilon(x)$ and $\lim_{r\to 0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y) - f^*(x)|dy = 0$.

d) . Let us consider an element $t$ of $(dp(\frac{1}{\alpha} - \frac{1}{d}), d-1]$. We notice that $\beta = \varphi^{-1}(t)$ satisfies $0 < \frac{1}{\beta} < \frac{1}{d} < \frac{1}{\alpha}$. Thus, by Proposition 4.2, $H^t(N_t) = 0$ and therefore $H^t(N) = 0$.

. As $H^{d-1}(N) = 0$, we have $H^t(N) = 0$ for any real number $t > d-1$.

□

For $1 \leq \theta \leq \infty$, the elements of $T^{\infty,\theta}(\mathbb{R}^d)$ are absolutely continuous with respect to the $d(1 - \frac{1}{\theta})$-dimensional Hausdorff measure as stated belove.

**Proposition 4.3.** *Let us assume that* $1 \leq \theta \leq \infty$ *and* $\nu$ *is a non negative element of* $T^{\infty,\theta}(\mathbb{R}^d)$. *Then for any* $\nu$*-measurable subset* $A$ *of* $\mathbb{R}^d$ *satisfying* $H^{d(1-\frac{1}{\theta})}(A) = 0$, *we have* $\nu(A) = 0$.

*Proof.* Let $A$ be a $\nu$-measurable subset of $\mathbb{R}^d$ satisfying $H^{d(1-\frac{1}{\theta})}(A) = 0$.
For any real number $\varepsilon > 0$ there is a family $\{B(x_i, r_i) \,/\, i \in I\}$ of balls of $\mathbb{R}^d$ satisfying

$$A \subset \bigcup_{i\in I} B(x_i, r_i) \text{ and } \sum_{i\in I} r_i^{d(1-\frac{1}{\theta})} < \varepsilon.$$

Therefore

$$\nu(A) \leq \sum_{i\in I} \nu(B(x_i, r_i)) \leq \sum_{i\in I} \nu(Q(x_i, 2r_i)) \leq 2^{d(1-\frac{1}{\theta})} \|\nu\|_{T^{\infty,\theta}} \sum_{i\in I} r_i^{d(1-\frac{1}{\theta})} \leq 2^{d(1-\frac{1}{\theta})} \|\nu\|_{T^{\infty,\theta}} \varepsilon.$$

Thus

$$\nu(A) = 0.$$

□

**Corollary 4.4.** *Let us assume that :* $1 < \alpha < d$, $0 \leq \frac{1}{\theta} < \frac{\alpha}{d}$, $\frac{\frac{1}{\alpha}-\frac{1}{d}}{1-\frac{1}{\theta}} < \frac{1}{p} \leq \frac{1}{\alpha}$, *$f$ belongs to $BV^{p,\alpha}(\mathbb{R}^d)$ and $\nu$ is a non negative element of $T^{\infty,\theta}(\mathbb{R}^d)$. Then for $\nu$-almost every element $x$ of $\mathbb{R}^d$*

$$\lim_{\varepsilon\to 0} f^{\varepsilon}(x) = f^*(x) = \lim_{r\to 0} f_{(x,r)} \in \mathbb{R}.$$

*Proof.* 1[rst]case $\quad 0 \leq \frac{1}{\theta} \leq \frac{1}{d}$.
By hypothesis $f$ belongs to $BV^{p,\alpha}(\mathbb{R}^d)$ and therefore to $BV_{\text{Loc}}(\mathbb{R}^d)$.
So, by Proposition 2.9 there is a subset $N$ of $\mathbb{R}^d$ such that

$$\begin{cases} \lim\limits_{\varepsilon\to 0} f^{\varepsilon}(x) = f^*(x) = \lim\limits_{r\to 0} f_{(x,r)} \in \mathbb{R}, & x \in \mathbb{R}^d \setminus N \\ H^{d-1}(N) = 0. \end{cases}$$

Furthermore $d - 1 \leq d(1 - \frac{1}{\theta})$ and $\nu$ is a non negative element of $T^{\infty,\theta}(\mathbb{R}^d)$. Thus, by Proposition 4.3, $\nu(N) = 0$.
2[nd]case $\quad \frac{1}{d} < \frac{1}{\theta} < \frac{\alpha}{d}$.
From inequalities $\frac{1}{d} < \frac{1}{\theta}$ and $\frac{\frac{1}{\alpha}-\frac{1}{d}}{1-\frac{1}{\theta}} < \frac{1}{p} \leq \frac{1}{\alpha}$, we get $0 < (\frac{1}{\alpha} - \frac{1}{d})\frac{d}{d-1} < \frac{\frac{1}{\alpha}-\frac{1}{d}}{1-\frac{1}{\theta}} < \frac{1}{p} \leq \frac{1}{\alpha}$.
Therefore, by Proposition 2.8, there is a subset $N$ of $\mathbb{R}^d$ such that :

$$\begin{cases} \lim\limits_{\varepsilon\to 0} f^{\varepsilon}(x) = f^*(x) = \lim\limits_{r\to 0} f_{(x,r)}, & x \in \mathbb{R}^d \setminus N \\ H^t(N) = 0, \quad t > dp(\frac{1}{\alpha} - \frac{1}{d}). \end{cases}$$

By hypothesis $dp(\frac{1}{\alpha} - \frac{1}{d}) < d(1 - \frac{1}{\theta})$. Therefore $H^{d(1-\frac{1}{\theta})}(N) = 0$ and thus, by Proposition 4.3, $\nu(N) = 0$. □

**Remark 4.5.** Let us assume that : $1 < \alpha < d$, $0 \leq \frac{1}{\theta} < \frac{\alpha}{d}$, $\frac{\frac{1}{\alpha}-\frac{1}{d}}{1-\frac{1}{\theta}} < \frac{1}{p} \leq \frac{1}{\alpha}$ and $f$ belongs to $BV^{p,\alpha}(\mathbb{R}^d)$. Let us define $f^*$ on $\mathbb{R}^d$ by:

$$f^*(x) = \begin{cases} \lim\limits_{r\to 0} f_{(x,r)} & \text{when this limit exists in } \mathbb{R} \\ 0 & \text{if} \quad \text{not.} \end{cases}$$

By the Lebesgue differentiation Theorem and Proposition 2.8, $f^*$ is a representative of the element $f$ of $L^1_{\text{loc}}(\mathbb{R}^d)$ (the so called precise representative). Actually, by Corollary 4.4, if $\nu$ is a non negative element of $T^{\infty,\theta}(\mathbb{R}^d)$ then $f^*$ is a representative of $f$ as element of $L^1_{\text{loc}}(\mathbb{R}^d, \nu)$.

## 5 A Poincaré Inequality in $BV^{p,\alpha}(\mathbb{R}^d)$

In [3] an inequality similar to (2.2) was established for a function $f$ whose variation measure $|Df|$ belongs to $M^{p,\alpha}(\mathbb{R}^d)$ [ a space of Radon measures containing $T^{p,\alpha}(\mathbb{R}^d)$ (see [5])] but under the supplementary hypotheses : $\frac{1}{\theta} < \frac{1}{d}$ and $\frac{1-\frac{1}{d}}{1-\frac{1}{\theta}} < \frac{1}{p}$. The proof of point 2) of Theorem 2.2 given below is strongly based on ideas developed in [3]. We start by establishing a norm inequality for the maximal fractional operator $m_\beta (1 \le \beta \le \infty)$ defined in the beginning of Section 4.

**Proposition 5.1.** *Let us assume that :* $0 \le \frac{1}{\beta} < \frac{1}{\alpha} \le 1$, $0 \le \frac{1}{\theta} \le \frac{\alpha}{\beta}$, $\frac{1}{p} = \frac{\frac{1}{\alpha}-\frac{1}{\beta}}{1-\frac{1}{\theta}}$ *and* $\mu$ *and* $\nu$ *are non negative Radon measures on* $\mathbb{R}^d$*. Then*

$$ {}_\nu\|m_\beta\mu\|^*_{p,\infty} \quad \le \quad 5^{d(\frac{1}{\alpha}-\frac{1}{\beta})}\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|\mu\|_{T^{p,\alpha}}. \tag{5.1}$$

*Proof.* The claim is trivially true when $\|\mu\|_{T^{p,\alpha}} = \infty$. Thus we may assume that $\|\mu\|_{T^{p,\alpha}} < \infty$.
Let us consider a real number $\lambda > 0$ and set $E_\lambda = \{x \in \mathbb{R}^d \ / \ m_\beta\mu(x) > \lambda\}$.
An argument similar to point a) of the proof of Proposition 4.2 shows that there exists a family $\{Q_i\}_{i\in I}$ of pairwise disjoint cubes of $\mathbb{R}^d$ satisfying :

$$E_\lambda \subset \bigcup_{i\in I} 5Q_i \quad \text{and} \quad 1 < \lambda^{-1}|Q_i|^{\frac{1}{\beta}-1}\mu(Q_i) \quad \text{for} \quad i \in I.$$

Therefore we have

$$\begin{aligned}
\nu(E_\lambda) &\le \sum_{i\in I}\nu(5Q_i) \le \sum_{i\in I}\nu(5Q_i)[\lambda^{-1}|Q_i|^{\frac{1}{\beta}-1}\mu(Q_i)]^p \\
\nu(E_\lambda) &\le \lambda^{-p}5^{d(1-\frac{1}{\theta})}\sum_{i\in I}|5Q_i|^{\frac{1}{\theta}-1}\nu(5Q_i)[|Q_i|^{\frac{1}{\alpha}-1}\mu(Q_i)]^p \\
\nu(E_\lambda) &\le \lambda^{-p}5^{d(1-\frac{1}{\theta})}\|\nu\|_{T^{\infty,\theta}}\|\mu\|^p_{T^{p,\alpha}} \\
\lambda\nu(E_\lambda)^{\frac{1}{p}} &\le 5^{d(\frac{1}{\alpha}-\frac{1}{\beta})}\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|\mu\|_{T^{p,\alpha}}.
\end{aligned}$$

This inequality being true for any real number $\lambda > 0$, we have

$$ {}_\nu\|m_\beta\mu\|^*_{p,\infty} \le 5^{d(\frac{1}{\alpha}-\frac{1}{\beta})}\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|\mu\|_{T^{p,\alpha}}.$$

□

*Proof of Proposition 2.6.* We notice that

$$\begin{cases} 0 < \frac{\frac{1}{\alpha}-\gamma}{1-\frac{1}{\theta}} < \frac{\frac{1}{\alpha}-\frac{1}{\beta}}{1-\frac{1}{\theta}} \quad , \quad \beta > \frac{1}{\gamma} \\ \lim_{\beta\to\frac{1}{\gamma}} \frac{\frac{1}{\alpha}-\frac{1}{\beta}}{1-\frac{1}{\theta}} = \frac{\frac{1}{\alpha}-\gamma}{1-\frac{1}{\theta}} \quad \text{and} \quad \lim_{\beta\to\frac{1}{\gamma}} \frac{\alpha}{\beta} = \alpha\gamma > \frac{1}{\theta} \end{cases}$$

Therefore there is a real number $\beta_0 > \frac{1}{\gamma}$ such that

$$\frac{1}{\theta} < \frac{\alpha}{\beta} \quad \text{and} \quad \frac{\frac{1}{\alpha} - \frac{1}{\beta}}{1 - \frac{1}{\theta}} < \frac{1}{p}, \quad \beta \in (\frac{1}{\gamma}, \beta_0].$$

Let us consider a fixed element $\beta$ of $(\frac{1}{\gamma}, \beta_0]$. We have $0 \leq \frac{1}{\beta} < \gamma < \frac{1}{\alpha} \leq 1$.
Thus, by Welland Inequality (see[10]) there is a real number $D > 0$ not depending on $\mu$, such that :

$$\begin{aligned} I_\gamma\mu(x) &\leq D[m_\beta\mu(x)]^{\frac{\frac{1}{\alpha}-\gamma}{\frac{1}{\alpha}-\frac{1}{\beta}}}[m_\alpha\mu(x)]^{\frac{\gamma-\frac{1}{\beta}}{\frac{1}{\alpha}-\frac{1}{\beta}}}, \quad x \in \mathbb{R}^d \\ I_\gamma\mu(x) &\leq D[m_\beta\mu(x)]^{\frac{\frac{1}{\alpha}-\gamma}{\frac{1}{\alpha}-\frac{1}{\beta}}}\|\mu\|_{T^{\infty,\alpha}}^{\frac{\gamma-\frac{1}{\beta}}{\frac{1}{\alpha}-\frac{1}{\beta}}}, \quad x \in \mathbb{R}^d. \end{aligned}$$

Therefore

$$\nu(\{x \in \mathbb{R}^d \ / \ I_\gamma\mu(x) > t\}) \leq \nu(\{x \in \mathbb{R}^d \ / \ m_\beta\mu(x) > (D^{-1}t)^{\frac{\frac{1}{\alpha}-\frac{1}{\beta}}{\frac{1}{\alpha}-\gamma}}\|\mu\|_{T^{\infty,\alpha}}^{-\frac{\gamma-\frac{1}{\beta}}{\frac{1}{\alpha}-\gamma}}\}), t \in (0,\infty).$$

From this inequality and Proposition 5.1 we deduce

$$\begin{aligned} \nu(\{x \in \mathbb{R}^d \ / \ I_\gamma\mu(x) > t\}) &\leq 5^{d(1-\frac{1}{\theta})}\|\nu\|_{T^{\infty,\theta}}(Dt^{-1})^q\|\mu\|_{T^{\infty,\alpha}}^{q\frac{\gamma-\frac{1}{\beta}}{\frac{1}{\alpha}-\frac{1}{\beta}}}\|\mu\|^{\frac{1-\frac{1}{\theta}}{\frac{1}{\alpha}-\frac{1}{\beta}}}_{T^{\frac{1-\frac{1}{\theta}}{\frac{1}{\alpha}-\frac{1}{\beta}},\alpha}}, \ t \in (0,\infty) \\ \nu(\{x \in \mathbb{R}^d \ / \ I_\gamma\mu(x) > t\}) &\leq 5^{d(1-\frac{1}{\theta})}\|\nu\|_{T^{\infty,\theta}}(Dt^{-1})^q\|\mu\|^q_{T^{\frac{1-\frac{1}{\theta}}{\frac{1}{\alpha}-\frac{1}{\beta}},\alpha}}, \ t \in (0,\infty). \end{aligned}$$

Hence,

$${}_\nu\|I_\gamma\mu\|^*_{q,\infty} \leq 5^{d(\frac{1}{\alpha}-\gamma)}D\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\|\mu\|_{T^{\frac{1-\frac{1}{\theta}}{\frac{1}{\alpha}-\frac{1}{\beta}},\alpha}}.$$

Furthermore we have $p < \frac{1-\frac{1}{\theta}}{\frac{1}{\alpha}-\frac{1}{\beta}}$ and therefore $\| \ \|_{T^{\frac{1-\frac{1}{\theta}}{\frac{1}{\alpha}-\frac{1}{\beta}},\alpha}} \leq \| \ \|_{T^{p,\alpha}}$.

Thus

$${}_\nu\|I_\gamma\mu\|^*_{q,\infty} \leq B\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\|\mu\|_{T^{p,\alpha}},$$

where $B = 5^{d(\frac{1}{\alpha}-\gamma)}D$. □

Beside the Welland inequality used in the proof of Proposition 2.6 there is another control of the fractional integral operator by the fractional maximal operator. Actually the following result was established in the proof of Proposition 2.7 of [3].

**Proposition 5.2.** *Let us assume that :*
$0 < \gamma < \frac{1}{\alpha} \leq 1$, $0 \leq \frac{1}{\theta} < 1$, $\frac{1}{p} \leq \min(1, \frac{1-\gamma}{1-\frac{1}{\theta}})$ *and* $\nu$ *a non negative element of* $T^{\infty,\theta}(\mathbb{R}^d)$ *satisfying the condition* $(A_\infty)$ *in Proposition 2.7. Then there is a real number* $C > 0$ *such that, for any non negative Radon measure* $\mu$ *on* $\mathbb{R}^d$

$${}_\nu\|I_\gamma\mu\|^*_{p,\infty} \leq C \, {}_\nu\|m_{\frac{1}{\gamma}}\mu\|^*_{p,\infty}. \tag{5.2}$$

*Proof of Proposition 2.7.* The desired result is obtained by a direct application of Proposition 5.1 and Proposition 5.2. □

A weighted form of the Hardy-Littlewood-Sobolev inequality for fractional integral in Lebesgue spaces is obtained from Proposition 2.7 as follows.

**Corollary 5.3.** *Let us assume that :* $0 < \gamma < \frac{1}{\alpha} \leq 1$, $0 \leq \frac{1}{\theta} \leq \gamma\alpha$, $\frac{1}{p} = \frac{\frac{1}{\alpha}-\gamma}{1-\frac{1}{\theta}}$ *and* $\nu$ *is a non negative element of* $T^{\infty,\theta}(\mathbb{R}^d)$ *satisfying the condition* $(A_\infty)$ *in Proposition 2.7. Then, for any element* $f$ *of* $F(1,p,\alpha)$

*a)* $I_\gamma f(x) = \int_{\mathbb{R}^d} |x-y|^{d(\gamma-1)} f(y)dy$ *converges (absolutely) for* $\nu$*-almost every element* $x$ *of* $\mathbb{R}^d$*;*

*b)*

$$_\nu\|I_\gamma f\|^*_{p,\infty} \quad \leq \quad C\,\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|f\|_{F(1,p,\alpha)} \leq C\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|f\|_\alpha. \tag{5.3}$$

*Proof.* Let us consider an element $f$ of $L^1_{loc}(\mathbb{R}^d)$.
The Radon measure $\mu_f$, defined by $d\mu_f(x) = f(x)dx$, satisfies : $I_\gamma\mu_{|f|} = I_\gamma|f|$
Therefore by Proposition 2.7

$$_\nu\|I_\gamma|f|\,\|^*_{p,\infty} \quad \leq \quad C\,\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|\mu_{|f|}\|_{T^{p,\alpha}} = C\,\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|f\|_{F(1,p,\alpha)}. \tag{5.4}$$

where $C$ is a real number not depending on $f$.

a) Let us suppose that $f$ belongs to $F(1,p,\alpha)$.
By the inequality (5.4) we have, for any real number $t > 0$

$$\nu(\{x \in \mathbb{R}^d \,/\, I_\gamma|f|(x) > t\}) \quad \leq \quad t^{-p}C^p\|\nu\|_{T^{\infty,\theta}}\|f\|^p_{F(1,p,\alpha)}.$$

Furtheremore

$$C^p\|\nu\|_{T^{\infty,\theta}}\|f\|^p_{F(1,p,\alpha)} < \infty$$

Therefore, for $\nu$-almost element $x$ of $\mathbb{R}^d$

$$I_\gamma|f|(x) = \int_{\mathbb{R}^d} |x-y|^{d(\gamma-1)}|f(y)|dy < \infty$$

and so,
$I_\gamma f(x) = \int_{\mathbb{R}} |x-y|^{d(\gamma-1)} f(y)dy$ converges absolutely and satisfies

$$|I_\gamma f(x)| \leq I_\gamma|f|(x).$$

b) From the inequality above, inequality (5.4) and point 3) of Proposition 2.4, we get

$$\|I_\gamma f\|^*_{p,\infty} \leq \|I_\gamma|f|\|^*_{p,\infty} \leq C\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|f\|_{F(1,p,\alpha)} \leq C\|\nu\|^{\frac{1}{p}}_{T^{\infty,\theta}}\|f\|_\alpha.$$

□

**Corollary 5.4.** *Let us assume that :* $0 < \gamma < \frac{1}{\alpha} \leq 1, 0 \leq \frac{1}{\theta} < \gamma\alpha, \frac{1}{p} = \frac{\frac{1}{\alpha}-\gamma}{1-\frac{1}{\theta}}$ *and* $\nu$ *is a non negative element of* $T^{\infty,\theta}(\mathbb{R}^d)$ *satisfying the condition* $(A_\infty)$ *in Proposition 2.7. Then there is a real number* $C > 0$ *such that*

$$_\nu\|I_\gamma f\|_p \quad \leq \quad C\|\nu\|_{T^{\infty,\theta}}^{\frac{1}{p}}\|f\|_\alpha, \quad f \in L^\alpha(\mathbb{R}^d). \tag{5.5}$$

*Proof.* Let us consider $\alpha_0$ and $\alpha_1$ such that $\frac{1}{\theta} < \gamma\alpha_1 < \gamma\alpha < \gamma\alpha_0 \leq 1$ and set $\frac{1}{p_i} = \frac{\frac{1}{\alpha_i}-\gamma}{1-\frac{1}{\theta}}$ for $i \in \{0, 1\}$.
By Corollary 5.3, for $i \in \{0, 1\}$ there is a real number $C_i > 0$ such that

$$_\nu\|I_\gamma f\|_{p_i,\infty}^* \leq C_i\|\nu\|_{T^{\infty,\theta}}^{\frac{1}{p_i}}\|f\|_{\alpha_i}, \quad f \in L^{\alpha_i}(\mathbb{R}^d).$$

Furthermore there is an element $s$ of $(0, 1)$ such that

$$\frac{1}{\alpha} = \frac{1-s}{\alpha_0} + \frac{s}{\alpha_1} \text{ and } \frac{1}{p} = \frac{1-s}{p_0} + \frac{s}{p_1}.$$

Therefore, by Marcinckiewicz interpolation theorem, there is a real number $D > 0$ such that

$$_\nu\|I_\gamma f\|_p \leq D(C_0\|\nu\|_{T^{\infty,\theta}}^{\frac{1}{p_0}})^{1-s}(C_1\|\nu\|_{T^{\infty,\theta}}^{\frac{1}{p_1}})^s\|f\|_\alpha = C\|\nu\|_{T^{\infty,\theta}}^{\frac{1}{p}}\|f\|_\alpha, \quad f \in L^\alpha(\mathbb{R}^d).$$

□

In the proof of Theorem 2.2 we shall use the following result.

**Lemma 5.5.** *Let us assume that* $f$ *belongs to* $BV_{loc}(\mathbb{R}^d)$ *and* $1 \leq \alpha \leq p \leq \infty$*. Then*

$$\|\,|\nabla f^\varepsilon|\,\|_{F(1,p,\alpha)} \quad \leq \quad \|\,|Df|\,\|_{T^{p,\alpha}(\mathbb{R}^d)}, \quad \varepsilon \in (0, \infty).$$

*Proof.* Let us consider a real number $\varepsilon > 0$.

1) By Lemma 3.1, we have

$$\int_{\Omega(x,r)} |\nabla f^\varepsilon(y)|dy \leq \int_{\mathbb{R}^d} |Df|(\Omega(x,r) - y)\rho_\varepsilon(y)dy, \quad (x,r) \in \mathbb{R}^d \times (0, \infty).$$

Therefore

$$\int_{Q(x,r)} |\nabla f^\varepsilon(y)|dy = \int_{\Omega(x,r)} |\nabla f^\varepsilon(y)|dy \leq \int_{\mathbb{R}^d} |Df|(Q(x,r)-y)\rho_\varepsilon(y)dy, \quad (x,r) \in \mathbb{R}^d \times (0, \infty).$$

Thus for any cube $Q$ of $\mathbb{R}^d$, we have

$$|Q|^{\frac{1}{\alpha}-1}\|\,|\nabla f^\varepsilon|\chi_Q\|_1 \quad \leq \quad \int_{\mathbb{R}^d} |Q|^{\frac{1}{\alpha}-1}|Df|(Q - y)\rho_\varepsilon(y)dy \tag{5.6}$$

$$|Q|^{\frac{1}{\alpha}-1}\|\,|\nabla f^\varepsilon|\chi_Q\|_1 \leq \|\,|Df|\,\|_{T^{\infty,\alpha}} \int_{\mathbb{R}^d} \rho_\varepsilon(y)dy = \|\,|Df|\,\|_{T^{\infty,\alpha}}.$$

That is

$$\|\,|\nabla f^\varepsilon|\,\|_{F(1,\infty,\alpha)} \leq \|\,|Df|\,\|_{T^{\infty,\alpha}}.$$

2) Let us suppose that $p < \infty$ and consider a family $\{Q_i\}_{i\in I}$ of mutually disjoint cubes of $\mathbb{R}^d$. We have, by inequality (5.6)

$$\left\{\sum_{i\in I}[|Q_i|^{\frac{1}{\alpha}-1}\| \, |\nabla f^{\varepsilon}|\chi_{Q_i}\|_1]^p\right\}^{\frac{1}{p}} \leq \left\{\sum_{i\in I}[|Q_i|^{\frac{1}{\alpha}-1}\int_{\mathbb{R}^d}|Df|(Q_i-y)\rho_{\varepsilon}(y)dy]^p\right\}^{\frac{1}{p}}$$

And therefore, by Minkowski inequality

$$\begin{aligned}\left\{\sum_{i\in I}[|Q_i|^{\frac{1}{\alpha}-1}\| \, |\nabla f^{\varepsilon}|\chi_{Q_i}\|_1]^p\right\}^{\frac{1}{p}} &\leq \int_{\mathbb{R}^d}\{\sum_{i\in I}[|Q_i|^{\frac{1}{\alpha}-1}|Df|(Q_i-y)]^p\}^{\frac{1}{p}}\rho_{\varepsilon}(y)dy \\ &\leq \int_{\mathbb{R}^d}\| \, |Df| \, \|_{T^{p,\alpha}}\rho_{\varepsilon}(y)dy = \| \, |Df| \, \|_{T^{p,\alpha}}.\end{aligned}$$

Thus

$$\| \, |\nabla f^{\varepsilon}| \, \|_{F(1,p,\alpha)} \leq \| \, |Df| \, \|_{T^{p,\alpha}}.$$

□

*Proof of Theorem 2.2.* 1) The assertion is just point 2) of Proposition 3.4.

2) Let us set $g = f - f_{(\infty)}$.
We notice that $g$ belongs to $BV^{p,\alpha}(\mathbb{R}^d)$ with $|Dg| = |Df|$ and $g_{(\infty)} = 0$.
Let us consider two real numbers $\varepsilon > 0$ and $r > 0$.

a) We know that $g^{\varepsilon} = \rho_{\varepsilon} * g$ belongs to $C^1_c(\mathbb{R}^d)$ and therefore satisfies :

$$|g^{\varepsilon} - g^{\varepsilon}{}_{(0,r)}|\chi_{Q(0,r)} \leq d^{\frac{d}{2}-1}I_{\frac{1}{d}}(|\nabla g^{\varepsilon}|\chi_{Q(0,r)})\chi_{Q(0,r)} \leq d^{\frac{d}{2}-1}I_{\frac{1}{d}}(|\nabla g^{\varepsilon}|)$$

(see Lemma 7.16 of [7]).
From the inequality above and Proposition 2.6 we deduce that there is a real number $C > 0$, not depending on $\nu$, $f$, $\varepsilon$ and $r$, such that

$$_{\nu}\|(g^{\varepsilon} - g^{\varepsilon}{}_{(0,r)})\chi_{Q(0,r)}\|^*_{q,\infty} \leq C\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\| \, |\nabla g^{\varepsilon}| \, \|_{F(1,p,\alpha)}$$

and therefore, by Lemma 5.5

$$_{\nu}\|(g^{\varepsilon} - g^{\varepsilon}{}_{(0,r)})\chi_{Q(0,r)}\|^*_{q,\infty} \leq C\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\| \, |Dg| \, \|_{T^{p,\alpha}} = C\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\| \, |Df| \, \|_{T^{p,\alpha}}.$$

b) Let us consider $u$ and $\upsilon$ such that $1 \leq u < q$ and $\frac{1}{\upsilon} = \frac{1}{u} - \frac{1}{q}$ and a $\nu$-measurable subset $E$ of $\mathbb{R}^d$ satisfying $0 < \nu(E) < \infty$.
By the inequality above and Kolmogorov condition (see Lemma 2.8 on page 485 of [6]) we have

$$_{\nu}\|(g^{\varepsilon} - g^{\varepsilon}{}_{(0,r)})\chi_{E\cap Q(0,r)}\|_u \leq K(\frac{q}{q-u})^{\frac{1}{u}}\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\nu(E)^{\frac{1}{\upsilon}}\| \, |Df| \, \|_{T^{p,\alpha}}$$

where $K$ is a real number not depending on $\nu$, $f$, $\varepsilon$, $u$, $\upsilon$ and $E$.
We notice that

$$|g^{\varepsilon}|\chi_{E\cap Q(0,r)} \leq |g^{\varepsilon} - g^{\varepsilon}_{(0,r)}|\chi_{E\cap Q(0,r)} + |g^{\varepsilon}{}_{(0,r)}|\chi_{E\cap Q(0,r)}$$

and therefore
${}_{\nu}\|g^{\varepsilon}\chi_{E\cap Q(0,r)}\|_u \leq K(\frac{q}{q-u})^{\frac{1}{u}}\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\nu(E)^{\frac{1}{v}}\|\,|Df|\,\|_{T^{p,\alpha}} + |g^{\varepsilon}{}_{(0,r)}|\nu(E)^{\frac{1}{u}}.$
By Proposition 3.5, $g^{\varepsilon}{}_{(\infty)} = g_{(\infty)} = 0$. From the inequality above and the monotone convergence Theorem, we get

$$ {}_{\nu}\|g^{\varepsilon}\chi_E\|_u \leq K(\frac{q}{q-u})^{\frac{1}{u}}\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\nu(E)^{\frac{1}{v}}\|\,|Df|\,\|_{T^{p,\alpha}}. $$

By Corollary 4.4,
$\lim_{\varepsilon\to 0} g^{\varepsilon}(x) = g^*(x) = f^*(x) - f_{(\infty)},\;\; \nu-$almost every $x \in \mathbb{R}^d$.
Therefore from the inequality above and Fatou Lemma, we obtain

$$ {}_{\nu}\|(f^* - f_{(\infty)})\chi_E\|_u \leq K(\frac{q}{q-u})^{\frac{1}{u}}\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\nu(E)^{\frac{1}{v}}\|\,|Df|\,\|_{T^{p,\alpha}}; $$

that is, by Remark 4.5,

$$ {}_{\nu}\|(f - f_{(\infty)})\chi_E\|_u \leq K(\frac{q}{q-u})^{\frac{1}{u}}\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\nu(E)^{\frac{1}{v}}\|\,|Df|\,\|_{T^{p,\alpha}}. $$

Thus by Kolmogorov condition we have

$$ {}_{\nu}\|(f - f_{(\infty)})\chi_E\|^*_{q,\infty} \leq A\|\nu\|^{\frac{1}{q}}_{T^{\infty,\theta}}\|\,|Df|\,\|_{T^{p,\alpha}} $$

where $A$ is a real number not depending on $f$ and $\nu$.

□

*Proof of Theorem 2.3.* The result is obtained by almost the same argumentation than that in the Proof of Theorem 2.2. We need only to

i) use Proposition 2.7 instead of Proposition 2.6 [ clearly the Lebesgue measure on $\mathbb{R}^d$ is a non negative element of $T^{\infty,\infty}(\mathbb{R}^d)$ satisfying the condition $(A_\infty)$ in Proposition 2.7]

ii) notice that, by the Lebesgue differentiation theorem we have

$$ \lim_{\varepsilon\to 0}(f - f_{(\infty)})^{\varepsilon} = \lim_{\varepsilon\to 0} f^{\varepsilon} - f_{(\infty)} = f - f_{(\infty)} $$

Lebesgue almost everywhere.

□

## 6 Banach space structure of $BV^{p,\alpha}(\mathbb{R}^d)$

We assume that : $1 < \alpha < d$ and $\frac{1}{\alpha} - \frac{1}{d} = \frac{1}{p}$.

**Notation 6.1.** For any element $f$ of $BV^{p,\alpha}(\mathbb{R}^d)$

$$ \|f\|_{BV^{p,\alpha}} = \|\,|Df|\,\|_{T^{p,\alpha}} + |f_{(\infty)}|. \tag{6.1} $$

**Proposition 6.2.** *$(BV^{p,\alpha}(\mathbb{R}^d), \|\cdot\|_{BV^{p,\alpha}})$ is a Banach space.*

*Proof.* • From the definition of Sobolev spaces and Proposition 2.4 it follows easily that

$$W^{1,\alpha}(\mathbb{R}^d) = \{f \in L^1_{\text{loc}}(\mathbb{R}^d) \ / \ |\nabla f| \in L^\alpha(\mathbb{R}^d)\} \subset BV^{p,\alpha}(\mathbb{R}^d) \subset BV_{loc}(\mathbb{R}^d)$$

and therefore $BV^{p,\alpha}(\mathbb{R}^d)$ is non void.
It is easy to see that for any element $f$ of $BV^{p,\alpha}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$,
$\|f\|_{BV^{p,\alpha}} \in \mathbb{R}$ , $\|f\|_{BV^{p,\alpha}} = 0 \Leftrightarrow f = 0$ and $\|\lambda f\|_{BV^{p,\alpha}} = |\lambda|\|f\|_{BV^{p,\alpha}}$.
Let $f$ and $g$ be elements of $BV^{p,\alpha}(\mathbb{R}^d)$.
An application of the definition of variation measure yields, for any open subset $O$ of $\mathbb{R}^d$

$$|D(f+g)|(O) \leq |Df|(O) + |Dg|(O).$$

As $|Df|$ and $|Dg|$ are non negative Radon measures, the inequality above induces that, for any cube $Q$ of $\mathbb{R}^d$,

$$|D(f+g)|(Q) \leq |Df|(Q) + |Dg|(Q).$$

Let $\{Q_i\}_{i\in I}$ be an element of $S$. We have

$$|Q_i|^{\frac{1}{\alpha}-1}|D(f+g)|(Q_i) \leq |Q_i|^{\frac{1}{\alpha}-1}|Df|(Q_i) + |Q_i|^{\frac{1}{\alpha}-1}|Dg|(Q_i), \quad i \in I$$

and therefore

$$\left\{\sum_{i\in I}[|Q_i|^{\frac{1}{\alpha}-1}|D(f+g)|(Q_i)]^p\right\}^{\frac{1}{p}} \leq \left\{\sum_{i\in I}[|Q_i|^{\frac{1}{\alpha}-1}|Df|(Q_i)]^p\right\}^{\frac{1}{p}} + \left\{\sum_{i\in I}[|Q_i|^{\frac{1}{\alpha}-1}|Dg|(Q_i)]^p\right\}^{\frac{1}{p}}$$

$$\left\{\sum_{i\in I}[|Q_i|^{\frac{1}{\alpha}-1}|D(f+g)|(Q_i)]^p\right\}^{\frac{1}{p}} \leq \|\,|Df|\,\|_{T^{p,\alpha}} + \|\,|Dg|\,\|_{T^{p,\alpha}}.$$

Thus

$$\|\,|D(f+g)|\,\|_{T^{p,\alpha}} \leq \|\,|Df|\,\|_{T^{p,\alpha}} + \|\,|Dg|\,\|_{T^{p,\alpha}}.$$

This implies clearly

$$\|\,|D(f+g)|\,\|_{BV^{p,\alpha}} \leq \|\,|Df|\,\|_{BV^{p,\alpha}} + \|\,|Dg|\,\|_{BV^{p,\alpha}}.$$

So $(BV^{p,\alpha}(\mathbb{R}^d), \|\|_{BV^{p,\alpha}})$ is a normed space.
• Let $(f_n)_{n\geq 0}$ be a Cauchy sequence in $BV^{p,\alpha}(\mathbb{R}^d)$ and let us set $g_n = f_n - f_{n(\infty)}$ for all $n \geq 0$.
Clearly we have, for any integer $n \geq 0$

$$|Dg_n| = |Df_n| \quad \text{and} \quad g_{n(\infty)} = 0$$

and therefore

$$\begin{cases} \|g_n\|_{BV^{p,\alpha}} = \|\,|Df_n|\,\|_{T^{p,\alpha}} \text{ and } \|g_n\|^*_{p,\infty} \leq A\|\,|Df_n|\,\|_{T^{p,\alpha}}, & n \in \mathbb{N} \\ \|g_n - g_m\|_{BV^{p,\alpha}} = \|\,|D(f_n - f_m)|\,\|_{T^{p,\alpha}}, & n \in \mathbb{N} \end{cases}$$

where $A$ is a real number not depending on $(f_n)_{n\geq 0}$ (see Theorem 2.3).

a) From the remarks made above, it is clear that $(g_n)_{n\geq 0}$ is Cauchy sequence in the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^d)$ which is complete. Therefore there is an element $g$ of $L^{p,\infty}(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \|g - g_n\|^*_{p,\infty} = 0.$$

Furthermore $(g_n)_{n\geq 0}$ is Cauchy sequence in $BV^{p,\alpha}(\mathbb{R}^d)$ and therefore

$$\sup_{n\geq 0} \| \, |Dg_n| \, \|_{T^{p,\alpha}} = \sup_{n\geq 0} \|g_n\|_{BV^{p,\alpha}} = M < \infty.$$

b) Let $O$ be a bounded open subset of $\mathbb{R}^d$ and $Q$ a cube of $\mathbb{R}^d$ such that : $O \subset Q$.
The sequence $(g_n)_{n\geq 0}$ converges to $g$ in $L^1(O)$ and therefore
$|Dg|(O) \leq \liminf_{n\to\infty} |Dg_n|(O)$ (see Theorem 1 in Section 5.2.1 [4]).
We have also, for any integer $n \geq 0$,

$$|Dg_n|(O) \leq |Dg_n|(Q) \leq |Q|^{1-\frac{1}{\alpha}} \| \, |Dg_n| \, \|_{T^{p,\alpha}} = |Q|^{1-\frac{1}{\alpha}} M.$$

Therefore we have

$$|Dg|(O) \leq |Q|^{1-\frac{1}{\alpha}} M < \infty \quad \text{and} \quad g \in BV(O).$$

For any integer $n$, $(g_m - g_n)_{m\geq 0}$ converges to $g - g_n$ in $L^1(O)$ and therefore

$$|D(g - g_n)|(O) \leq \liminf_{m\to\infty} |D(g_m - g_n)|(O).$$

Furthermore

$$|D(g_m - g_n)|(O) \quad \leq \quad |Q|^{1-\frac{1}{\alpha}} \| \, |D(g_m - g_n)| \, \|_{T^{p,\alpha}} = |Q|^{1-\frac{1}{\alpha}} \| \, |D(g_m - g_n)| \, \|_{BV^{p,\alpha}}, \ n \geq 0, \ m \geq 0,$$

and $(g_n)_{n\geq 0}$ is a Cauchy sequence in $BV^{p,\alpha}(\mathbb{R}^d)$.
Therefore, for any real number $\varepsilon > 0$ there is $n_\varepsilon \geq 0$ such that

$$|D(g_m - g_n)|(O) < \varepsilon, \quad n \geq n_\varepsilon \quad \text{and} \quad m \geq n_\varepsilon$$

$$\sup_{m\geq n_\varepsilon} |D(g_m - g_n)|(O) < \varepsilon, \quad n \geq n_\varepsilon$$

$$|D(g - g_n)|(O) \leq \liminf_{m\to\infty} |Df(g_m - g_n)|(O) \leq \varepsilon, \quad n \geq n_\varepsilon.$$

Thus we have

$$\lim_{n\to\infty} |D(g - g_n)|(O) = 0.$$

We notice also that

$$|Dg_n|(O) \leq |D(g_n - g)|(O) + |Dg|(O), \quad n \geq 0.$$

$$\limsup_{n\to\infty} |Dg_n|(O) \leq \lim_{n\to\infty} |D(g_n - g)|(O) + |Dg|(O) = |Dg|(O).$$

Combining this result with inequality

$$|Dg|(O) \leq \liminf_{n\to\infty} |Dg_n|(O),$$

we get

$$|Dg|(O) = \lim_{n\to\infty} |Dg_n|(O).$$

c) Let $Q$ be a cube of $\mathbb{R}^d$. For any integers $n \geq 0$ and $m \geq 0$ we have

$$||Dg_n|(Q) - |Dg_m|(Q)| \leq |D(g_n - g_m)|(Q) \quad \leq \quad |Q|^{1-\frac{1}{\alpha}} \| \, |D(g_n - g_m)| \, \|_{T^{p,\alpha}}$$

and therefore $(|Dg_n|(Q))_{n\geq 0}$ converges in $\mathbb{R}$.
Let us consider a real number $\varepsilon > 0$. There is a sequence $(O_n)_{n\geq 0}$ of open subset of $\mathbb{R}^d$ satisfying :

$$\begin{cases} Q \subset O_1, \ |Dg|(O_1) < |Dg|(Q) + \varepsilon \ and \ |Dg_1|(O_1) < |Dg_1|(Q) + 2^{-1}\varepsilon \\ Q \subset O_{n+1} \subset O_n, \quad |Dg_{n+1}|(O_{n+1}) < |Dg_{n+1}|(Q) + 2^{-n-1}\varepsilon, \quad n \geq 0. \end{cases}$$

We have :

$$|Dg|(Q) \leq |Dg|(O_n) \leq |D(g - g_n)|(O_n) + |Dg_n|(O_n) \quad \leq \quad |D(g - g_n)|(O_1) + |Dg_n|(Q) + 2^{-n}\varepsilon, \forall n \geq 0$$

and therefore

$$|Dg|(Q) \leq \lim_{n\to\infty} |Dg_n|(Q).$$

Furthermore

$$|Dg_n|(Q) \leq |Dg_n|(O_1), \quad n \geq 0$$

and therefore, by the result obtained in b),

$$\lim_{n\to\infty} |Dg_n|(Q) \leq \lim_{n\to\infty} |Dg_n|(O_1) = |Dg|(O_1) < |Dg|(Q) + \varepsilon.$$

Thus

$$|Dg|(Q) = \lim_{n\to\infty} |Dg_n|(Q).$$

d) Let $\{Q_i\}_{i\in I}$ be element of $S$ with $I$ finite. We have

$$\begin{aligned} \left\{ \sum_{i\in I} [|Q_i|^{\frac{1}{\alpha}-1} |Dg|(Q_i)]^p \right\}^{\frac{1}{p}} \quad &:= \quad \left\{ \sum_{i\in I} [|Q_i|^{\frac{1}{\alpha}-1} \lim_{n\to\infty} |Dg_n|(Q_i)]^p \right\}^{\frac{1}{p}} = \lim_{n\to\infty} \left\{ \sum_{i\in I} [|Q_i|^{\frac{1}{\alpha}-1} |Dg_n|(Q_i)|]^p \right\}^{\frac{1}{p}} \\ &\leq \quad \limsup_{n\to\infty} \| \, |Dg_n| \, \|_{T^{p,\alpha}} = M < \infty. \end{aligned}$$

Thus

$$\| \, |Dg| \, \|_{T^{p,\alpha}} \leq \limsup_{n\to\infty} \| \, |Dg_n| \, \|_{T^{p,\alpha}} = M < \infty$$

and so $g$ belongs to $BV^{p,\alpha}(\mathbb{R}^d)$.

e) Let us consider a real number $\varepsilon > 0$. There is an integer $n_\varepsilon \geq 0$ such that

$$\|D(g_n - g_m)\|_{T^{p,\alpha}} < \varepsilon, \qquad n \geq n_\varepsilon \quad \text{and} \quad m \geq n_\varepsilon.$$

Let us fix $n \geq n_\varepsilon$ and set $h_m = g_n - g_m$, for any integer $m \geq 0$.
The sequence $(h_m)_{m\geq 0}$ of elements of $BV^{p,\alpha}(\mathbb{R}^d)$ satisfies :

$$\|h_{m'} - h_{m''}\|_{BV^{p,\alpha}} = \|g_{m'} - g_{m''}\|_{BV^{p,\alpha}} \quad , m' \geq 0 \quad , \quad m'' \geq 0$$

and therefore is a Cauchy sequence in $BV^{p,\alpha}(\mathbb{R}^d)$.
By applying the argumentations held in points a), b), c) and d), with $(h_m)_{m\geq 0}$ in place of $(g_m)_{m\geq 0}$, we obtain

$$\|D(g-g_n)\|_{T^{p,\alpha}} \leq \lim_{n\to\infty} \sup \|D(g_m-g_n)\|_{T^{p,\alpha}} \leq \varepsilon$$

that is

$$\|g-g_n\|_{BV^{p,\alpha}} \leq \varepsilon.$$

So $(g_n)_{n\geq 0}$ converges to $g$ in $BV^{p,\alpha}(\mathbb{R}^d)$ and therefore $(f_n)_{n\geq 0}$ converges to $f = g + \lim_{n\to\infty} f_{n(\infty)}$ in $BV^{p,\alpha}(\mathbb{R}^d)$. □

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